Topological phases of matter, modular tensor categories and operator algebras

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Study of topological phases in terms of operator algebras

Formulate physical notions of topological phases, gapped domain walls between them and their composition in condensed matter physics in terms of subfactors in operator algebra theory.

Outline of the talk:

1. Modular tensor categories
2. Topological phases
3. Operator algebras and subfactors
4. Physical conjecture of Lan-Wang-Wen
5. Composition of gapped domain walls
6. Anyon condensation and boundary-bulk duality

Tensor categories

A finite group G consists of the following ingredients.

1. An associative multiplication
2. The identity element
3. The inverse elements

For a finite group G, the set of its finite dimensional unitary representations have the following structures.

1. Irreducible decomposition into finitely many ones
2. An associative tensor product
3. The identity representation
4. The contragredient (dual) representation

Abstract axiomatization of such a set gives a notion of a tensor category.

Braiding and modular tensor categories

For two representations π and σ of a group G, the two tensor products $\pi \otimes \sigma$ and $\sigma \otimes \pi$ are trivially unitarily equivalent. The corresponding equivalence for a general tensor category does not hold.

We have an important class of tensor categories for which the above commutativity of tensor products holds in some mathematically nice way. Such commutativity is called braiding because it is similar to switching two wires.

A braiding naturally comes in a pair — overcrossing and undercrossing. It is more interesting if these two are really different. If this is the case, we say that the tensor category is modular.

Topological phase

A topological phase is a certain 2-dimensional status of matters and a typical example is a thin liquid on a large plane. A point on the plane can have a special status by excitation. An excited point behaves like a particle and is called an anyon.

Suppose we have finitely many anyons and study exchanges among them. The natural group for exchanges is the braid group and we have braid group statistics. (If the original dimension is 3, then the natural group representing such an exchange is the permutation group, and we have a boson or a fermion.)

A modular tensor category gives a mathematical description of such a system of anyons.

Examples of modular tensor categories

The Kitaev toric code gives one example of a modular tensor category. For this example, we have four anyons. All have dimensions 1, and the tensor product rules (fusion rules) are given by the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (The group structure can give a trivial braiding, but here we have a different, nontrivial braiding structure.)

The Fibonacci category has two anyons, the trivial one labeled as $\mathbf{1}$ and another one labeled as τ. The fusion rules are given by $\tau^2 = \mathbf{1} \oplus \tau$. This is expected to be related to fractional quantum Hall liquids. This is also related to the Jones polynomial at the deformation parameter $q = \exp(2\pi i/5)$.

Operator algebras and subfactors

In quantum mechanics, an observable is represented by a (self-adjoint) operator on a Hilbert space. We can add and multiply operators. A set of operators which is closed under addition and multiplication is called an operator algebra. We usually also require that an operator algebra is closed under a certain topology. An important class of operator algebras consists of von Neumann algebras.

An inclusion $N \subset M$ of certain von Neumann algebras is called a subfactor. (A factor is a name for a certain von Neumann algebra.) Theory of subfactors produced the Jones polynomial for knots, which is expected to be useful for quantum computations. Subfactors also give a powerful tool to study (modular) tensor categories.

Modular invariants

Suppose we have a modular tensor category having n irreducible objects. The braiding produces an n-dimensional unitary representation π of $SL(2,\mathbb{Z})$, the modular group. The matrices $\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\pi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are represented as S and T, respectively.

A matrix Z is called a modular invariant if it satisfies the following, where the index 0 means the vacuum sector.
(1) $Z_{\lambda\mu} \in \{0, 1, 2, \dots\}$.
(2) $Z_{00} = 1$.
(3) $ZS = SZ$, $ZT = TZ$.
They are sometimes classified completely.

Gapped domain walls

A topological phase is described with a modular tensor category where each anyon corresponds to an irreducible objects. Suppose we have two topological phases described with two modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$. The exterior tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$, where "opp" means reversing the braiding, gives a new modular tensor category.

We have a physical notion of a gapped domain wall between the two topological phases and it is mathematically defined to be an irreducible local Lagrangian Frobenius algebra with the generator $\bigoplus Z_{\lambda\mu}\lambda \boxtimes \bar{\mu}$ in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$, where $Z_{\lambda\mu} = 0, 1, 2, \ldots$ gives a modular invariant.

Conjecture of Lan-Wang-Wen

In a paper [Phys. Rev. Lett. 2015], Lan, Wang and Wen conjectured that if we have a modular invariant matrix Z with $Z_{00} = 1$ for modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$, and the entries of matrix Z satisfy some inequalities about multiplicities, then there would exist a corresponding irreducible local Lagrangian Frobenius algebra.

However, subfactor theory easily disproves this conjecture. Actually, the charge conjugation matrix $(\delta_{\lambda\bar{\mu}})$ gives a counterexample for some modular tensor category $\mathcal{C}_1 = \mathcal{C}_2$ with a recent work of Davydov. We have also given a correct form of the conjecture using Witt equivalence. (K 2015)

Composition of gapped domain walls

In physics literature, we have a notion of composition of two gapped domain walls and its irreducible decomposition. We would like to formulate this notion mathematically.

Suppose we have three topological phases described with three modular tensor categories $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. We further assume to have two irreducible local Lagrangian Frobenius algebras with generators $\bigoplus Z^1_{\lambda\mu}\lambda \boxtimes \bar{\mu}$ in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$ and $\bigoplus Z^2_{\mu\nu}\mu \boxtimes \bar{\nu}$ in $\mathcal{C}_2 \boxtimes \mathcal{C}_3^{\mathrm{opp}}$.

We would like to have a new Frobenius algebra with the generator $\bigoplus(\sum_\mu Z^1_{\lambda\mu} Z^2_{\mu\nu})\lambda \boxtimes \bar{\nu}$. That is, the matrix part is just given by a matrix multiplication.

Locality and modular invariance of composition

We can construct a Frobenius algebra with the generator $\bigoplus(\sum_\mu Z^1_{\lambda\mu} Z^2_{\mu\nu})\lambda \boxtimes \bar{\nu}$ by considering a tensor product functor and taking an intermediate Frobenius algebra but this is reducible in general. We have a notion of irreducible decomposition of a Frobenius algebra corresponding to irreducible decomposition of an operator algebra.

We show that after irreducible decomposition, each Frobenius algebra is local and Lagrangian. (K 2017) Being Lagrangian is shown to be equivalent to modular invariance property. (Müger, K-Longo)
On the matrix level, we thus have a decomposition $\sum_\mu Z^1_{\lambda\mu} Z^2_{\mu\nu} = \sum_i Z^{3,i}_{\lambda\nu}$.

The α-induction construction

For a modular tensor category $\mathcal{C}$ and a Frobenius algebra, we have a machinery of α-induction, similar to the induction procedure in the classical representation theory. This process depends on a choice of a braiding, and we use symbols $\alpha^{\pm}_{\lambda}$ for the induced objects arising from λ in the modular tensor category $\mathcal{C}$. This corresponds to condensation of anyons. (Branching rule of Bais-Slingerland)

The α-induction machinery produces a modular invariant from $Z_{\lambda\mu} = \langle \alpha^{+}_{\lambda}, \alpha^{-}_{\mu} \rangle$ (Böckenhauer-Evans-K) and an irreducible local Lagrangian Frobenius algebra in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}}$ (Rehren). The latter coincides with the one of Fröhlich-Fuchs-Runkel-Schweigert. (Bischoff-K-Longo)

Boundary-bulk duality and modular tensor category

A general tensor category has no braiding, but we have a general procedure called the Drinfeld center which produces a new modular tensor category from a given tensor category.

The α-induction produces a new tensor category for one choice of $+/-$ braidings. Its Drinfeld center is given by the tensor product of the original modular tensor category and the extended one. (Böckenhauer-Evans-K) This mathematical result coincides with a result called boundary-bulk duality in physical literature.

We have recently extended the above result to the relative case. (K 2018) This is related to a notion of an orbifold subfactor.