Extinction and blowup of weak solutions to the Navier-Stokes equations

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KE: Navier-Stokes equations, Leray equations, blowup, extinction

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Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}$$

with smooth initial data with finite energy in $\mathbb{R}^3$

J. Leray, Acta Math. **63**, 193 (1934).

Motivation

- “log” blowup in critical norms

  characterisation of possible blowup via extinction

- dynamic scaling and turbulent solutions, known to exist in 3D & 4D

  strong form energy inequality holds (except for singular points)

Plan

1. Scale-invariance and criticality

2. Constraint on the extinction phenomenon

3. Weak form of Leray's dynamic scaling (*attempt*)

4. 4D N-S and generalised dissipativity

5. Summary

1. Scale-invariance and criticality

$$\boldsymbol{x} \to \boldsymbol{x}/\lambda, t \to t/\lambda^2$$

$$\boldsymbol{u}(\boldsymbol{x}, t) \to \boldsymbol{U}(\boldsymbol{\xi}, \tau) \equiv \lambda^{-1}\boldsymbol{u}(\boldsymbol{x}/\lambda, t/\lambda^2)$$

$$p(\boldsymbol{x}, t) \to P(\boldsymbol{\xi}, \tau) \equiv \lambda^{-2}p(\boldsymbol{x}/\lambda, t/\lambda^2)$$

example:

$$\boldsymbol{x} = \lambda\boldsymbol{\xi}, \boldsymbol{t} = \lambda^2\boldsymbol{\tau}, \boldsymbol{u} = \lambda^{-1}\boldsymbol{U}$$

$$\int |\boldsymbol{u}|^q d\boldsymbol{x} = \lambda^{n-q} \int |\boldsymbol{U}|^q d\boldsymbol{\xi},$$

$q < n$: super-critical, $q = n$: critical, $q > n$: sub-critical
n =spatial dimension

$$\int_{\mathbb{R}^4} |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} = [\nu^2],\ \int_{\mathbb{R}^2} |\boldsymbol{u}|^2 d\boldsymbol{x} = [\nu^2]$$

3D blowup criterion with $H^{1/2}$-norm $[\nu^2]$ $\quad (\Lambda \equiv (-\triangle)^{1/2} \leftrightarrow |\boldsymbol{k}|)$

$$\int |\Lambda^{1/2}\boldsymbol{u}|^2 d\boldsymbol{x} \equiv \int \boldsymbol{u}\cdot\Lambda\boldsymbol{u} d\boldsymbol{x} \le \left(\int |\boldsymbol{u}|^2 d\boldsymbol{x} \int |\boldsymbol{\omega}|^2 d\boldsymbol{x}\right)^{1/2}$$

It has the same physical dimension as

$$\text{Helicity} \quad \int \boldsymbol{u}\cdot\boldsymbol{\omega} d\boldsymbol{x} \le \left(\int |\boldsymbol{u}|^2 d\boldsymbol{x} \int |\boldsymbol{\omega}|^2 d\boldsymbol{x}\right)^{1/2}$$

$$\frac{1}{2}|\Lambda^{1/2}\boldsymbol{u}|^2 \le \boldsymbol{u}\cdot\Lambda\boldsymbol{u}, \quad \text{Cordoba-Cordoba(2003)}$$

Notations

energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^3)}$$

enstrophy

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{\omega}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{H^1(\mathbb{R}^3)}$$

"helicity"

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\Lambda^{1/2}\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \frac{1}{2}\|\boldsymbol{u}\|^2_{H^{1/2}(\mathbb{R}^3)}$$

Leray's attempt to build self-similar blowup weak solutions (1934)

$$\boldsymbol{u}(\boldsymbol{x},t) = \begin{cases} \dfrac{\boldsymbol{U}(\boldsymbol{\xi})}{[2a(t_* - t)]^{1/2}} \ (0 \le t < t_*), \\ 0 \quad (t \ge t_*) \end{cases}$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{[2a(t_* - t)]^{1/2}}$$

Leray equations

$$\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}} P + \nu \triangle_{\boldsymbol{\xi}} \boldsymbol{U}, \ \ \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{U} = 0$$

If $\boldsymbol{U} \in L^3(\mathbb{R}^3)$ then $\boldsymbol{U} \equiv 0$ Nečas, Růžička and Šverák (1996), also Tsai (1998)

3D N-S eqs: $H^{1/2}$-critical

$$\int_{\mathbb{R}^3} |\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \sqrt{2a(t_* - t)} \int_{\mathbb{R}^3} |\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$

$$\int_{\mathbb{R}^3} |\Lambda^{1/2}\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} = \int_{\mathbb{R}^3} |\Lambda_{\boldsymbol{\xi}}^{1/2}\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$

$$\int_{\mathbb{R}^3} |\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2 d\boldsymbol{x} \frac{1}{\sqrt{2a(t_* - t)}} \int_{\mathbb{R}^3} |\nabla_{\boldsymbol{\xi}}\boldsymbol{U}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}$$

Blowup criteria

$$\text{Leray bound} \quad \int |\boldsymbol{\omega}|^2 d\boldsymbol{x} \geq C \frac{\nu^{3/2}}{\sqrt{t_* - t}}$$

$$\|\boldsymbol{u}\|_{L^3(\mathbb{R}^3)} \to \infty \quad \text{as} \quad t \to t_*$$

Escauriaza, Seregin & Šverák (2003). Hence

$$\|\boldsymbol{u}\|_{H^{1/2}(\mathbb{R}^3)} \to \infty \quad \text{as} \quad t \to t_*$$

$$\sqrt{\|\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{H^1}} \geq \|\boldsymbol{u}\|_{H^{1/2}} \geq \|\boldsymbol{u}\|_{L^3}$$

cf. With $\epsilon = \frac{p-3}{2p}$

$$\sqrt{\|\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}\|_{H^{1+4\epsilon}}} \geq \|\boldsymbol{u}\|_{H^{(1+4\epsilon)/2}} \geq \|\boldsymbol{u}\|_{L^{3/(1-2\epsilon)}} \geq 2C \frac{\nu\epsilon}{\{\nu(t_* - t)\}^{\epsilon}}$$

2. Constraint on the extinction phenomenon

Observation (weak solution)

$$\text{If blowup :} \; c\frac{\nu^{3/2}}{\sqrt{t_* - t}} \leq Q(t) \Rightarrow c\frac{\nu^{3/2}}{\sqrt{t_* - t}} E(t) \leq E(t)Q(t)$$

$$\text{Assume } E(t)Q(t) \leq \; \nu^4 \text{'} \log \frac{1}{t - t_*} \text{'}$$

$$E(t) \leq c\nu^{5/2}\sqrt{t_* - t}\text{'} \log \frac{1}{t - t_*} \text{'}$$

$$\lim_{t \to t_*} E(t) = 0, \;\; \text{extinction}$$

$$E(t) = 0 \;\; \text{for} \;\; t \geq t_*, \; \text{a trivial solution}$$

Physicists do not (cannot) distinguish

$$EQ = O(\nu^4): \text{ small initial data}$$

and

$$EQ = O\left(\nu^4 \log \frac{1}{t_* - t}\right): \text{ critical blowup}$$

to leading-order

Enstrophy equation

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\omega} d\boldsymbol{x} - \nu \int_{\mathbb{R}^3} |\nabla \times \boldsymbol{\omega}|^2$$

well-known bound $\frac{dQ}{dt} \leq C\frac{Q^3}{\nu^3} - \frac{5\nu}{4}\frac{Q^2}{E}$

With $\frac{dE}{dt} = -2\nu Q \;\Rightarrow\; \frac{d}{dt}\log(EQ) \leq C\frac{Q^2}{\nu^3} = C\frac{(EQ)^2}{\nu^3 E^2}$

Note:

$$E(t)Q(t) \leq E(0)Q(0)\exp\left(\frac{C}{\nu^3}\int_0^t Q(t')^2 dt'\right)$$

$$Q(t) \approx \frac{c\nu^{3/2}}{\sqrt{t_* - t}} \Rightarrow \text{power-law upperbound,}$$

$$Q(t) \approx \frac{c\nu^{3/2}}{\sqrt{(t_* - t)\log\frac{1}{t_* - t}}} \Rightarrow \text{log upperbound}$$

Recasting the known bound

$$\frac{1}{C\nu^5}\frac{E^2}{Q}\frac{dQ}{dt} \leq f^2 - \frac{5}{4C}f, \quad \text{where} \quad f(t) \equiv \frac{E(t)Q(t)}{\nu^4}$$

quadratic inequality in f

$$f \geq \frac{1}{2}\left(\frac{5}{4C} + \sqrt{\left(\frac{5}{4C}\right)^2 + \frac{4E^2}{C\nu^5 Q}\frac{dQ}{dt}}\right)$$

Essentially,

$$f \geq c\frac{E}{\nu^{5/2}}\sqrt{\frac{d}{dt}\log Q}$$

Question:

small $EQ \Rightarrow$ global regularity

mildly singular $EQ \Rightarrow$ constraints on the evolution ?

A simple analysis

Assume f blows up in a mild fashion: e.g. $f(t) \leq$ ' $\log \frac{1}{t-t_*}$'

More precisely, $\lim_{t \to t_*} (t_* - t)^{\epsilon} f(t) = 0$, $\forall \epsilon (> 0)$

$$\frac{dQ}{dt} \leq C\nu^5 \frac{Q}{E^2} \times \text{'} \log \frac{1}{t-t_*} \text{'}$$

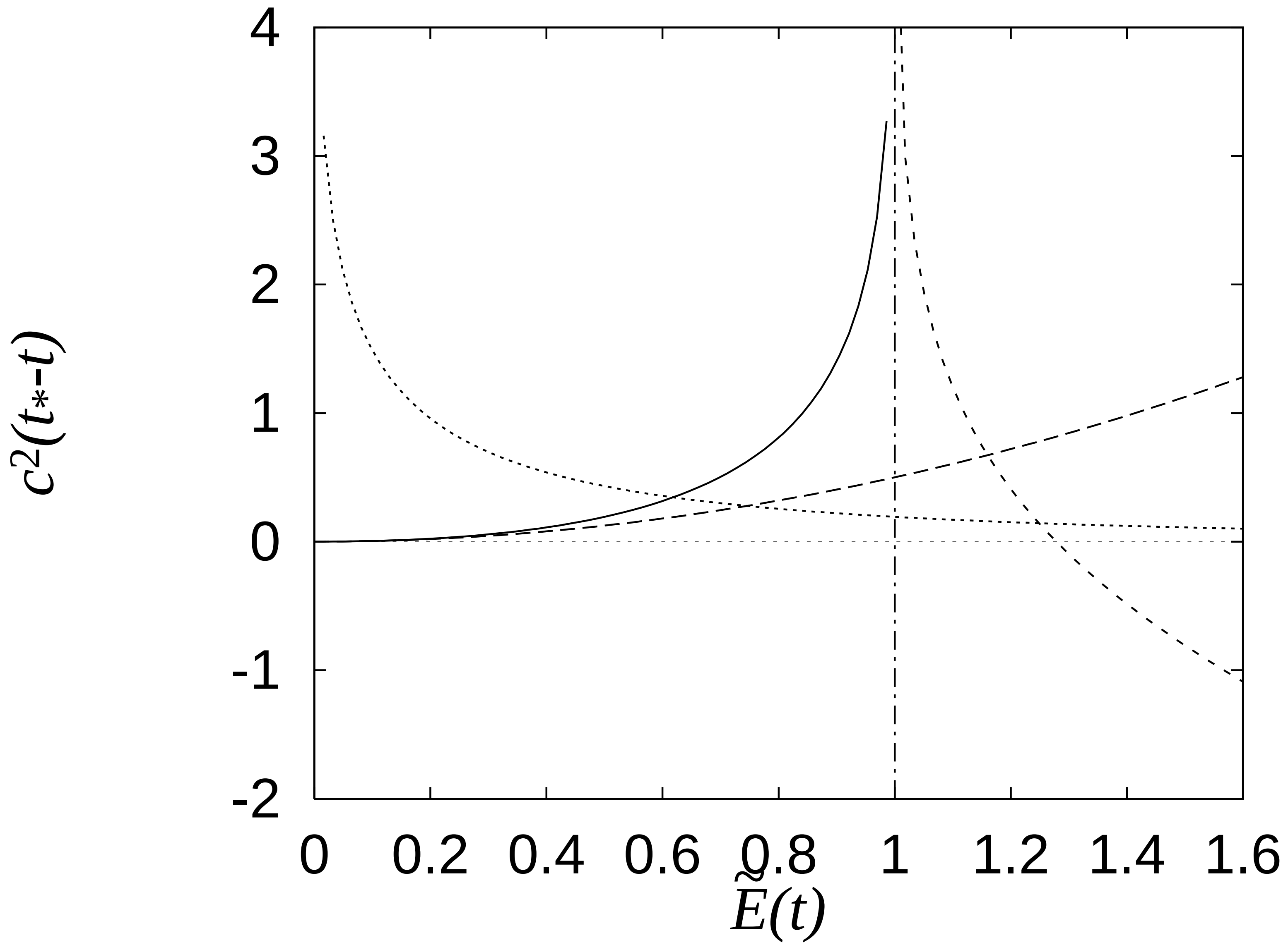
$$\begin{cases} \dfrac{dQ}{dt} \lesssim C\nu^5 \dfrac{Q}{E^2}, \\ \dfrac{dE}{dt} = -2\nu Q \end{cases}$$

By $\tilde{E}(t) \equiv \dfrac{E(t)}{C^{1/2}\nu^{5/2}}$

$$\log|1 - c\,\tilde{E}(t)| + c\,\tilde{E}(t) \gtrsim c^2\,(t - t_*)$$

(t_* : constant)

Taylor expansion around $\tilde{E} = 0$: $\tilde{E}(t) \lesssim \sqrt{2(t_* - t)}$


4
3
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
$c^2(t_*-t)$
$\tilde{E}(t)$

3. Weak form of Leray's dynamic scaling(*attempt*)

Assuming N-S solutions blows up at $t = t_*$ apply dynamics rescaling.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\sqrt{2a(t_* - t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau)$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2a(t_* - t)}},\ \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a}\log\frac{t_*}{t_* - t}$$

$$\left(\lambda(t) = \sqrt{2a(t_* - t)},\ \ t = \frac{1 - e^{-2a\tau}}{2a},\ \ t_* = \frac{1}{2a}\right)$$

non-stationary Leray equations

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + a(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}}P + \nu\triangle_{\boldsymbol{\xi}}\boldsymbol{U},\ \ \nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U} = 0$$

Dynamic scaling has been used to discuss the first singularity t_*

Asymptotic self-similar blowup ruled out.

$$\lim_{\tau\to\infty} \|\boldsymbol{U}(\boldsymbol{\xi},\tau) - \overline{\boldsymbol{U}}(\boldsymbol{\xi})\|_{L^p} = 0,\ \overline{\boldsymbol{U}} \in L^p,\ p \geq n$$

$\Rightarrow \overline{\boldsymbol{U}}$ is a steady solution to the Leray equations.
$\overline{\boldsymbol{U}} \equiv 0$ by NRS (1996)

Chae (2007), Hou & Li (2007)

Leray's structure theorem

"Time singularities are quantised (digitised)"

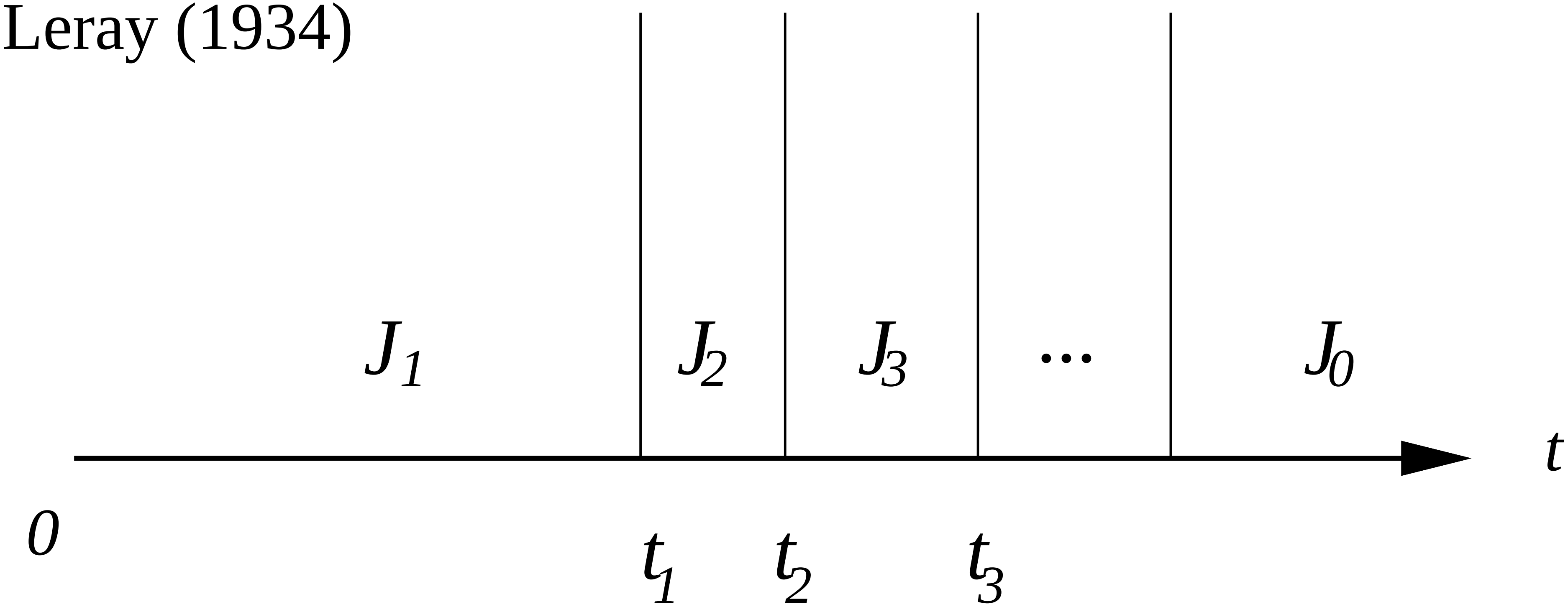
Disjoint, countable time intervals $J_k,\ k = 0, 1, 2, \ldots$

$$\exists T > 0, J_0 = [T, \infty)$$

$$|R^+ - \cup_{k=0}^{\infty} J_k| = 0$$

$$\sum_{k=1}^{\infty} |J_k|^{1/2} < \infty$$

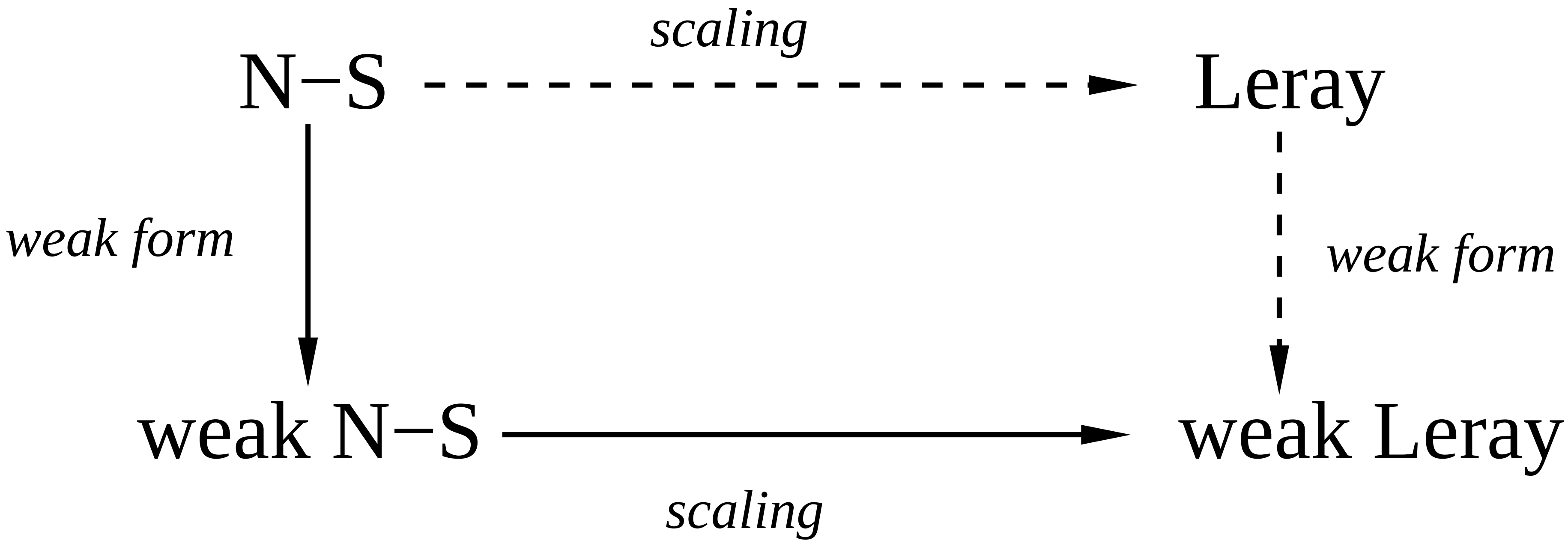
Leray (1934)



4D: Kato (1984)

Two operations: dynamic scaling & weak formulation
Structure theorem valid in 3D and 4D

To apply dynamic scaling to singular points $t_1, t_2, t_3 \ldots$, it is necessary for them to commute.


scaling
N−S
Leray
weak form
weak form
weak N−S
weak Leray
scaling

Route 1

Navier-Stokes equations $\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}$

Weak form

$$\int_0^T \left[-\left(\boldsymbol{u}, \frac{\partial \boldsymbol{\Phi}}{\partial t} \right) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\Phi}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\Phi}) \right] dt = (\boldsymbol{b}, \boldsymbol{\Phi})$$

then scale

$$\int_0^{T'} e^{-(n-1)a\tau} \left[-\left(\boldsymbol{U}, \frac{\partial \boldsymbol{\Phi}}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right)' + (\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + \nu(\nabla \boldsymbol{U}, \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi})' \right] d\tau$$

$$= (\boldsymbol{b}, \boldsymbol{\Phi})'$$

More pricesely $\boldsymbol{\Phi}(\boldsymbol{x}, t) = \boldsymbol{\Phi}(e^{-a\tau}\boldsymbol{\xi}, \frac{1-e^{-2a\tau}}{2a}) \equiv \tilde{\boldsymbol{\Phi}}(\boldsymbol{\xi}, \tau)$

$$\int_0^T \left[- \underbrace{(\boldsymbol{u}, \boldsymbol{\Phi}_t)}_{=(1)} + \underbrace{(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\Phi})}_{=(2)} + \nu \underbrace{(\nabla \boldsymbol{u}, \nabla \boldsymbol{\Phi})}_{=(3)} \right] dt = \underbrace{(\boldsymbol{b}, \boldsymbol{\Phi})}_{=(4)}$$

$$\nabla = \frac{1}{\sqrt{2a(t_* - t)}} \nabla_{\boldsymbol{\xi}}, \quad \frac{\partial}{\partial t} = \frac{1}{2a(t_* - t)} \left(\frac{\partial}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \right)$$

$$(1) = \int \frac{1}{\sqrt{2a(t_* - t)}} \boldsymbol{U} \frac{1}{2a(t_* - t)} \left(\frac{\partial \boldsymbol{\Phi}}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right) (2a(t_* - t))^{n/2} d\boldsymbol{\xi}$$

$$= (2a(t_* - t))^{(n-3)/2} \int \boldsymbol{U} \left(\frac{\partial \boldsymbol{\Phi}}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right) d\boldsymbol{\xi}$$

$$(2) = \int \frac{1}{\sqrt{2a(t_* - t)}} U_j \frac{1}{\sqrt{2a(t_* - t)}} \left(\frac{\partial}{\partial \xi_j} \frac{1}{\sqrt{2a(t_* - t)}} U_i \right) \Phi_i (2a(t_* - t))^{n/2} d\boldsymbol{\xi}$$

$$= (2a(t_* - t))^{(n-3)/2} \int U_j \frac{\partial U_i}{\partial \xi_j} \Phi_i d\boldsymbol{\xi}$$

$$(3) = \nu \int \left(\frac{1}{\sqrt{2a(t_* - t)}} \frac{\partial}{\partial \xi_i} \frac{1}{\sqrt{2a(t_* - t)}} U_j \right) \left(\frac{1}{\sqrt{2a(t_* - t)}} \frac{\partial}{\partial \xi_i} \Phi_j \right) (2a(t_* - t))^{n/2} d\boldsymbol{\xi}$$

$$= (2a(t_* - t))^{(n-3)/2} \nu \int \frac{\partial U_j}{\partial \xi_i} \frac{\partial \Phi_j}{\partial \xi_i} d\boldsymbol{\xi}$$

$$(4) = \int \boldsymbol{b} \boldsymbol{\Phi} (2a(t_* - t))^{n/2} d\boldsymbol{\xi} \Big|_{t=0} = (\boldsymbol{b}, \boldsymbol{\Phi})'$$

Route 2

Leray equations

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}} P + \nu \triangle_{\boldsymbol{\xi}} \boldsymbol{U}$$

then weak form

$$\int_0^{T'} e^{-(n-1)a\tau} \left[-\left(\boldsymbol{U}, \frac{\partial \boldsymbol{\Phi}}{\partial \tau} + a\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right)' + (\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + \nu(\nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi})' \right] d\tau$$

$$= (\boldsymbol{b}, \boldsymbol{\Phi})'$$

$$(\boldsymbol{U}_\tau, \boldsymbol{\Phi})' + (\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + \underbrace{a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})'}_{= -na(\boldsymbol{U}, \boldsymbol{\Phi})' - a(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})'} + a(\boldsymbol{U}, \boldsymbol{\Phi})'$$

$$= -\underbrace{(\nabla_{\boldsymbol{\xi}} P, \boldsymbol{\Phi})'}_{=0} + \underbrace{\nu(\triangle_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})'}_{= -\nu(\nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi})'}$$

$\int_0^{T'} e^{-(n-1)a\tau} (\boldsymbol{U}_\tau, \boldsymbol{\Phi})' d\tau$

$$= \left[e^{-(n-1)a\tau} (\boldsymbol{U}, \boldsymbol{\Phi})' \right]_0^{T'} + (n-1)a \int e^{-(n-1)a\tau} (\boldsymbol{U}, \boldsymbol{\Phi})' d\tau - \int e^{-2a\tau} (\boldsymbol{U}, \boldsymbol{\Phi}_\tau)' d\tau$$

They match.

They formally commute in any dimensions.

For $t > t_1$, non-uniqueness may be an obstacle for weak formulation of Leray equations; it is ambiguous whether/how we may consider e.g.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\sqrt{2a(t_2-t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau)\ (0 \le t < t_2,\ t \ne t_1)$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2a(t_2-t)}},\ \tau = \frac{1}{2a}\log\frac{t_2}{t_2-t}.$$

4. 4D N-S and generalised dissipativity $(P(t) = \frac{1}{2}\|\boldsymbol{u}\|_{H^2}^2)$

$$\frac{dQ}{dt} \le CPQ^{1/2} - 2\nu P$$

$$Q(t)^2 \le E(t)P(t) \equiv \nu^4 g(t)$$

$$\frac{E}{\nu^5}\frac{dQ}{dt} \le Cg(t)\left(g(t)^{1/4} - 2\right)$$

$$g(t) \le C' \log\frac{1}{t-t_*}, \;\Rightarrow\; Q(t) \lesssim Q(0) + C\nu^5 \int_0^t \frac{dt'}{E(t')}$$

Osgood-type condition for blowup

$$\lim_{t\to t_*}\int_0^t \frac{dt'}{E(t')} = \infty \;\; \text{(extinction)}$$

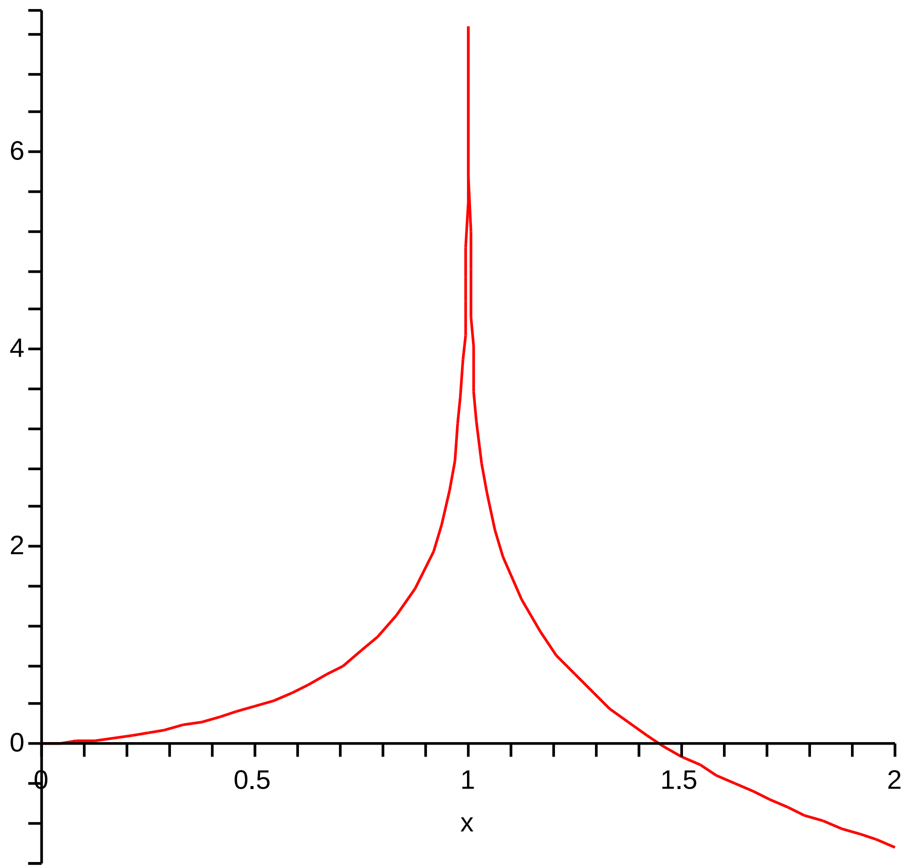
$$\frac{E}{\nu^5}\frac{dQ}{dt} \le Cg(t)\left(g(t)^{1/4} - 2\right)\frac{Q(t)}{\nu^2}, \quad \left(\frac{Q(t)}{\nu^2} \ge 1\right)$$

$$\text{or, } \frac{E}{\nu^3}\frac{d\log Q}{dt} \le Cg(t)\left(g(t)^{1/4} - 2\right)$$

$$g(t) \le C' \log\frac{1}{t - t_*}, \Rightarrow \frac{dQ}{dt} \lesssim C\nu^3\frac{Q}{E}$$

$$\text{By } \frac{dE}{dt} = -2\nu Q, \quad \tilde{E} \equiv \frac{E}{C\nu^3}, \quad \mathrm{Li}(c\tilde{E}(t)) \gtrsim c(t - t_*)$$

$$\text{Logarithmic integral } \mathrm{Li}(x) \equiv \int_0^x \frac{du}{\log u}, \quad (x < 1), \quad \equiv \fint_0^x \frac{du}{\log u}, \quad (x > 1)$$



$$x = \tilde{E}(t),\ y = c(t_* - t)$$

Alors H.Lebesgue, consulté, déclara: "Ne consacrez pas trop de temp à une question aussi rebelle. Faites autre chose!"

J.Leray,
"Aspects de la mécanique théoreique des fluides,"
La vie des Sciences, Compte Rendus,série générale **11** 287(1994).

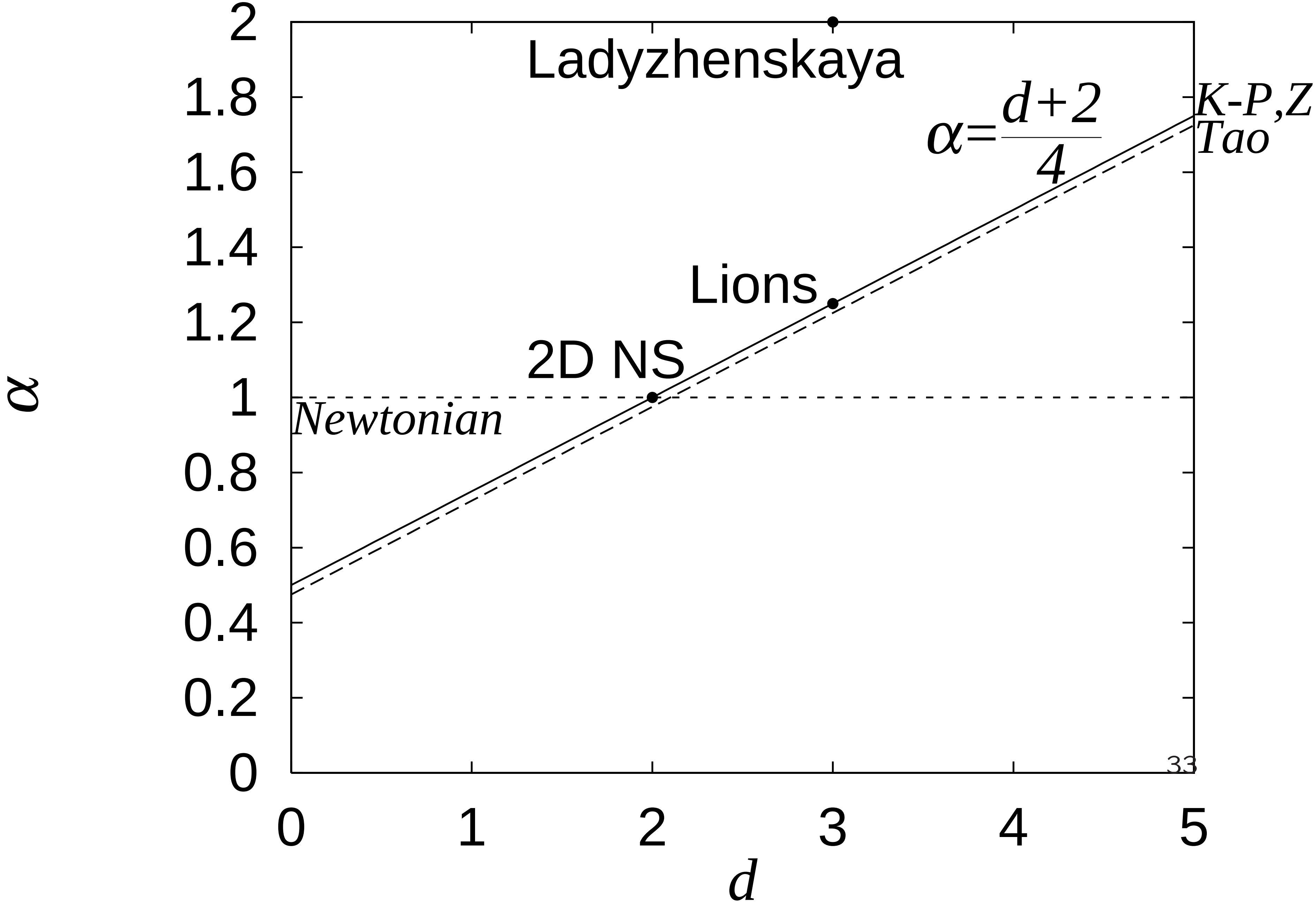
generalised dissipativity

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p - \nu_\alpha(-\triangle)^\alpha \boldsymbol{u}, \quad (\alpha > 0)$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\alpha > (<) 1: \text{ hyper-(hypo-)viscosity}$$

If $\alpha \geq \dfrac{d+2}{4}$ global regularity Lions (1969)
If $\alpha \geq \frac{5}{4}$, global regularity in 3D


Ladyzhenskaya
$\alpha=\frac{d+2}{4}$
K-P,Z
Tao
Lions
2D NS
Newtonian
α
d
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5

An explanation for the Lions condition

$$Q_\alpha \equiv \frac{1}{2}\int |\Lambda^\alpha \boldsymbol{u}|^2 d\boldsymbol{x}$$

$$\frac{dQ_\alpha}{dt} \le c\nu_\alpha^3 \left(\frac{Q_\alpha}{\nu_\alpha^2}\right)^m$$

$$m \equiv \frac{8\alpha - d - 2}{6\alpha - d - 2} \le 2 \iff \alpha \ge \frac{d+2}{4}$$

$$\frac{dQ_\alpha}{dt} \le \frac{c}{\nu_\alpha} Q_\alpha^2 \quad \text{on} \quad \alpha = \frac{d+2}{4}$$

Another (lower) branch: $6\alpha - d - 2 = 0$ or $\alpha = \dfrac{d+2}{6}$

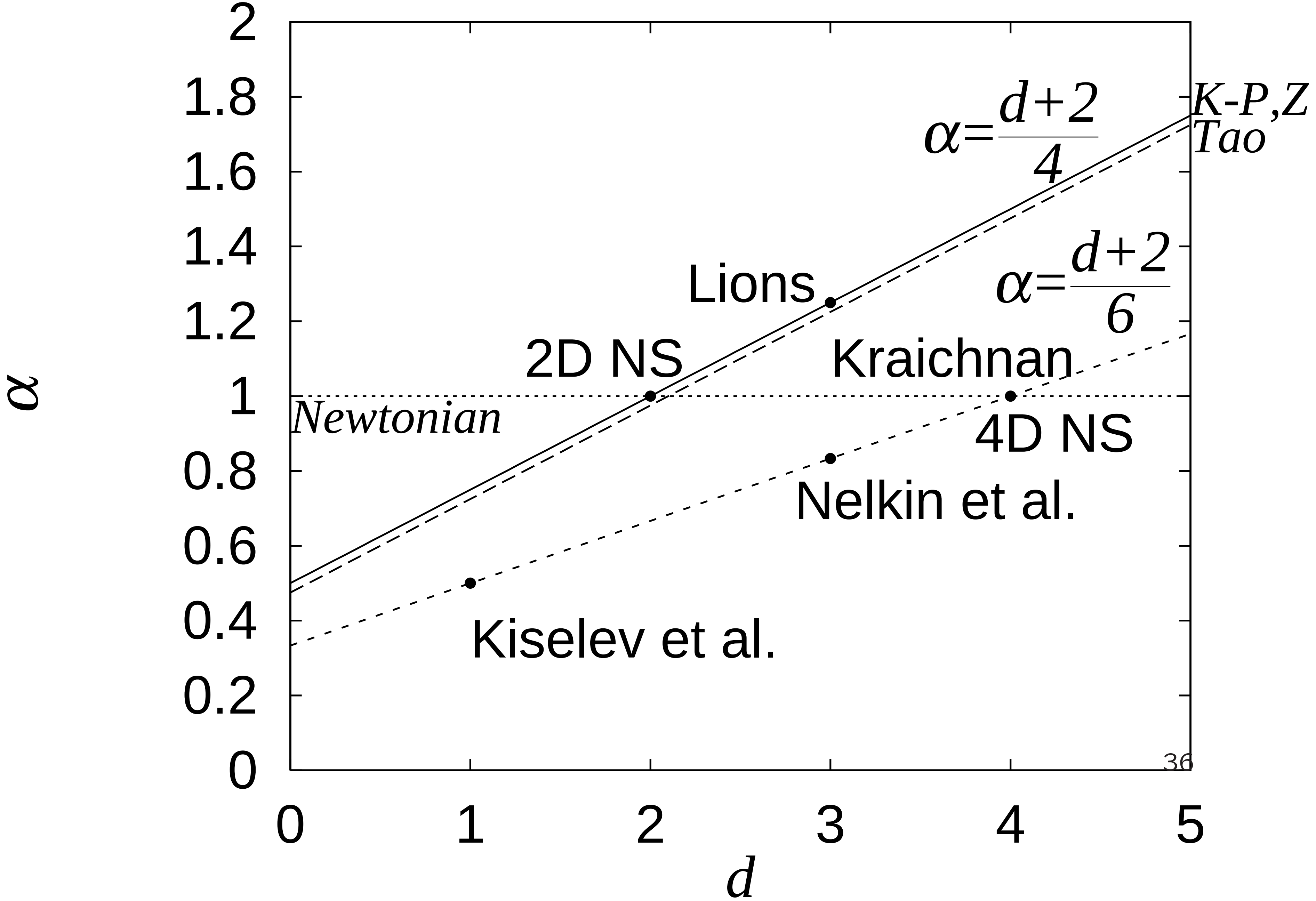
The modified equations scale-invariant under

$$\boldsymbol{x} \to \lambda \boldsymbol{x}, t \to \lambda^{2\alpha} t, \quad (\alpha > 0)$$

$$\boldsymbol{u}(\boldsymbol{x}, t) \to \boldsymbol{u}_\lambda(\boldsymbol{x}, t) \equiv \lambda^{2\alpha-1} \boldsymbol{u}(\lambda \boldsymbol{x}, \lambda^{2\alpha} t)$$

$$\|\boldsymbol{u}_\lambda\|_{H^s}^2 = \lambda^{2s+2(2\alpha-1)-d} \|\boldsymbol{u}\|_{H^s}^2, \text{ critical if } s = \frac{d+2-4\alpha}{2}$$

Ex. $s = 0$: Energy (L^2-norm) at $\alpha = \frac{d+2}{4}$
$s = \alpha$: Energy dissipation (H^α-norm) at $\alpha = \frac{d+2}{6}$



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
α
0
1
2
3
4
5
d
K-P,Z
Tao
α=(d+2)/4
α=(d+2)/6
Lions
2D NS
Kraichnan
Newtonian
4D NS
Nelkin et al.
Kiselev et al.
36

Dimensional analysis on enstrophy bounds

$$\frac{dQ_\alpha}{dt} \leq c\nu_\alpha^3 \underbrace{\left(\frac{Q_\alpha}{\nu_\alpha^2}\right)^{\frac{8\alpha-d-2}{6\alpha-d-2}}}_{=[L^{-2\alpha}]}.$$

Note that $[Q_\alpha] = [\nu_\alpha^2] = \left(\frac{L^{2\alpha}}{T}\right)^2$ at $\alpha = (d+2)/6$

A possible way to recover a spatial scale

$$\frac{dQ_\alpha}{dt} \leq c\nu_\alpha^3 \frac{Q_\alpha}{E}$$

5. Summary

*If N-S solution breaks down with a logarithmic blow-up in critical norm, the energy must vanish (extinction), as $\lesssim \sqrt{t_* - t}$.

*constraints on extinction

3D: $\frac{E(t)Q(t)}{\nu^4} \lesssim C,\ \frac{dQ}{dt} \lesssim C\nu^5 \frac{Q}{E^2}$

$$\log |1 - c\,\tilde{E}(t)| + c\,\tilde{E}(t) \gtrsim c^2\,(t - t_*)$$

4D: $\frac{E(t)P(t)}{\nu^4} \lesssim C,\ \frac{dQ}{dt} \lesssim C\nu^3 \frac{Q}{E}$

$$\mathrm{Li}(c\tilde{E}(t)) \gtrsim c(t - t_*)$$

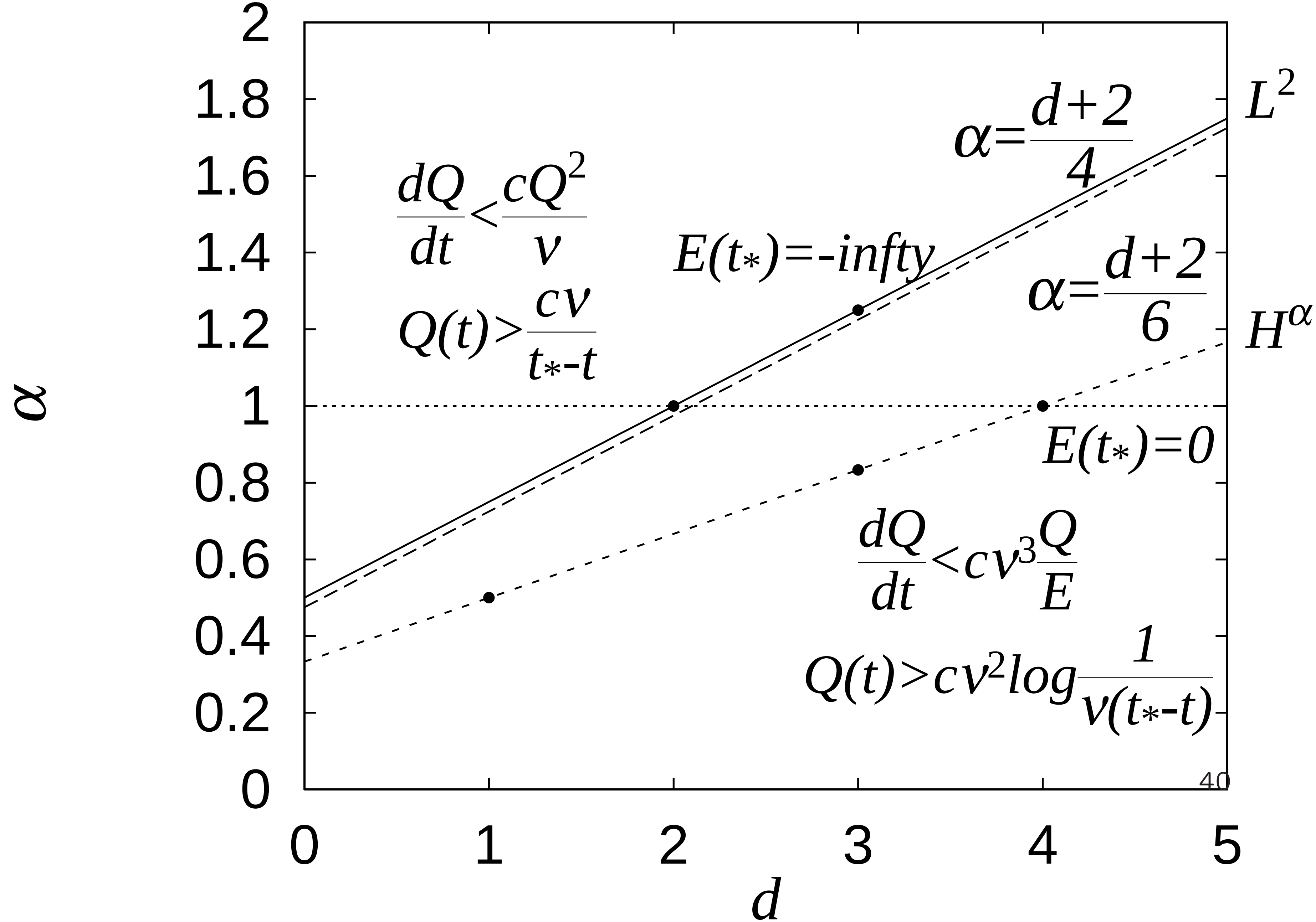
*Modified viscosity 4D NS ≈ 3D with $(-\triangle)^{5/6}$

$$H^{5/6}(\mathbb{R}^3) \subset L^{9/2}(\mathbb{R}^3)$$

*Time scales of smoothness recovery appear as prefactors

$$3\text{D}: \quad \frac{E^2}{C\nu^5}\frac{d}{dt}\log Q \leq f^2 - \frac{5}{4C}f, \quad f(t) = \frac{E(t)Q(t)}{\nu^4}$$

$$4\text{D}: \quad \frac{E}{C\nu^3}\frac{d}{dt}\log Q \leq g^{5/4} - 2g, \quad g(t) = \frac{E(t)P(t)}{\nu^4}$$


α
d
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5
L^2
H^α
α=(d+2)/4
α=(d+2)/6
dQ/dt<cQ^2/ν
Q(t)>cν/(t*-t)
E(t*)=-infty
E(t*)=0
dQ/dt<cν^3 Q/E
Q(t)>cν^2 log 1/(ν(t*-t))

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