Anomalous Fluctuations in One-Dimensional, Conservative Systems

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The model: deterministic part

- $\eta \in \Omega = \mathbb{R}^{\mathbb{Z}} \to$ volume configuration
- Base dynamics: system of ODE's

$$\frac{d}{dt}\eta_t^0(x) = \eta_t^0(x+1) - \eta_t^0(x-1)$$

- $f : \Omega \to \mathbb{R}$ local, $\frac{d}{dt} f(\eta_t^0) = Af(\eta_t^0)$, where

$$Af(\eta) = \sum_{x \in \mathbb{Z}} \Big(\eta(x+1) - \eta(x-1)\Big) \frac{\partial f(\eta)}{\partial \eta(x)}$$

The model: stochastic part

- Stochastic evolution: stirring process $\to$ whiteboard!
- For $f : \Omega \to \mathbb{R}$ local,

$$Sf(\eta) = \sum_{x \in \mathbb{Z}} \nabla_{x,x+1} f(\eta).$$

- $L = S + A$, generator of a Markov process $\{\eta_t; t \geq 0\}$

The model: stationary properties

- Gaussian invariant measures:

$$\mu_{\rho,\beta}(d\eta) = \prod_{x\in\mathbb{Z}} \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2}(\eta(x)-\rho)^2} d\eta(x)$$

- Two (formally) conserved quantities:

$$\sum_{x\in\mathbb{Z}} \eta(x) \longrightarrow \text{ the volume}$$

$$\sum_{x\in\mathbb{Z}} \eta(x)^2 \longrightarrow \text{ the energy}$$

Energy correlations

- Take $\eta_0 \sim \mu_{\beta,\rho}$ for some $\beta > 0$, $\rho \in \mathbb{R}$ and WLG we take $\rho = 0$.
- Energy correlation function:

$$S_t(x) = \mathbb{E}_{\mu_{\beta,0}}\left[\left(\eta_t(x)^2 - \tfrac{1}{\beta}\right)\left(\eta_0(0)^2 - \tfrac{1}{\beta}\right)\right]$$

Energy correlations

Theorem

For any regular functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} f\left(\tfrac{x}{n}\right) g\left(\tfrac{y}{n}\right) S_{tn^{3/2}}(x-y) = \iint f(x)g(y)P_t(x-y)dxdy,$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the fractional heat equation

$$\partial_t u = \left\{-\left(-\Delta\right)^{3/4} + \nabla\left(-\Delta\right)^{1/4}\right\}u.$$

Energy correlations

- $1:2:3$ KPZ-like space-time scale
- Linear evolution $\longrightarrow$ Gaussian fluctuations of various observables of the energy (current, occupation variables, etc...)
- The fractional exponent $3/4$ is universal; skewness is not
- Result is robust with respect to modifications of the model (no stochastic integrability required)
- Aims to a complete description of the FPU-β universality class

Ideas of proof

- The quadratic volume field:

$$Q_t(x,y) = \mathbb{E}_{\mu_{\beta,0}}\Big[\eta_t(x)\eta_t(y)\Big(\eta_0(0)^2 - \frac{1}{\beta}\Big)\Big]$$

for $x \neq y \in \mathbb{Z}$.

- Right space-time scale for Q is *not* super-diffusive

The quadratic filedd

- Hyperbolic scaling: $Q_{tn}(\frac{x}{n}, \frac{y}{n}) \longrightarrow$ solution of a linear transport equation
- Characteristic velocity $v = (2, 2)$
- Along characteristics, diffusive scaling:

$$Q_{tn^2}\Big(\frac{x}{n} - 2nt, \frac{y}{n} - 2nt\Big) \longrightarrow \text{ solution of a heat equation}$$

An extension problem

- Take g solution of the Laplace problem

$$\begin{cases} \partial_y^2 g + \partial_x g = 0, & x \in \mathbb{R}, y \geq 0 \\ \partial_y g(x,0) = f'(x), & x \in \mathbb{R}. \end{cases}$$

Theorem (Extension problem)

$$\partial_x g(x,0) = \left\{ -(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4} \right\} f(x)$$

Fractional integration by parts

- Microscopic formulation:

$$\frac{d}{dt}\sum_{x\in\mathbb{Z}} S_{tn^{3/2}}(x) f\left(\tfrac{x}{n}\right) = \sqrt{n}\sum_{x\in\mathbb{Z}} Q_{tn^{3/2}}(x,x+1) f'\left(\tfrac{x}{n}\right)$$

plus error terms

- RHS computed using the extension problem
- Boundary effects (akin to renormalization):

$$\text{RHS} \longrightarrow \sum_{x\in\mathbb{Z}} S_{tn^{3/2}}(x)\partial_x g\left(\tfrac{x}{n},0\right).$$

The test function

- Define

$$h_n(x,y) = g\left(\tfrac{x+y}{2n}, \tfrac{x-y}{2\sqrt{n}}\right)$$

Then

$$\frac{d}{dt}\sum_{x,y} Q_{tn^{3/2}}(x,y)h_n(x,y) = \sqrt{n}\sum_{x\in\mathbb{Z}} Q_{tn^{3/2}}(x,x+1)f'\left(\tfrac{x}{n}\right) + \sum_{x} S_{tn^{3/2}}(x)Lf\left(\tfrac{x}{n}\right)$$

plus lower order terms