Invariant Measures for Stochastic Conservation Laws

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We study the first-order scalar conservation law with stochastic forcing

$$du + \mathrm{div}(A(u))dt = \Phi(u)dW(t), \quad t \in (0, T).$$

We consider a periodic space variable x: $x \in \mathbb{T}^N$.

The flux function $A \in C^2(\mathbb{R};\mathbb{R}^N)$, A and its derivatives have at most polynomial growth.

The noise is constructed thanks to a cylindrical Wiener process: $W = \sum_{k\geq 1} \beta_k e_k$

- β_k are independent brownian processes
- $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space $L^2(\mathbb{T}^N)$.
- $\Phi(u) \in \mathcal{L}(L^2(\mathbb{T}^N))$, $\Phi(u)dW = \sum_{k\in\mathbb{N}} \Phi(u)e_k d\beta_k$.

Pioneering works:

- E, Khanin, Mazel & Sinai have studied the stochastic Burgers equation with additive noise: $A(u) = u^2$, $\Phi(u)e_k = \phi_k$, $d = 1$.

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  $\longrightarrow$ They use a Lax-Oleinik formula to construct solutions and show that there exists a unique invariant measure:

$$u(t,x) = \frac{\partial}{\partial x} \inf_{\xi(t)=x} \left\{ \mathcal{A}_{0,t}(\xi) + \int_0^{\xi(0)} u_0(x)dx \right\}$$

with

$$\mathcal{A}_{0,t}(\xi) = \frac{1}{2}\int_0^t \dot{\xi}(s)^2 ds + \sum_k \int_0^t \phi_k(\xi(s))d\beta_k(s)$$

A minimizer ξ satisfies:

$$\dot{\xi}(s) = v(s),\ dv(s) = \phi_k(\xi(s))d\beta_k(s).$$

and $u(t,x) = \dot{\xi}(t)$.

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- Generalization by Iturriaga and Khanin.
- The ideas have been used by Dirr and Souganidis for general Hamilton-Jacobi equations under some assumptions on the Hamiltonian. The essential ingredients are:
  1. The deterministic equation has an attractor which reduces to a single trajectory.
  2. When the noise is small, the stochastic solution is close to the deterministic one, uniformly with respect to the initial data.
  3. The noise is small on long time intervals with positive probability.
  4. With an additive noise, the distance between two solutions cannot increase

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- Bakhtin, Cator and Khanin have considered the Burgers equation on the real line with Poisson noise.
- Boritchev has obtained very fine estimates on the moments of solutions of the visous Burgers equation in terms of the viscosity.

Kinetic formulation

In this work we use the kinetic formulation introduced by Lions, Perthame & Tadmor to prove existence and uniqueness of entropy solutions in any space dimension.

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$\longrightarrow$ introduce a new variable ξ and consider the equation satisfied by

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Let $\varphi \in C^\infty(\mathbb{R})$ with compact support and set $\theta(\xi) = \int_{-\infty}^{\xi} \varphi(\zeta) d\zeta$, we have

$$(\mathbf{1}_{u>\xi}, \varphi) = \int_{\mathbb{R}} \mathbf{1}_{u>\xi} \theta'(\xi) d\xi = \int_{-\infty}^{u} \theta'(\xi) d\xi = \theta(u)$$

$$(\delta_{u=\xi}, \theta) = \theta(u)$$

By Itô Formula, we deduce

$$d(\mathbf{1}_{u>\xi},\varphi)=d\theta(u)$$

$$=\theta'(u)(-a(u)\cdot\nabla u dt+\Phi(u)dW)+\frac{1}{2}\theta''(u)\mathbf{G}^2(u)dt$$

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- $$\begin{aligned} \theta'(u)a(u)\cdot\nabla u &= \mathrm{div}\left(\int_{-\infty}^{u} a(\xi)\theta'(\xi)d\xi\right) = \mathrm{div}_x(a\mathbf{1}_{u>\xi}, \varphi) \\ &= (a\cdot\nabla_x \mathbf{1}_{u>\xi}, \varphi) \end{aligned}$$

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We obtain the kinetic formulation:

$$d\mathbf{1}_{u>\xi} + a(\xi)\cdot\nabla\mathbf{1}_{u>\xi}dt = \delta_{u=\xi}\Phi(\xi)dW - \partial_\xi(\frac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi})dt.$$

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An additional measure accounting for the schocks of u has to be added. We say that u is a kinetic solution if $f(x,\xi,t) = \mathbf{1}_{u(x,t)>\xi}$ satisfies:

$$df + a(\xi)\cdot\nabla f dt = \delta_{u=\xi}\Phi(\xi)dW - \partial_\xi(\frac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi})dt + \partial_\xi m\, dt$$

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Given a kinetic solution, we recover u by :

$$u(x,t) = \int_{\mathbb{R}} \mathbf{1}_{u(x,t)>\xi} - \mathbf{1}_{0>\xi}d\xi.$$

The viscous approximation:

$$\begin{cases} du^\eta + \mathrm{div}(A(u^\eta))dt - \eta\Delta u^\eta dt = \Phi(u^\eta)dW(t), & t > 0, x \in \mathbb{T}^N, \\ u^\eta(x,0) = u_0(x), & x \in \mathbb{T}^N. \end{cases}$$

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Energy inequality:

$$\mathbb{E}\left(\|u^\eta(t)\|^p_{L^p(\mathbb{T}^N)}\right)+\eta\mathbb{E}\int_0^T\int_{\mathbb{T}^N}|u^\eta(t,x)|^{p-2}|\nabla u^\eta(t)|^2dxdt \leq C(p,u_0,T).$$

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Estimate of m^η: $\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}}|\xi|^{p-2}dm^\eta(x,t,\xi) \le C_p.$

We also have the improved estimate, for $p \ge 0$,

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Estimate on $\nu^\eta=\delta_{u^\eta=\xi}$**:** $\mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p d\nu^\eta_{x,t}(\xi)dx\le C_p$,

$t\in(0,T)$.

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For a subsequence $(\eta_n)\downarrow 0$

1. $\nu^{\eta_n}\to\nu$ in the sense of Young measures indexed by $\Omega\times\mathbb{T}^N\times[0,T]$ and $f^{\eta_n}=\mathbf{1}_{u^{\eta_n}>\xi}\rightharpoonup f$ in $L^\infty(\Omega\times\mathbb{T}^N\times(0,T)\times\mathbb{R})-weak-*$. Moreover $\nu_{x,t}=-\partial_\xi f$
2. $m^{\eta_n}\rightharpoonup m$ in $L^2(\Omega;\mathcal{M}_b)$-weak star, where $\mathcal{M}_b$ denote the space of bounded Borel measures over $\mathbb{T}^N\times[0,T]\times\mathbb{R}$

$$\longrightarrow \quad df+a\cdot\nabla f dt=\nu\Phi(\xi)dW-\frac{1}{2}\partial_\xi(\mathbf{G}^2_\eta\nu)dt+\partial_\xi m\,dt.$$

We say that f is a generalized kinetic solution.

$$\mathbb{E}\left|\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} |\xi|^{p-2} dm^{\eta}(x,t,\xi)\right|^2 \leq C_p, \mathbb{E}\int_{\mathbb{T}^N}\int_{\mathbb{R}} |\xi|^p d\nu^{\eta}_{x,t}(\xi)dx \leq C_p$$

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We say that f is a generalized kinetic solution.
We only have $f \in [0,1]$ and $\nu_{x,t}$ is a probability measure. To get a kinetic solution, we need $f \in \{0,1\}$ then necessarily $f = \mathbf{1}_{u>\xi}$ and $\nu_{x,t} = \delta_{u=\xi}$.

Left and right limits of generalized solution

Proposition Let f be a generalized solution with initial datum f_0. Then f admits almost surely left and right limits at all point $t_* \in [0, T]$. For all $t_* \in [0, T]$ there exists some kinetic functions $f^{*,\pm}$ on $\Omega \times \mathbb{T}^N \times \mathbb{R}$ such that $\mathbb{P}$-a.s.

$$\langle f(t_* - \varepsilon), \varphi \rangle \to \langle f^{*,-}, \varphi \rangle$$

and

$$\langle f(t_* + \varepsilon), \varphi \rangle \to \langle f^{*,+}, \varphi \rangle$$

as $\varepsilon \to 0$ for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$. Moreover, almost surely,

$$\langle f^{*,+} - f^{*,-}, \varphi \rangle = -\int_{\mathbb{T}^N \times [0,T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{1}_{\{t_*\}}(t) dm(x, t, \xi).$$

In particular, almost surely, the set of $t_* \in [0, T]$ such that $f^{*,-} \neq f^{*,+}$ is countable.

Doubling of variables

Proposition Let f_i, $i = 1, 2$, be generalized solution. Then, for $0 \le t \le T$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\mathbb{E} \iint_{\mathbb{R}^2 \times (\mathbb{T}^N)^2} \rho(x-y)\psi(\xi-\zeta) f_1^{\pm}(x,t,\xi)(1-f_2^{\pm}(y,t,\zeta)) d\xi d\zeta dx dy$$
$$\le \mathbb{E} \iint_{\mathbb{R}^2 \times (\mathbb{T}^N)^2} \rho(x-y)\psi(\xi-\zeta) f_{1,0}(x,\xi)(1-f_{2,0}(y,\zeta)) d\xi d\zeta dx dy$$
$$+ \mathrm{I}_\rho + \mathrm{I}_\psi,$$

where

$$\mathrm{I}_\rho = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x,s,\xi)\bar{f}_2(y,s,\zeta)(a(\xi)-a(\zeta))\psi(\xi-\zeta) d\xi d\zeta \cdot \nabla_x \rho(x-y) dx dy ds,$$

$$\mathrm{I}_\psi = \frac{1}{2} \int_{(\mathbb{T}^N)^2} \rho(x-y) \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi-\zeta) \times \sum_{k\ge1} |g_k(x,\xi) - g_k(y,\zeta)|^2 d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi,\zeta) dx dy ds.$$

Take $\rho = \rho_\delta$, $\psi = \psi_\varepsilon$ $\rightsquigarrow$ when $\delta \to 0$, $\varepsilon \to 0$:

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^{\pm}(t)(1 - f_2^{\pm}(t)) dx d\xi \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_{1,0}(1 - f_{2,0}) dx d\xi.$$

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If $f_1 = f_2 = f$ is a generalized solution with initial datum $\mathbf{1}_{u_0>\xi}$, we deduce $f^{\pm}(1-f^{\pm}) = 0$ a.e., *i.e.*

Take $\rho = \rho_\delta$, $\psi = \psi_\varepsilon$ $\rightsquigarrow$ when $\delta \to 0$, $\varepsilon \to 0$:

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If $f_1 = f_2 = f$ is a generalized solution with initial datum $\mathbf{1}_{u_0 > \xi}$, we deduce $f^{\pm}(1 - f^{\pm}) = 0$ a.e., *i.e.* $f^{\pm} \in \{0, 1\}$ a.e.
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If u_1 and u_2 are two solutions, we deduce from the identity

$$\int_{\mathbb{R}} \mathbf{1}_{u_1>\xi}(1-\mathbf{1}_{u_2>\xi})d\xi = (u_1-u_2)^+$$

the contraction property

$$\mathbb{E}\|(u_1^\pm(t)-u_2^\pm(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|(u_{1,0}-u_{2,0})^+\|_{L^1(\mathbb{T}^N)}.$$

This implies the L^1-contraction property, comparison and uniqueness of solutions.

Theorem: Let $u_0 \in L^\infty(\mathbb{T}^N)$. There exists a unique kinetic solution u with initial datum u_0. It is the strong limit of (u^η) as $\eta \to 0$: for every $T > 0$, for every $1 \leq p < +\infty$,

$$\lim_{\eta \to 0} \mathbb{E}\|u^\eta - u\|_{L^p(\mathbb{T}^N \times (0,T))} = 0.$$

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Assumptions on the coefficient of the noise:
We assume that $\Phi(u)e_k(x) = g_k(x,u)$, $x \in \mathbb{T}^N$ and

- $\mathbf{G}^2(x,u) = \sum_{k\ge 1} |g_k(x,u)|^2 \le D_0(1+|u|^2)$
- $\sum_{k\ge 1} |g_k(x,u) - g_k(y,v)|^2 \le D_1(|x-y|^2 + |u-v|h(|u-v|))$,

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Remark: The result can be extended to initial data in $L^1(\mathbb{T}^N)$and we obtain renomalized solutions.

Other works on scalar conservation laws:

- Kim and Vallet & Wittbold consider an additive noise.
  $\longrightarrow$ setting $v = u - \Phi W$, the equation transtorms into

$$\frac{dv}{dt} + \mathrm{div}(\widetilde{A}(v,t)) = 0,$$

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- Hofmanova has treated the case of a degenerate parabolic equation.

Invariant measures

- Is it possible to use the dissipation due to the shocks to prove existence of an invariant measure ?
- This is done in the work of E, Khanin, Mazel and Sinai thanks to the Lax-Oleinik formula but it is not available in general ...
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- We cannot consider any type of noise:

$$du + \mathrm{div} A(u)dt = \Phi(u)dW$$

Integrate in x: $d\int_{\mathbb{T}^N} u(x,t)dx = \int_{\mathbb{T}^N} \phi(u)dWdx.$

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Integrate in x: $d\int_{\mathbb{T}^N} u(x,t)dx = \int_{\mathbb{T}^N} \phi(u)dWdx.$

$\rightarrow$ We need the right hand side to vanish. No realistic noise depending on u satisfies this ... (except for noise in divergence form which are not covered by our theory but by Lions, Perthame, Souganidis).

$\rightarrow$ We restrict to additive noise with zero spatial average.

Dissipation through averaging Lemma:

We have constructed a kinetic solution:

$$df + a(\xi)\cdot\nabla f dt = \delta_{u=\xi}\Phi dW - \partial_\xi\left(\tfrac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi}\right)dt$$

$$+\partial_\xi m\, dt$$

If the system is sufficiently nonlinear, we expect some smoothing, which is related to dissipation, for $u = \int_{\mathbb{R}} f d\xi$.

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Introduce: $B = \gamma(-\Delta)^\alpha + \delta I$ and rewrite the equation:

$$df + a(\xi)\cdot\nabla f dt + Bf = Bf + \delta_{u=\xi}\Phi(\xi)dW - \partial_\xi\left(\frac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi}\right)dt$$

$$+\partial_\xi m\, dt$$

(Bouchut, Desvillettes)

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Let $S_{\gamma,\delta}(t)$ be the semigroup associated to $B + a(\xi)\cdot\nabla$:

$$\begin{aligned} f(t) &= S_{\gamma,\delta}(t)f_0 + \int_0^t S_{\gamma,\delta}(t-s)Bfds + \int_0^t S_{\gamma,\delta}\delta_{u=\xi}\Phi(\xi)dW \\ &+ \int_0^t S_{\gamma,\delta}\partial_\xi(m - \frac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi}))ds. \end{aligned}$$

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Integrate with respect to ξ:

$$\begin{aligned} u(t) &= \int_\xi S_{\gamma,\delta}(t)f_0 d\xi + \int_\xi\int_0^t S_{\gamma,\delta}(t-s)Bfdsd\xi \\ &+ \int_\xi\int_0^t S_{\gamma,\delta}(t-s)\delta_{u=\xi}\Phi(\xi)dWd\xi \\ &+ \int_\xi\int_0^t S_{\gamma,\delta}\partial_\xi(m - \frac{1}{2}\mathbf{G}^2(\xi)\delta_{u=\xi}))dsd\xi. \end{aligned}$$

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$$\leq c_4(\gamma,\delta) \left(D_0 \mathbb{E} \int_0^T \left\| a'(u) \right\|_{L^1(\mathbb{T}^N)} dt + \mathbb{E} \int_{\mathbb{T}^N} \Theta(u_0) dx + D_0 \right)$$

for $\lambda < \alpha$, μ depending on N, p, α, D_0 the intensity of the noise and $\Theta(u) = \int_0^u \int_0^v |a'(\xi)| d\xi dv$.

Assume

$$|a'(\xi)| \leq C(|\xi| + 1)$$

then

$$\frac{1}{T}\mathbb{E}\int_0^T \|u\|_{W^{\lambda,p}(\mathbb{T}^1)} \leq C(N, \alpha, \lambda, p, \gamma, \delta, D_0)(\frac{1}{T}\mathbb{E}\|u_0\|^3_{L^3(\mathbb{T}^1)} + 1)$$

with λ and p depending on the dimension.

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It is supported in $L^p(\mathbb{T}^N)$ for $p < 2 + \frac{b}{2}$ if $N = 1$ and $p < \frac{N}{N-1}$. otherwise.

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Note that for a stationary solution:

$$\mathbb{E}\int_{\mathbb{T}^N\times[0,T]\times\mathbb{R}} dm(x,t,\xi) = T\, D_0.$$

Uniqueness

We generalize the strategy used in Dirr and Souganidis:

1. The deterministic equation has an attractor which reduces to a single trajectory.
2. When the noise is small, the stochastic solution is close to the deterministic one, **depending on the size of the initial data**. $\rightsquigarrow$ we need to assume a' is bounded.
3. The noise is small on long time intervals with positive probability.
4. **The solution enter in a fixed ball in a finite time**.
5. With an additive noise, the distance between two solutions cannot increase

Small noise yields small solutions

For any $\varepsilon > 0$, there exists $T > 0$ and $\delta > 0$ such that:

$$\frac{1}{T}\int_0^T \|u(s)\|_{L^1(\mathbb{T}^N)} ds \leq \frac{\varepsilon}{2}$$

if

$$\|u(0)\|_{L^1(\mathbb{T}^N)} \leq 2\kappa_0 \text{ and } \sup_{t\in[0,T]} |W|_{W^{1,\infty}} \leq \delta.$$

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$\rightarrow$ Use the averaging technique for $v = u - W$:

$$\frac{d}{dt}v + \mathrm{div} A(v + W) = 0.$$

Set $g(t,x,\xi) = \mathbf{1}_{v(t,x)>\xi}$:

$$\begin{aligned}\frac{d}{dt}g + a(\xi)\cdot\nabla g + Bg = \quad & Bg - a'(\xi + W)\cdot\nabla W\,\delta_{u=\xi} \\ & +(a(\xi) - a(\xi + W))\cdot\nabla g + \partial_\xi m.\end{aligned}$$

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$$
\begin{aligned}
v(t) &= \int_\xi S_{\gamma,\delta}(t) f_0 d\xi + \int_\xi \int_0^t S_{\gamma,\delta}(t-s) Bg ds d\xi \\
&+ \int_\xi \int_0^t S_{\gamma,\delta}(t-s) a'(\xi + W) \cdot \nabla W(s) \delta_{u=\xi} d\xi \\
&+ \int_\xi \int_0^t S_{\gamma,\delta}(t-s)(a(\xi) - a(\xi + W)) \cdot \nabla g d\xi \\
&+ \int_\xi \int_0^t S_{\gamma,\delta}(t-s) \partial_\xi m ds d\xi.
\end{aligned}
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$$
\|u(0)\|_{L^1(\mathbb{T}^1)} \leq 2\kappa_0, \qquad \sup_{t \in [0,T]} |W|_{W^{1,\infty}} \leq \delta.
$$

$\rightarrow$ We need to assume a' bounded.

The solution enters in a fixed ball in a finite time

$$\frac{1}{T}\mathbb{E}\int_0^T \|u\|_{L^1(\mathbb{T}^N)}dt \le \kappa_0\left(\frac{1}{T}\mathbb{E}\|u_0\|^2_{L^3(\mathbb{T}^N)} + 1\right)$$

$$\mathbb{E}\left(\int_t^{t+T} \|u(s)\|_{L^1(\mathbb{T}^N)}ds \middle| \mathcal{F}_t\right) \le \kappa_0(\|u(t)\|^3_{L^3(\mathbb{T}^N)} + T)$$

$$\begin{aligned}\mathbb{E}\left(\int_t^{t+T} \|u(s)\|_{L^1(\mathbb{T}^N)}ds \middle| \mathcal{F}_t\right) \le \quad & \kappa_0(\|u_0\|^3_{L^3(\mathbb{T}^N)} + 3D_0\int_0^t \|u(s)\|_{L^1(\mathbb{T}^N)}ds \\ & +3\int_0^t ((u(s))^2, \Phi dW(s))_{L^2(\mathbb{T}^N)} + T)\end{aligned}$$

The solution enters in a fixed ball in a finite time

Define recursively the sequences of times $(t_k)_{k\geq 0}$ and $(t_k)_{k\geq 0}$:

$$t_0 = 0,$$

$$t_{k+1} = t_k + r_k,$$

where $(r_k)_{k\geq 0}$ will be chosen below. And the events:

$$A_k = \{ \inf_{s\in[t_\ell, t_{\ell+1}]} \|u(s)\|_{L^1(\mathbb{T}^N)} \geq 2\kappa_0,\ \ell = 0, \dots, k\}.$$

Then, for all $k \geq 0$,

$$\begin{aligned}
&\mathbb{P}\left(\inf_{s\in[t_k, t_{k+1}]} \|u(s)\|_{L^1(\mathbb{T}^N)} \geq 2\kappa_0 \middle| \mathcal{F}_{t_k} \right) \\
&\leq \mathbb{P}\left(\frac{1}{r_k} \int_{t_k}^{t_k+r_k} \|u(s)\|_{L^1(\mathbb{T}^N)} ds \geq 2\kappa_0 \middle| \mathcal{F}_{t_k} \right) \\
&\leq \frac{1}{2r_k} (\|u_0\|^3_{L^3(\mathbb{T}^N)} + 3D_0 \int_0^{t_k} \|u(s)\|_{L^1(\mathbb{T}^N)} + 1) + \frac{1}{2} \\
&+ \frac{3}{2r_k} \int_0^{t_k} ((u(s))^2, \Phi dW(s))_{L^2(\mathbb{T}^N)}.
\end{aligned}$$

The solution enters in a fixed ball in a finite time

Then, for all $k \geq 0$,

$$\mathbb{P}\left(\inf_{s\in[t_k,t_{k+1}]} \|u(s)\|_{L^1(\mathbb{T}^N)} \geq 2\kappa_0 \middle| \mathcal{F}_{t_k}\right)$$
$$\leq \mathbb{P}\left(\frac{1}{r_k}\int_{t_k}^{t_k+r_k} \|u(s)\|_{L^1(\mathbb{T}^N)} ds \geq 2\kappa_0 \middle| \mathcal{F}_{t_k}\right)$$
$$\leq \frac{1}{2r_k}(\|u_0\|^3_{L^3(\mathbb{T}^N)} + 3D_0\int_0^{t_k} \|u(s)\|_{L^1(\mathbb{T}^N)} + 1) + \frac{1}{2}$$
$$+\frac{3}{2r_k}\int_0^{t_k}((u(s))^2, \Phi dW(s))_{L^2(\mathbb{T}^N)}.$$

Multiply this inequality by $\mathbf{1}_{A_k}$ and take the expectation:

$$\mathbb{P}(A_{k+1}) \leq \frac{5}{8}\mathbb{P}(A_k) + \frac{3D_0}{2r_k}\mathbb{E}\left(\int_0^{t_k} \|u(s)\|_{L^1(\mathbb{T}^N)} ds \mathbf{1}_{A_k}\right)$$
$$+\frac{3}{2r_k}\mathbb{E}\left(\int_0^{t_k}((u(s))^2, \Phi dW(s))_{L^2(\mathbb{T}^N)}\mathbf{1}_{A_k}\right).$$

The solution enter in a fixed ball in a finite time

For r_k large enough:

$$\mathbb{P}(A_{k+1}) \leq \frac{3}{4}\mathbb{P}(A_k) + \left(\frac{3}{4}\right)^k.$$

We then define the stopping time

$$\tau^{u_0} = \inf\{t \geq 0 \mid \|u(t)\|_{L^1(\mathbb{T}^1)} \leq 4\kappa_0\}.$$

and by Borel-Cantelli $\tau^{u_0} < \infty$ almost surely.
It follows that for $T > 0$ the following stopping times are also almost surely finite:

$$\tau_\ell = \inf\{t \geq \tau_{\ell-1} + T \mid \|u(t)\|_{L^1(\mathbb{T}^1)} \leq 4\kappa_0\}, \; \tau_0 = 0.$$

Conclusion

Take two solutions u^1, u^2 starting from u_0^1, u_0^2:

$$\mathbb{P}(\frac{1}{T}\int_{\tau_\ell}^{T+\tau_\ell} \|u^1(s)-u^2(s)\|_{L^1(\mathbb{T}^1)}ds \geq \varepsilon \,|\mathcal{F}_{\tau_\ell}) \leq (1-\eta).$$

By Borel-Cantelli and the fact that $\|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^1)}$ decreases:

$$\mathbb{P}(\lim_{t\to\infty} \|u^1(t)-u^2(t)\|_{L^1(\mathbb{T}^1)} \geq \varepsilon) = 0.$$

Thanks for your attention.