Finite time singularity of the nematic liquid crystal flow in dimension three

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Outline

1. Introduction
2. First Example of finite time singularity
3. Second Example of finite time singularity

Mathematical modeling of Nematic Liquid Crystals [1]

Static theory

- Oseen-Frank energy
- Q-tensor/ Landau-de Gennes energy
- ......

Hydrodynamic theory

- Ericksen-Leslie system
- Dynamic system of Q-tensor
- ......

[1] F. H. Lin and C. Y. Wang 2014

Dynamic Model of Nematic Liquid Crystals

Ericksen-Leslie

$$\frac{d}{dt}\int|\mathbf{u}|^2+\chi|\mathbf{n}_t|^2+W(\mathbf{n},\nabla\mathbf{n})=-2\int|\nabla\mathbf{u}|^2+|\mathbf{n}_t|^2$$

- $\mathbf{u}$ velocity field of underlying incompressible fluid, $\operatorname{div}\mathbf{u}=0$
- $\mathbf{n}$ direction field of nematic liquid crystal molecules with $|\mathbf{n}|=1$
- Oseen-Frank potential for nematic liquid crystal:

  $$W(\mathbf{n},\nabla\mathbf{n})=\alpha|\mathbf{n}\times(\nabla\times\mathbf{n})|^2+\beta(\nabla\cdot\mathbf{n})^2+\gamma(\mathbf{n}\cdot\nabla\times\mathbf{n})^2+\eta\left[\mathrm{tr}(\nabla\mathbf{n})^2-(\nabla\cdot\mathbf{n})^2\right]$$

  α, β, γ and η are all positive viscosity constants
- When $\alpha=\beta=\gamma=\eta=1$

  $$W(\mathbf{n},\nabla\mathbf{n})=|\nabla\mathbf{n}|^2$$

$$\frac{d}{dt}\int |\mathbf{u}|^2 + \chi|\mathbf{n}_t|^2 + |\nabla \mathbf{n}|^2 = -2\int |\nabla \mathbf{u}|^2 + |\mathbf{n}_t|^2$$

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{n}\odot\nabla\mathbf{n} - \frac{1}{2}|\nabla\mathbf{n}|^2\mathbb{I}_3) \\ \nabla\cdot\mathbf{u} = 0 \\ \chi\frac{D^2}{Dt^2}\mathbf{n} + \mathbf{n}_t + \mathbf{u}\cdot\nabla\mathbf{n} = \Delta\mathbf{n} + \left(|\nabla\mathbf{n}|^2 - |\mathbf{n}_t|^2\right)\mathbf{n} \end{cases}$$

- $\frac{D}{Dt}$ material derivative
- $\chi \equiv 0$ simplified model of liquid crystal flows

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{n}\odot\nabla\mathbf{n} - \frac{1}{2}|\nabla\mathbf{n}|^2\mathbb{I}_3) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{n}_t + \mathbf{u}\cdot\nabla\mathbf{n} = \Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n} \end{cases}$$

$$\frac{d}{dt}\int |\mathbf{u}|^2 + \chi|\mathbf{n}_t|^2 + |\nabla \mathbf{n}|^2 = -2\int |\nabla \mathbf{u}|^2 + |\mathbf{n}_t|^2$$

- $\mathbf{n} \equiv$ constant vector $\Rightarrow$ Navier-Stokes equations

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = 0 \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

- $\mathbf{u} \equiv \mathbf{0} \Rightarrow$ critical wave equations

$$\chi\mathbf{n}_{tt} + \mathbf{n}_t = \Delta\mathbf{n} + \left(|\nabla\mathbf{n}|^2 - \chi|\mathbf{n}_t|^2\right)\mathbf{n}$$

- $\mathbf{u} \equiv \mathbf{0}$, $\chi \equiv 0 \Rightarrow$ heat flow of harmonic maps to the unit sphere

$$\mathbf{n}_t = \Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n}$$

Hydrodynamic flow of nematic liquid crystals

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_3) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} \end{cases} \tag{1}$$

- $\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$ bounded smooth domain
- $\mathbf{d} : \Omega \times [0, T) \to \mathbb{S}^2$ director field of nematic liquid crystal molecules
- $\nabla\mathbf{d}\odot\nabla\mathbf{d} = \left(\langle \frac{\partial\mathbf{d}}{\partial\mathbf{x}_i}, \frac{\partial\mathbf{d}}{\partial\mathbf{x}_j}\rangle\right)_{1\le i,j\le 3} \in \mathbb{R}^{3\times 3}$ stress tensor induced by d
- Simplified version of Ericksen-Leslie system modeling the hydrodynamics of liquid crystal materials derived by Lin (89')

Main difficulties

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot\left(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_3\right) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} \end{cases}$$

- A strongly coupled system between the Navier-Stokes equations and the (transported) heat flow of harmonic maps into two sphere
- Scaling

$$\mathbf{u}_\lambda(\mathbf{x},t) = \lambda\mathbf{u}(\lambda\mathbf{x},\lambda^2 t),\ P_\lambda(\mathbf{x},t) = \lambda^2 P(\lambda\mathbf{x},\lambda^2 t),\ \mathbf{d}_\lambda(\mathbf{x},t) = \mathbf{d}(\lambda\mathbf{x},\lambda^2 t)$$

- Energy inequality

$$\sup_t \int |\mathbf{u}|^2 + |\nabla\mathbf{d}|^2\,d\mathbf{x} + \int\int |\nabla\mathbf{u}|^2 + \left|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\right|^2 d\mathbf{x}dt \leq \int |\mathbf{u}_0|^2 + |\nabla\mathbf{d}_0|^2\,d\mathbf{x}$$

Navier-Stokes equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

- Leary (34'), Hopf (50')
  Existence of global Leray-Hopf weak solutions
- Caffarelli-Kohn-Nirenberg (82'), Lin (98')
  Partial regularity of suitable weak solutions in 3D
- Regularity (singularity) and uniqueness of weak solutions in 3D
  OPEN!

Heat flow of harmonic maps

$$\partial_t \mathbf{d} - \Delta \mathbf{d} = |\nabla \mathbf{d}|^2 \mathbf{d}$$

- Struwe (85'), Chang (89')
  Existence of unique global weak solution with finitely many singular points in 2D

- Chang-Ding-Ye (92')
  Example of finite time blowup in 2D

$$\varphi_t = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin\varphi\cos\varphi}{r^2}$$

Heat flow of harmonic maps

$$\partial_t \mathbf{d} - \Delta \mathbf{d} = |\nabla \mathbf{d}|^2 \mathbf{d}$$

- Chen-Struwe (89'), Chen-Lin (93')
  Existence of global weak solutions with partial regularity in 3D
- Coron-Ghidaglia (89') Chen-Ding (90')
  Example of finite time blowup in 3D

Hydrodynamic flow of nematic liquid crystals

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 \mathbb{I}_3) \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \end{cases}$$

- Wang (11')
  Global wellposedness if

$$\|\mathbf{u}_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} + \|\mathbf{d}_0\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq \epsilon$$

- Unique (regular) global solution in 2D
  Xu-Zhang (12'): $\exp\left(216(\|\mathbf{u}_0\|_{L^2}^2 + \frac{1}{16})^2\right) \|\nabla \mathbf{d}_0\|_{L^2}^2 < \frac{1}{16}$
  Lei-Li-Zhang (14'): third component $d_0^3 > 0$

Hydrodynamic flow of nematic liquid crystals

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_3) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} \end{cases}$$

- Lin-Lin-Wang (10')
  Existence of global weak solutions for initial and boundary problem, with finitely many possible singular times in 2D

  Finitely many possible singular points: Open!
- Lin-Wang (10')
  Uniqueness of weak solutions in 2D

Hydrodynamic flow of nematic liquid crystals

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = -\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2}|\nabla \mathbf{d}|^2 \mathbb{I}_3) \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \end{cases}$$

- Lin-Wang (14')
  Existence of global weak solutions in 3D when $d_0(x) \in \mathbb{S}^2_+$
- TH (13')
  Uniqueness of weak solutions if

$$\mathbf{u}, \nabla \mathbf{d} \in L^p_t L^q_x(\mathbb{R}^{n+1}_+), \quad \frac{2}{p} + \frac{n}{q} = 1$$

- For 3D, existence of weak solutions for general initial data OPEN!

Finite time singularity in dimension three: example 1

Theorem 1 (TH-Lin-Liu-Wang 15')

The short time smooth solution $(\mathbf{u}, \mathbf{d})$ *to the system (1) must blow up at finite time* T_0, *if*

$$\mathbf{u}_0(\mathbf{x}) = (x, y, -2z)$$

$$\mathbf{d}_0(\mathbf{x}) = \Big(\frac{x}{r}\sin\varphi_0(r), \frac{y}{r}\sin\varphi_0(r), \cos\varphi_0(r)\Big)$$

where $\mathbf{x} = (x, y, z) \in \Omega = B_1^2 \times [0, 1]$, $r = \sqrt{x^2 + y^2}$, $\varphi_0 \in C^\infty([0, 1])$, *with* $\varphi_0(0) = 0$ *and* $|\varphi_0(1)| > \pi$

$$\begin{cases} \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}) & \mathbf{x} \in \partial\Omega,\ t > 0, \\ \mathbf{d}(\mathbf{x}, t) = \mathbf{d}_0(\mathbf{x}) & \mathbf{x} \in \partial B_1^2 \times [0, 1],\ t > 0, \\ \frac{\partial \mathbf{d}}{\partial z}(\mathbf{x}, t) = \mathbf{0} & \mathbf{x} \in B_1^2 \times \{0, 1\},\ t > 0. \end{cases}$$

- Examples of finite time singularities in dimension two: Open!

Axisymmetric solutions

- Axisymmetric forms

$$\begin{aligned}\mathbf{f}(\mathbf{x},t) =&(f^1(\mathbf{x},t), f^2(\mathbf{x},t), f^3(\mathbf{x},t))\\ =&f^r(r,\theta,z,t)\mathbf{e}^r + f^\theta(r,\theta,z,t)\mathbf{e}^\theta + f^3(r,\theta,z,t)\mathbf{e}^3\end{aligned}$$

where

$$\mathbf{e}^r = (\cos\theta, \sin\theta, 0), \quad \mathbf{e}^\theta = (-\sin\theta, \cos\theta, 0), \quad \mathbf{e}^3 = (0,0,1)$$

- Axisymmetric forms without swirls,

$$\mathbf{u}(r,\theta,z,t) = u^r(r,z,t)\mathbf{e}^r + u^3(r,z,t)\mathbf{e}^3$$

$$\mathbf{d}(r,\theta,z,t) = \sin\varphi(r,z,t)\mathbf{e}^r + \cos\varphi(r,z,t)\mathbf{e}^3$$

Axisymmetric solutions without swirl

$$\begin{aligned}
\frac{\widetilde{D}u^r}{Dt} - \widetilde{\Delta}u^r + \frac{1}{r^2}u^r + P_r &= -\,(\widetilde{\Delta}\varphi - \frac{\sin(2\varphi)}{2r^2})\varphi_r \\
\frac{\widetilde{D}u^3}{Dt} - \widetilde{\Delta}u^3 + P_z &= -\,(\widetilde{\Delta}\varphi - \frac{\sin(2\varphi)}{2r^2})\varphi_z \\
\frac{1}{r}(ru^r)_r + (u^3)_z &= 0 \\
\frac{\widetilde{D}\varphi}{Dt} - \widetilde{\Delta}\varphi &= -\,\frac{\sin(2\varphi)}{2r^2}
\end{aligned}$$

where

$$\frac{\widetilde{D}}{Dt} := \partial_t + u^r\partial_r + u^3\partial_z, \quad \widetilde{\Delta} := \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$$

- Global existence of weak solutions: Open!

Axisymmetric solutions for heat flow of harmonic maps

$$\varphi(r,t) \Rightarrow \varphi_t = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin\varphi\cos\varphi}{r^2}$$

- Chang-Ding (90')
  Gobal smooth solutions if $|\varphi_0| \leq \pi$
- Chang-Ding-Ye (92')
  Finite time blowup if $|\varphi_0(1)| > \pi$
- Grotowski (91', 93')

$$\varphi(r,z,t) \Rightarrow \varphi_t = \varphi_{rr} + \frac{\varphi_r}{r} + \varphi_{zz} - \frac{\sin\varphi\cos\varphi}{r^2}$$

Axisymmetric solutions for Navier-Stokes equations

$$\begin{aligned}
\frac{\widetilde{D}u^r}{Dt} - \widetilde{\Delta}u^r + \frac{1}{r^2}u^r + P_r =& 0 \\
\frac{\widetilde{D}u^3}{Dt} - \widetilde{\Delta}u^3 + P_z =& 0 \\
\frac{1}{r}(ru^r)_r + (u^3)_z =& 0
\end{aligned}$$

- Ladyzheskaya (68'), Leonardi-Málek-Nečas-Pokorný (99')
  Gobal smooth axisymmetric solutions without swirl
- 2D case: $u^r = u^r(r,t)$, $(u^3)_z \equiv 0 \quad \Rightarrow u^r(r,t) \sim \frac{C}{r}$

Liquid crystal flows

Lei-Dong (12'): global smooth solution in 2D in forms of

- $\mathbf{u}$ rotational

$$\mathbf{u}(r,t) = u^{\theta}(r,t)(-\sin\theta, \cos\theta, 0)$$

$$\nabla = \mathbf{e}^r \partial_r + r^{-1}\mathbf{e}^{\theta}\partial_{\theta} \Rightarrow \operatorname{div}\mathbf{u} = 0$$

- p, $\mathbf{d}$ symmetric

$$p = p(r,t)$$

$$\mathbf{d}(r,t) = (\sin\psi(r,t)\cos\phi(r,t), \sin\psi(r,t)\sin\phi(r,t), \cos\psi(r,t))$$

$0 < \delta_1 < \psi_0(r) < \pi - \delta_1$, and $\phi_0 = \Phi(\psi_0)$ for some smooth function Φ

Simplified case

$(r, \theta, z) \in [0,1] \times [0, 2\pi] \times [0,1]$ and $t \geq 0$

$$\begin{cases} \mathbf{u}(r,\theta,z,t) := v(r,t)\mathbf{e}^r + w(z,t)\mathbf{e}^3 \\ \mathbf{d}(r,\theta,z,t)(= \mathbf{d}(r,t)) := \sin\varphi(r,t)\mathbf{e}^r + \cos\varphi(r,t)\mathbf{e}^3 \\ P(r,\theta,z,t) := Q(r,t) + R(z,t) \end{cases}$$

$$\begin{aligned} v_t + v v_r - v_{rr} - \frac{v_r}{r} + \frac{1}{r^2} v + Q_r =& - \left(\varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}\right)\varphi_r \\ w_t + w w_z - w_{zz} + R_z =& 0 \\ \frac{1}{r}(rv)_r + w_z =& 0 \\ \varphi_t + v\varphi_r =& \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2} \end{aligned}$$

Simplified case

- Initial condition

$$v|_{t=0} = v_0(r) = r, \quad w|_{t=0} = w_0(z) = -2z, \;\; \varphi|_{t=0} = \varphi_0(r)$$

for some $\varphi_0 \in C^\infty([0,1])$ with $\varphi_0(0) = 0$ and

$$\mathbf{u}_0(r,\theta,z,t) := r(\cos\theta, \sin\theta, 0) - 2z(0,0,1)$$

$$\frac{1}{r}(rv_0)_r + (w_0)_z = 0$$

- Boundary condition

$$\begin{aligned} v(0,t) =& 0, \quad w(0,t) = 0, \quad \varphi(0,t) = 0 \\ v(1,t) =& 1, \quad w(1,t) = -2, \quad \varphi(1,t) = \varphi_0(1) \end{aligned}$$

Local existence of smooth solutions

Claim: (v, w) is a stationary fluid field

$$v(r,t) = v_0(r) = r, \quad w(z,t) = w_0(z) = -2z$$

$$\frac{1}{r}(rv)_r + w_z = 0$$

- Differentiate with respect to $z \Rightarrow w_{zz}(z,t) = 0$
  Solve ODE $\Rightarrow w(z,t) = a_1(t)z + a_2(t)$
  Initial and boundary condition $\Rightarrow w(z,t) = -2z$
- Differentiate with respect to $r \Rightarrow \left(\frac{1}{r}(rv)_r\right)_r(r,t) = 0$
  Solve ODE $\Rightarrow rv(r,t) = b_1(t)r^2 + b_2(t)$
  Initial and boundary condition $\Rightarrow v(r,t) = r$

Local existence of smooth solutions

$$
\begin{aligned}
v_t + vv_r - \left(v_{rr} + \frac{v_r}{r} - \frac{1}{r^2}v\right) + Q_r =& - \left(\varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}\right)\varphi_r \\
w_t + ww_z - w_{zz} + R_z =& 0 \\
\frac{1}{r}(rv)_r + w_z =& 0 \\
\varphi_t + v\varphi_r =& \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}
\end{aligned}
$$

$$\Downarrow$$

$$R(z,t) = -2z^2 + c(t)$$

$$Q_r = -\left(\varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}\right)\varphi_r - r,$$

$$\varphi_t + r\varphi_r = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}$$

Local existence of smooth solutions

$$\varphi_t + r\varphi_r = \ \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}$$

- Step 1: Local existence of a unique smooth solution to

$$\mathbf{d}_t + \mathbf{x} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}$$

  by contraction mapping theorem (e.g. Lin-Lin-Wang (10'))

- Step 2: $d(x, y, t)$ is axisymmetric

$$d(x, y, t) = (\sin\varphi(r)\cos\theta, \sin\varphi(r)\sin\theta, \cos(r)\varphi)$$

  if the initial and boundary data are axisymmetric.

Gobal existence

Theorem 2

Suppose $\varphi_0 \in C^\infty([0,1])$ *and* $|\varphi_0(r)| < \pi$ *for all* $r \in [0,1]$. *Then there is a unique, global smooth solution to*

$$\varphi_t + r\varphi_r = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}$$

- Comparison principle
- Maximum principle
- Chang-Ding (90')

Finite time singularity

Theorem 3

The local smooth solution to

$$\varphi_t + r\varphi_r = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin(2\varphi)}{2r^2}$$

must blowup at finite time T_0 *for some*

$$\varphi_0 \in C^\infty([0,1]), \quad \varphi_0(0) = 0, \quad |\varphi_0(1)| > \pi.$$

And

$$\varphi_r(0,t) \to \infty \text{ as } t \to T_0^-.$$

Main idea of proof

$$\varphi_t + r\varphi_r = \varphi_{rr} + \frac{\varphi_r}{r} - \frac{\sin\varphi\cos\varphi}{r^2}$$

- $\dfrac{d}{dt} r(R,t) = r(R,t), \quad r(R,0) = R \Rightarrow r = Re^t$
- $\dfrac{\partial(r,t)}{\partial(R,t)} = \begin{pmatrix} e^t & Re^t \\ 0 & 1 \end{pmatrix}$
- $\varphi_t = e^{-2t}\left(\varphi_{RR} + \dfrac{\varphi_R}{R} - \dfrac{\sin\varphi\cos\varphi}{R^2}\right)$
- Chang-Ding-Ye (92'): construct subsolution $f(r,t)$ with

$$f(0,t) = 0, \quad \lim_{t\uparrow T_0^-} f_r(0,t) = +\infty$$

Remarks

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_3) \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{d}_t + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d} \end{cases}$$

- For subsolution

$$\widetilde{d} = \big(\sin f(r,t)\cos\theta, \sin f(r,t)\sin\theta, \cos f(r,t)\big)$$

$$\Rightarrow \quad (\nabla\widetilde{d}\odot\nabla\widetilde{d} - \frac{1}{2}|\nabla\widetilde{d}|^2\mathbb{I}_2)\Big|_{r=0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- It is unclear whether the solution (u, d, P) constructed in this example enjoys the energy inequality

Finite time singularity: example 2

Theorem 4 (TH-Lin-Liu-Wang 15')

There exists $\epsilon_0 > 0$ *such that if* $\mathbf{u}_0 \in C^\infty_{0,\mathrm{div}}(B_1^3, \mathbb{R}^3)$ *and* $\mathbf{d}_0 \in C^\infty_{\mathbf{e}}(B_1^3, \mathbb{S}^2)$ *satisfies that* $\mathbf{d}_0$ *is not homotopic to the constant map* $\mathbf{e} : B_1^3 \to \mathbb{S}^2$ *relative to* ∂B_1^3*, and*

$$E(\mathbf{u}_0, \mathbf{d}_0) := \frac{1}{2} \int_{B_1^3} (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) \leq \epsilon_0^2.$$

Then the short time smooth solution $(\mathbf{u}, \mathbf{d}, P)$ *to the nematic liquid crystal flow must blow up before time* $T = 1$*.*

$$C^\infty_{0,\mathrm{div}}(B_1^3, \mathbb{R}^3) := \Big\{ v \in C^\infty(B_1^3, \mathbb{R}^3) \ \Big| \ \nabla \cdot v = 0 \Big\},$$

$$C^\infty_{\mathbf{e}}(B_1^3, \mathbb{S}^2) := \Big\{ d \in C^\infty(B_1^3, \mathbb{S}^2) \ \Big| \ d = \mathbf{e} \text{ on } \partial B_1^3 \Big\}.$$

$$\mathbf{e} = (0, 0, 1) \in \mathbb{S}^2$$

Remarks

- The solution satisfies the global energy inequality.

$$E(u(t), d(t)) + \int_0^t \int_{B_1^3} \left(|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2\right) \leq E(u_0, d_0)$$

- Let

$$H(z, w) = (|z|^2 - |w|^2, 2zw) : \mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{R} \times \mathbb{C}$$

be the Hopf map. Then H is not homotopic to the constant map $\mathbf{e}$. Let $\Phi : B_1^3 \to \mathbb{S}^3 \setminus \{\mathbf{e}\}$ be a differmorphism sending ∂B_1^3 to $\mathbf{e}$ and $D_\lambda : \mathbb{S}^3 \to \mathbb{S}^3$ be a $\lambda-$dilation centered at $\mathbf{e}$. Then

$$\mathbf{d}_0 = H \circ D_\lambda \circ \Phi : B_1^3 \to \mathbb{S}^2$$

satisfies the assumption of the theorem with $\lambda << 1$.

Main idea of proof

- Consider approximated harmonic maps

$$\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} = \tau(\mathbf{d})$$

with the energy inequality:

$$E(\mathbf{u}(t), \mathbf{d}(t)) + \int_0^t \int_{B_1^3} \left(|\nabla \mathbf{u}|^2 + |\tau(\mathbf{d})|^2\right) \leq E(u_0, d_0) \leq \epsilon^2,$$

for all $0 \leq t \leq 1$.

- By Fubini's theorem, there exists $t_1 \in (\frac{1}{2}, 1)$

$$E(\mathbf{u}(t_1), \mathbf{d}(t_1)) + \int_{B_1^3} \left(|\nabla \mathbf{u}(t_1)|^2 + |\tau(\mathbf{d}(t_1))|^2\right) \leq 8\epsilon^2.$$

Main idea of proof

- ϵ-a priori estimate inspired by Ding-Wang (07') and Lin-Wang (15')

$$\left[d(t_1)\right]_{C^{\frac{1}{2}}(B_1^3)} \leq C\sqrt{\epsilon}.$$

- $\mathbf{d}(t_1)(B_1^3) \subset B_{C\epsilon}^3(\mathbf{e}) \cap \mathbb{S}^2 \Rightarrow \mathbf{d}(t_1)$ homotopic to $\mathbf{e}$ relative to ∂B_1^3

  $\Rightarrow \mathbf{d}_0$ homotopic to $\mathbf{e}$ relative to ∂B_1^3

  $\Rightarrow$ Contradiction!

- We say f is homotopic to g relative to ∂B_1^3 if there exists a continuous map $\Phi \in C\left(\overline{B_1^3} \times [0,1], \mathbb{S}^2\right)$ such that
  (i) $\Phi(\cdot, t) = f(\cdot) = g(\cdot)$ on ∂B_1^3, for all $0 \leq t \leq 1$; and
  (ii) $\Phi(\cdot, 0) = f(\cdot)$ and $\Phi(\cdot, 1) = g(\cdot)$ in B_1^3.

Thank you!