Invariant measure selection by noise : a toy example.

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joint with J. Mattingly

Introduction 0

Very roughly, the question we want to address in this talk can be formulated as follows.

Take a dynamical system (an ODE) whose large time behavior depends dramatically upon the initial condition, e.g. because of some conserved quantities.

Could it be that when adding a very small noise (together with some small damping term, so that the perturbed system is ergodic), the system forgets its initial condition, and becomes ergodic, in such a way that this remains true in the small noise limit (i.e. those invariant measures would converge to a uniquely selected invariant measure of the dynamical system).

Introduction 1

- Our work is motivated by the following open problem. Consider a $2D$ Navier–Stokes equation with additive white noise on the torus $\mathbb{T}^2$ of the form

$$\dot{u} - \varepsilon\Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\varepsilon}\dot{W}, \quad \mathrm{div}(u) = 0,$$

  where W is an $L^2(\mathbb{T}^2)$–valued BM such that $\forall \varepsilon > 0$, the above has a unique invariant measure μ_ε (see Hairer, Mattingly (06)). Kuksin (06) shows that $\{\mu_\varepsilon,\, \varepsilon > 0\}$ is tight, and that any limit of a converging subsequence is an invariant measure of the Euler equation. But does the whole sequence converge, and if yes, towards which particular invariant measure of the Euler equation ?
- We do not claim to solve this difficult problem. Rather, we consider a much simpler problem, namely a $3D$ SDE with damping of the order of ε and additive white noise multiplied by $\sqrt{\varepsilon}$. Our very simple *toy problem* has however in common with the *true problem* the property that the limiting deterministic undamped ODE possesses conserved quantities and infinitely many invariant measures.

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Introduction 2

- Consider the following $3D$ ordinary differential equation:

$$\dot{X}_t = Y_t Z_t$$
$$\dot{Y}_t = X_t Z_t$$
$$\dot{Z}_t = -2X_t Y_t,$$

- This equation has two conserved quantities : $2X_t^2 + Z_t^2$ and $2Y_t^2 + Z_t^2$.
- We consider, for $\varepsilon > 0$, the following damped/noisy version of the above ODE

$$\dot{X}_t^\varepsilon = Y_t^\varepsilon Z_t^\varepsilon - \varepsilon X_t^\varepsilon + \sigma_1 \sqrt{\varepsilon} \dot{B}_t$$
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Ergodicity for each $\varepsilon > 0$

- The respective scalings of the damping factor and of the noise are chosen in such a way that

$$\sup_{0<\varepsilon\le 1}\sup_{t\ge 0}\mathbb{E}\left[\|(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)\|^2\right] < \infty.$$

- Provided both $\sigma_1 > 0$ and $\sigma_2 > 0$, which we assume from now on, then the solution of the $3D$ SDE has a unique invariant measure μ_ε for each $\varepsilon > 0$.
- Our aim is to study the limit of μ_ε, as $\varepsilon \to 0$.

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Large time behavior of the solution of the ODE

- The existence of the two conserved quantities implies that all of the orbits of the ODE are bounded and most are closed orbits, topologically equivalent to a circle. All orbits live on the surface of a sphere whose radius is dictated by the values of the conserved quantities ($X_t^2 + Y_t^2 + Z_t^2$ is also a conserved quantity).
- To any initial point (X_0, Y_0, Z_0) on one of the closed orbits, we can associate a measure defined by the following limit

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \delta_{(X_s,Y_s,Z_s)} ds \, .$$

- Any such defined measure is an invariant measure for the ODE. Hence we see that the ODE has infinitely many invariant measures.
- Our result is that under the above conditions, there exists a unique invariant probability measure μ of the ODE, such that $\mu_\varepsilon \Rightarrow \mu$ as $\varepsilon \to 0$.

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Convergence on $[0, T]$

- We first note that as $\varepsilon \to 0$, the process $(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)$ converges to the solution of the ODE on any finite time interval.
- A simple calculation yields

$$\mathbb{E}\left(\|(X_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon)\|^2\right) = e^{-2\varepsilon t}\|(X_0, Y_0, Z_0)\|^2 + \|\sigma\|^2\left(1 - e^{-2\varepsilon t}\right)/2.$$

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which is consistent with the convergence towards the solution of the ODE, and the conservation of the norm along solutions of the ODE.

- However

$$\mathbb{E}\left(\|(X_{t/\varepsilon}^\varepsilon, Y_{t/\varepsilon}^\varepsilon, Z_{t/\varepsilon}^\varepsilon)\|^2\right)$$

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A different time scale

- This suggests to consider the "asymptotically constant quantities" in the time scale t/ε.
- We define

$$U_t^\varepsilon = 2(X_{t/\varepsilon}^\varepsilon)^2 + (Z_{t/\varepsilon}^\varepsilon)^2, \quad V_t^\varepsilon = 2(Y_{t/\varepsilon}^\varepsilon)^2 + (Z_{t/\varepsilon}^\varepsilon)^2.$$

- We have

$$dU_t^\varepsilon = 2[\sigma_1^2 - U_t^\varepsilon]dt + 4\sigma_1 X_{t/\varepsilon}^\varepsilon dB_t,$$
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- An important step of our work consists in showing that the limit (U_t, V_t) as $\varepsilon \to 0$ of $(U_t^\varepsilon, V_t^\varepsilon)$ satisfies the following SDE.

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The (U, V) equation

- $$(*)\begin{cases} dU_t = 2[\sigma_1^2 - U_t]dt + \sigma_1\sqrt{8\big(U_t - \Gamma(U_t, V_t)\big)}\,dB_t, \\ dV_t = 2[\sigma_2^2 - V_t]dt + \sigma_2\sqrt{8\big(V_t - \Gamma(U_t, V_t)\big)}\,dC_t. \end{cases}$$

- with
$$\Gamma(u, v) = \lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s^2 ds,$$
where (X_t, Y_t, Z_t) follows the ODE, starting from any point $(x, y, z) \in \mathbb{R}^3$ such that $(2x^2 + z^2, 2y^2 + z^2) = (u, v)$.

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- More explicitly

$$\Gamma(u,v) = u \wedge v \; \Lambda\left(\frac{u \wedge v}{u \vee v}\right),$$

- where $\Lambda(r)$ is a continuous and strictly increasing function on $[0,1]$ with $\Lambda(0) = \frac{1}{2}$ and $\Lambda(1) = 1$. Furthermore as $\varepsilon \to 0^+$,

$$\Lambda(\varepsilon) = \frac{1}{2} + \frac{1}{16}\epsilon + \frac{1}{32}\epsilon^2 + o(\epsilon^2)$$
$$\Lambda(1-\varepsilon) = 1 - \frac{2}{|\ln(\varepsilon)|} + o\left(\frac{1}{|\ln(\varepsilon)|}\right)$$

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- We first show that any solution of equation $(*)$ which starts from (U_0, V_0) satisfying $U_0 > 0$ and $V_0 > 0$ lives in $(0, \infty) \times (0, \infty)$ for all times.
- For that sake, we show that (U_t, V_t) cannot hit a point of the form $(u, 0)$ nor $(0, v)$ with $u, v > 0$, and also that $\sigma_1^{-2} U_t + \sigma_2^{-2} V_t$ cannot hit 0.
- Each of those three facts follow from

Lemma

Let $\{X_t,\ t \geq 0\}$ and $\{Y_t,\ t \geq 0\}$ be continuous $\mathbb{R}_+$–valued $\mathcal{F}_t$–adapted processes which satisfy $0 \leq Y_t \leq X_t$ for all $t \geq 0$, with $Y_0 > 0$,

$$dX_t = (a - bX_t)dt + \sqrt{cY_t}dW_t,$$
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where $b, c, x > 0$ and W is a standard BM. If $a \geq c/2$, then a.s. $X_t > 0$ for all $t \geq 0$.

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- One difficulty is that the (U, V) SDE does not have a unique weak solution.
- However we have

Theorem

If $(U,V) = \lim_n (U^{\varepsilon_n}, V^{\varepsilon_n})$ *for some subsequence* $\varepsilon_n \to 0$*, then*

$$\int_0^t \mathbf{1}_{\{U_s = V_s\}} ds = 0 \text{ for all } t > 0 \text{ almost surely}.$$

- Idea of proof : we show that if $J_t := U_t - V_t$, φ_δ such that $\varphi_\delta(0) = \varphi_\delta'(0) = 0$, $\varphi_\delta''(x) = \psi_\delta(x) = -\log(|x|)\mathbf{1}_{[-\delta,\delta]}(x)$,

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- Because $\Lambda(1-\varepsilon) = 1 - \frac{2}{|\log(\varepsilon)|} + \circ\left(\frac{1}{|\log(\varepsilon)|}\right)$, one can show that to any $c > 0$, we can associate $\delta > 0$ and $a > 0$ such that whenever $u, v \geq c > 0$, and $-\delta \leq k = u - v \leq \delta$,

$$4\left[\sigma_1^2(u - \Gamma(u,v)) + \sigma_2^2(v - \Gamma(u,v))\right]\log\left(\frac{1}{|k|}\right) \geq a > 0.$$

- Consequently, letting $\delta \to 0$ we deduce that

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Equation $(*)$ has a unique weak solution which spends zero time on the diagonal.

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Idea of the proof of the Theorem

- Let for $u, v > 0$,

$$F(u,v) = \begin{cases} 1 - \Lambda\left(\frac{u \wedge v}{u \vee v}\right), & \text{if } \frac{u \wedge v}{u \vee v} \geq \frac{1}{2}, \\ 1 - \Lambda\left(\frac{1}{2}\right), & \text{if } \frac{u \wedge v}{u \vee v} < \frac{1}{2}. \end{cases}$$

- We define the time change

$$A_t = \int_0^t F(U_s, V_s) ds, \; \eta_t = \inf\{s > 0, \, A_s > t\},$$

$$H_t = U_{\eta_t}, \quad \text{and } K_t = V_{\eta_t}.$$

- With $G(h,k) = F^{-1}(h,k)[1 - \Lambda(h \wedge k / h \vee k)]$, $\tilde{\sigma}_i = 2\sqrt{2}\sigma_i$,

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- The diffusion coefficient of the (H, K) equation is elliptic in $(0, \infty) \times (0, \infty)$. The drift is unbounded, but using the methodology in Portenko '90, we deduce uniqueness of the weak solution of the (H, K) equation.
- **Remark 1** The process (U, V), like its time change (H, K), does cross the diagonal in both directions, although the diffusion vanishes there, and the drift either is parallel to the diagonal, or else pushes either to $\{u > v\}$ or to $\{u < v\}$.
- **Remark 2** In the case $\sigma_1 = \sigma_2$, there is another solution which stays on the diagonal for ever, once it hits it. In fact even in the case $\sigma_1 \neq \sigma_2$, it seems that one can extend some arguments for one–dimensional SDEs to our case, and prove that there exist solutions which spend non–zero time on the diagonal, and live, after having hit the diagonal, either above or below it.
- **Invariant Probability Measure** It is not hard to show that the process (U, V), characterized as the unique solution of $(*)$ which spends zero time on the diagonal has a unique invariant probability measure $\lambda(du, dv) = \rho(u, v)dudv$, and $\rho(u, v) > 0$ for $u, v > 0$.

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The invariant measures of the ODE

- To each $(x, y, z) \in \mathbb{R}^3 \backslash \{0\}$, we attach $(u, v) = (2x^2 + z^2, 2y^2 + z^2) \in (0, +\infty)^2$.
- To each $(u, v) \in (0, +\infty)^2$, at least if $u \neq v$, one can associate two orbits of the ODE starting from (x, y, z), which, in addition to (u, v) depend only upon the sign of

$$\sigma(x, y, z) = \text{sign}(\mathbf{1}_{\{|x| \geq |y|\}} x + \mathbf{1}_{\{|x| < |y|\}} y).$$

- We denote by $\mathcal{O}(u, v, +1)$ and $\mathcal{O}(u, v, -1)$ those two orbits, and by $\nu_{(u,v,+1)}(dx, dy, dz)$ (resp. $\nu_{(u,v,-1)}(dx, dy, dz)$) the probability measure which is the mean over $(x, y, z) \in \mathcal{O}(u, v, +1)$ (resp. over $(x, y, z) \in \mathcal{O}(u, v, -1)$) of the Dirac masses at (x, y, z). In case $u = v$, those measures degenerate to two–point measures.

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- Define the probability measure μ on $\mathbb{R}^3$ by

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