The KPZ line ensemble: a marriage of integrability and probability

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Joint work with Alan Hammond
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Introduction to the KPZ equation

The Kardar-Parisi-Zhang (KPZ) stochastic PDE is

$$\partial_t \mathcal{H}(t,x) = \tfrac{1}{2}\partial_x^2 \mathcal{H}(t,x) + \tfrac{1}{2}\left(\partial_x \mathcal{H}(t,x)\right)^2 + \dot{\mathcal{W}},$$

where $\dot{\mathcal{W}}$ is space-time Gaussian white noise.

The equation models a large class of $1D$ interfaces which

- grow randomly
- in a direction perpendicular to the local slope
- and are subject to restoring forces such as surface tension.

The assertion of universality is demonstrated by extensive **numerics**, some **experimental evidence**, and a limited but growing body of **proofs**.

Experimental and numerical evidence for KPZ

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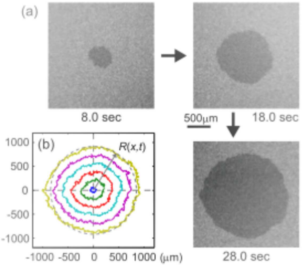
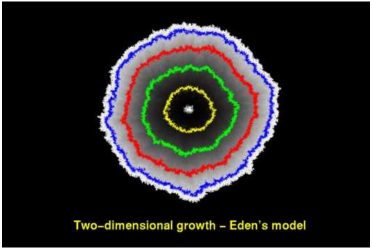


FIG. 1: (Color online) Growing DSM2 cluster. (a) Images. Indicated below is the elapsed time after the emission of laser pulses. (b) Snapshots of the interfaces taken every 5 s in the range $2\,\mathrm{s} \le t \le 27\,\mathrm{s}$. The gray dashed circle shows the mean radius of all the droplets at $t = 27\,\mathrm{s}$. The coordinate x at this time is defined along this circle.



Three eras of KPZ via integrability

- **Physics predictions (1986 - 1999)**: scaling exponents of 1/3 and 2/3 via non-rigorous analysis and numerics;
- **Rigorous zero temperature (1999 - 2009)**: certain totally asymmetric models such as TASEP, LPP and PNG via the RSK correspondence and determinantal point processes;
- **Rigorous positive temperature (2009 - present)**: ASEP, q-TASEP and KPZ via geometric lifting of RSK, Bethe ansatz and Macdonald processes. None of these processes are determinantal.

The solution of the KPZ equation

Define the **Hopf-Cole solution** of the KPZ equation as

$$\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)\,,$$

where $\mathcal{Z}(t,x)$ is the well-posed solution of the stochastic heat equation (SHE) with multiplicative white-noise

$$\partial_t \mathcal{Z}(t,x) = \tfrac{1}{2}\partial_x^2 \mathcal{Z}(t,x) + \dot{\mathcal{W}}(t,x)\mathcal{Z}(t,x)\,.$$

This is the physically relevant notion of solution to KPZ.

Narrow wedge initial data is given by $\mathcal{Z}(0,x) = \delta_{x=0}$. We write $\mathcal{Z}^{\mathrm{nw}}(t,x)$ and $\mathcal{H}^{\mathrm{nw}}(t,x)$ for the corresponding solutions.

Generally, $\mathcal{H}_0$ initial data for KPZ means $\mathcal{Z}(0,x) = e^{\mathcal{H}_0(x)}$.

The era of positive temperature: one-point information

Theorem (Balázs, Quastel and Seppäläinen 2011)

When $\mathcal{H}(0,x)$ is a two-sided Brownian motion, there exist $c_1, c_2 \in (0,\infty)$ for which

$$c_1 t^{1/3} \leq \sqrt{\mathrm{Var}\mathcal{H}(t,x)} \leq c_2 t^{1/3} \ \forall x \in \mathbb{R} \text{ and } t \geq 1 .$$

The scaled narrow wedge KPZ equation solution $\mathfrak{h}^t(x)$ is defined by

$$\mathcal{H}^{\mathrm{nw}}(t,x) = -\frac{t}{24} + t^{1/3}\mathfrak{h}^t\big(t^{-2/3}x\big) .$$

Theorem (Amir, C. and Quastel 2011)

The random variable $\mathfrak{h}^t(0)$ converges in law as $t \to \infty$.

Results with Alan Hammond: spatial regularity

Exploiting a new **line ensemble** structure which I will soon briefly describe, Alan Hammond and I have obtained new information about the solution of the KPZ equation.

Other recent advances in exact solvability of KPZ have concerned one-point distributions. The key contribution of our work is **spatial regularity** which is available even after the coordinate change to the $(1/3, 2/3)$-fluctuation scale.

Application 1: Uniformly Brownian spatial sample paths

Theorem

For each $C > 0$, the process $\mathfrak{h}^t : [-C, C] \to \mathbb{R}$ is absolutely continuous with respect to Brownian bridge, uniformly in $t \geq 1$.

Note that this inference is made in the $(1/3, 2/3)$ fluctuation scale.

Application 2: Universal 1/3 fluctuation exponent

Consider the solution $\mathcal{H}^{(t)}$ of the KPZ equation for some general initial data: $\mathcal{H}^{(t)}(0,x) = \mathcal{H}_0^{(t)}(x)$.

By linearity of the stochastic heat equation, we have that

$$\mathcal{H}^{(t)}(t,0) \overset{(d)}{=} \log \int_{-\infty}^{\infty} e^{\mathcal{H}^{\mathrm{nw}}(t,x)+\mathcal{H}_0^{(t)}(x)}\,\mathrm{d}x\,.$$

Setting $h_0^{(t)}(x) = t^{-1/3}\mathcal{H}_0^{(t)}\big(t^{2/3}x\big)$, we find that

$$\mathcal{H}^{(t)}(t,0) \overset{(d)}{=} -\frac{t}{24} + \frac{2}{3}\log t + \log \int_{-\infty}^{\infty} e^{t^{1/3}\left(\mathfrak{h}^{t}(x)+h_0^{(t)}(x)\right)}\,\mathrm{d}x\,.$$

Application 2: Universal 1/3 fluctuation exponent

The KPZ one-point distribution lives on scale $t^{1/3}$, under very weak hypotheses on the initial data.

Theorem

If the scaled initial data $h_0^{(t)}$

- *grows subparabolically*
- *and is bounded above and below near zero,*

then, for all $\epsilon > 0$, there exists $C_1 > 0$ such that, for all $t \geq 1$,

$$\mathbb{P}\Big(- C_1 t^{1/3} \leq \mathcal{H}^{(t)}(t,0) + \frac{t}{24} \leq C_1 t^{1/3} \Big) > 1 - \epsilon \, .$$

Application 2: Universal 1/3 fluctuation exponent

The same one-point distribution does not concentrate on a scale smaller than $t^{1/3}$.

Theorem

Suppose that the scaled initial data $h_0^{(t)}$ satisfies the same hypotheses.

For all $\epsilon > 0$, there exists $C_2 > 0$ such that, for all $\eta > 0$, $t \geq 1$ and $x \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{\mathcal{H}^{(t)}(t,0) + t/24}{t^{1/3}} \in \big(x, x+\eta\big) \right) \leq C_2 \eta + \epsilon \, .$$

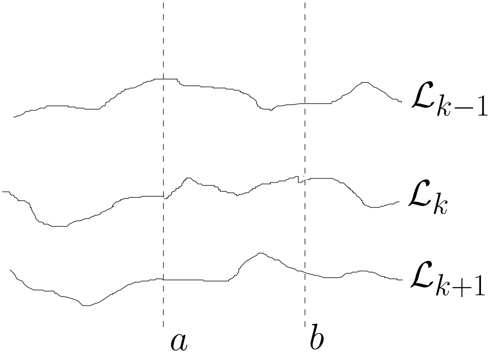
KPZ line ensemble: the structure underlying the results

We construct and analyze a family of $\mathbb{N}$-indexed ensembles of curves called KPZ_t line ensembles. The lowest indexed curve is distributed as the time t narrow wedge KPZ solution. The ensemble enjoys a Gibbsian resampling property. These ensembles display uniform regularity under $(1/3, 2/3)$ scaling as $t \to \infty$.

Existence of such ensembles is not obvious. Certain algebraic structure enjoyed by a prelimiting regularization of the KPZ equation is recast as a line ensemble with a resampling property. The algebraic structure does not clearly survive the limit, but we show that the line ensemble and its resampling property does.

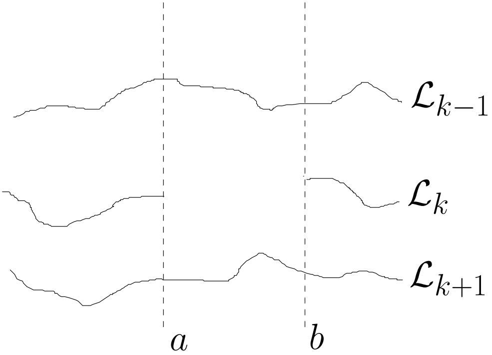
The $\mathbf{H}_\infty$-Brownian Gibbs property

The curve $\mathcal{L}_k$ around $[a, b]$ and the two adjacently indexed curves.



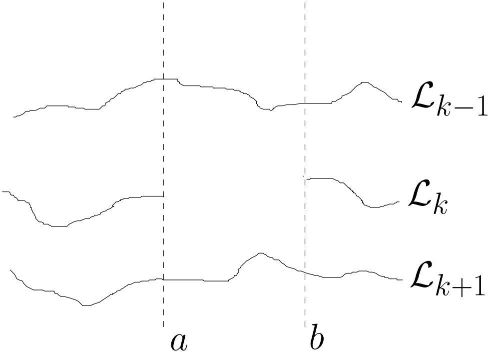
The $\mathbf{H}_\infty$-Brownian Gibbs property

Remove the curve $\mathcal{L}_k$ on $[a, b]$.



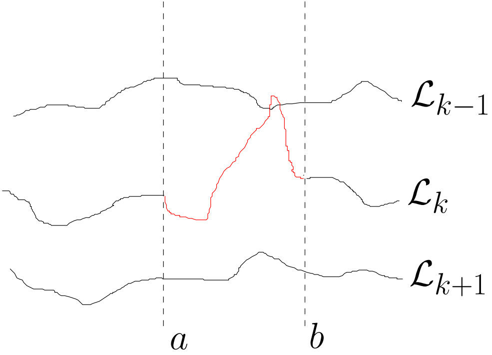
The $\mathbf{H}_\infty$-Brownian Gibbs property

What is the conditional law of $\mathcal{L}_k$ on $[a, b]$ after the removal?



The $\mathbf{H}_{\infty}$-Brownian Gibbs property

It is Brownian bridge conditioned to avoid the neighboring curves.



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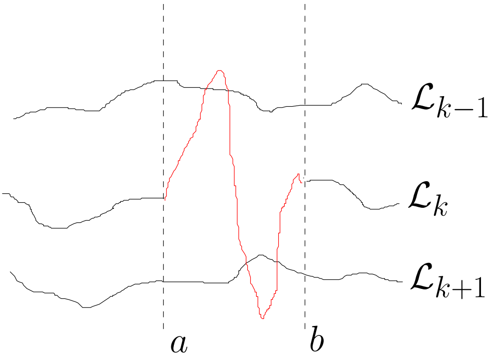

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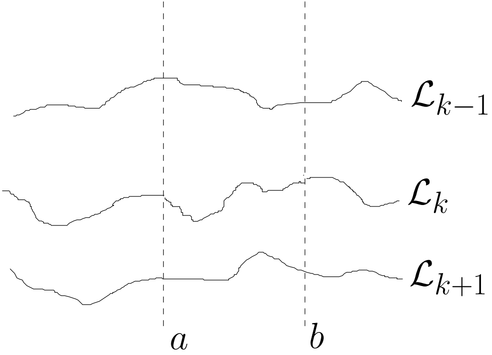
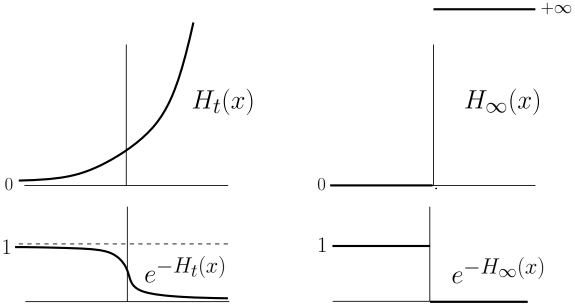


The $\mathbf{H}$-Brownian Gibbs property

The general definition of the $\mathbf{H}$-Brownian Gibbs property involves a continuous function $\mathbf{H} : \mathbb{R} \to [0, \infty)$.

We will focus on:

$\mathbf{H}_t(x) = e^{t^{1/3}x}$, with $t \in (0, \infty)$, and $\mathbf{H}_\infty(x) = +\infty \mathbf{1}_{x \geq 0}$.



The $\mathbf{H}$-Brownian Gibbs property

Definition

The line ensemble $\{\mathcal{L}_i : i \in \mathbb{N}\}$ has the $\mathbf{H}$-Brownian Gibbs property if: for any $k \in \mathbb{N}$ and $a, b \in \mathbb{R}$, $a < b$, the conditional distribution of $\mathcal{L}_k$ on $[a, b]$ given all other information has Radon-Nikodym derivative with respect to Brownian bridge $B : [a, b] \to \mathbb{R}$, $B(a) = \mathcal{L}_k(a)$, $B(b) = \mathcal{L}_k(b)$, given by

$$Z^{-1} \exp\left\{-\int_a^b \Big(\mathbf{H}\big(\mathcal{L}_{k+1}(s) - B(s)\big) + \mathbf{H}\big(B(s) - \mathcal{L}_{k-1}(s)\big)\Big)\,\mathrm{d}s\right\},$$

where $Z \in (0, \infty)$ is a normalization.

The main theorem: KPZ_t line ensemble

Recall that $\mathbf{H}_t(x) = e^{t^{1/3}x}$.

Theorem

There exists an $\mathbb{N}$-indexed line ensemble $\mathcal{H}^t = \{\mathcal{H}^t_n : n \in \mathbb{N}\}$ enjoying the $\mathbf{H}_1$-Brownian Gibbs property whose lowest indexed curve $\mathcal{H}^t_1 : \mathbb{R} \to \mathbb{R}$ has the law of the KPZ solution $\mathcal{H}^{\mathrm{nw}}(t,\cdot)$.

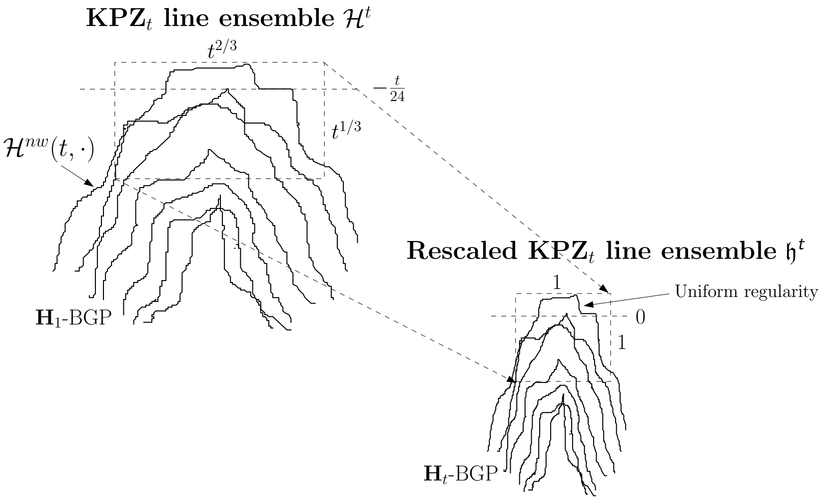
The scaled line ensemble $\{\mathfrak{h}^t_n : n \in \mathbb{N}\}$, defined by

$$\mathcal{H}^t_n(x) = -\frac{t}{24} + t^{1/3}\mathfrak{h}^t_n\left(t^{-2/3}x\right),$$

has the $\mathbf{H}_t$-Brownian Gibbs property.

Moreover, the curves $\mathfrak{h}^t_1$ (the scaled time t KPZ solutions) enjoys uniform comparison to Brownian bridge as $t \geq 1$ varies.

The KPZ_t line ensemble and its scaled version



The continuum directed random polymer

To see where the KPZ_t line ensemble comes from, we first observe a connection between KPZ and path integrals (directed polymers).

We may represent the narrow wedge $\mathcal{Z}(0,x) = \delta_{x=0}$ SHE solution

$$\partial_t \mathcal{Z}(t,x) = \tfrac{1}{2}\partial_x^2 \mathcal{Z}(t,x) + \dot{\mathcal{W}}(t,x)\mathcal{Z}(t,x)\,.$$

via Feynman-Kac as a polymer partition function:

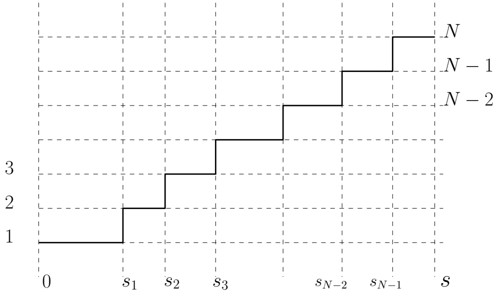
$$\mathcal{Z}^{\mathrm{nw}}(t,x) = \mathbb{E}\left[\delta_{B(t)=x} \; : \exp : \left\{ \int_0^t \dot{\mathcal{W}}\big(s, B(s)\big)\,\mathrm{d}s \right\}\right].$$

The expectation is over Brownian motions $B : [0,t] \to \mathbb{R}$ with $B(0) = 0$.

Brownian last passage percolation

For $N \in \mathbb{N}$ and $s > 0$, let $D_1^N(s)$ denote the set of non-decreasing surjective functions $\phi : [0, s] \to \mathbb{N} \cap [0, N]$.

For $\phi \in D_1^N(s)$, write $0 \le s_1 < \ldots < s_{N-1} \le s$, for ϕ's jump times.

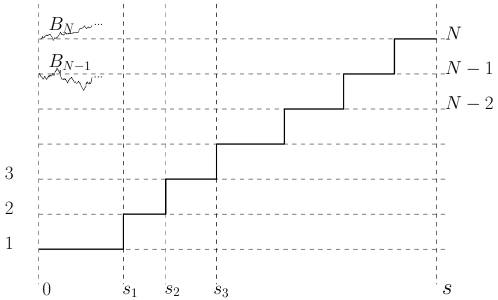


Brownian last passage percolation

Take N independent Brownian motions $B_i : [0,s] \to \mathbb{R}$, $1 \leq i \leq N$.

For $\phi \in D_1^N(s)$, define its weight $E(\phi)$ to be $\int_0^s \mathrm{d}B_{\phi(r)}(r)$, or

$$\big(B_1(s_1) - B_1(0)\big) + \big(B_2(s_2) - B_2(s_1)\big) + \cdots + \big(B_N(s) - B_N(s_{N-1})\big)\,.$$

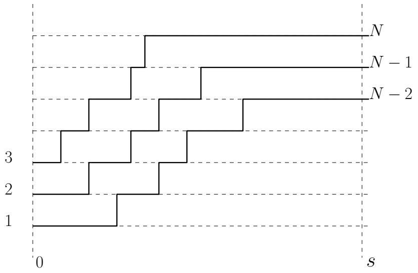


Brownian last passage percolation

Set
$$M_1^N(s) = \max_{\phi \in D_1^N(s)} E(\phi) .$$

Now let $1 \leq k \leq N$. Write $D_k^N(s)$ for the set of $(\phi_1, \ldots, \phi_k)$, where

- $\phi_j : [0, s] \to \mathbb{N} \cap [j, N - k + j]$ is a non-decreasing surjection;
- the paths ϕ_i are disjoint.



Brownian last passage percolation

Setting $$M_1^N(s) + \ldots + M_k^N(s) = \max_{(\phi_1,\ldots,\phi_k) \in D_k^N(s)} \sum_{i=1}^{k} E(\phi_i)\,,$$

we write $M^N(s) = \big(M_1^N(s), \ldots, M_N^N(s)\big)$.

Theorem (Baryshnikov (and Kuperberg) 2001)

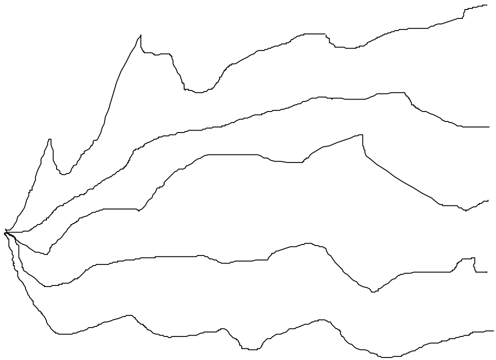
The process M^N is Dyson's Brownian motion: it is the diffusion on $\mathcal{W} = \big\{x \in \mathbb{R}^N : x_1 > \ldots > x_N\big\}$ with generator

$$\tfrac{1}{2} h^{-1} \Delta h,$$

where $h(x) = \prod_{1 \leq i < j \leq N} \big(x_i - x_j\big)$ is the Vandermonde determinant, and Δ is the Dirichlet Laplacian on $\mathcal{W}$.

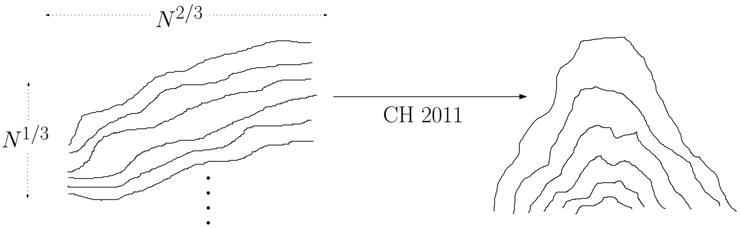
Brownian last passage percolation

The process M^N is equivalent to N Brownian motions on $\mathbb{R}$ started at 0 and conditioned on permanent non-intersection. In other words, it has the $\mathbf{H}_\infty$-Brownian Gibbs property.



Brownian LPP and the Airy line ensemble

As N and s tend to infinity together, the top edge of the Dyson Brownian motion (centered by the limit shape) has a limit when scaled vertically by $N^{1/3}$ and horizontally by $N^{2/3}$. This is the Airy line ensemble, which inherits the $\mathbf{H}_{\infty}$-Brownian Gibbs property.

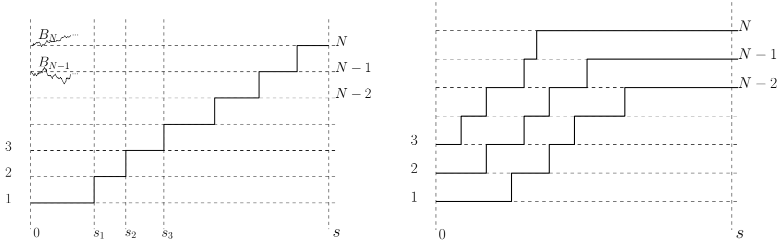


Brownian LPP ensemble
(edge of Dyson Brownian motion)

Airy line ensemble

Brownian last passage percolation

Recall Brownian last passage percolation,



and the maximization formula

$$M_1^N(s) + \ldots + M_k^N(s) = \max_{(\phi_1,\ldots,\phi_k)\in D_k^N(s)} \sum_{i=1}^{k} E(\phi_i) \, .$$

The O'Connell-Yor polymer line ensemble

Consider now a positive temperature analog: for $\beta = 1$ and $s \geq 0$,

$$Z_k^N(s) = \int_{D_k^N(s)} e^{\beta \sum_{i=1}^k E(\phi_i)} \, \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_k \, .$$

Define $\left\{ X_k^N(s) \right\}_{k=1}^N$ by

$$X_1^N(s) + \cdots + X_k^N(s) = \log Z_k^N(s) \, .$$

The O'Connell-Yor polymer line ensemble

The process $X^N = (X_1^N, \cdots, X_N^N)$ is a diffusion:

Theorem (O'Connell 2012)

The process X^N has generator $\frac{1}{2}\psi_0^{-1}\mathfrak{Q}\psi_0$, where ψ_0 is the class-one $\mathfrak{gl}_N$-Whittaker function, and where

$$\mathfrak{Q} = \Delta - 2\sum_{i=1}^{N-1} e^{x_{i+1}-x_i}$$

is the quantum Toda lattice Hamiltonian.

Corollary (Standard theory)

The process X^N has the H_1-Brownian Gibbs property.

Overview and conjecture