Local vs. Nonlocal Diffusions
— A Tale of Two Laplacians

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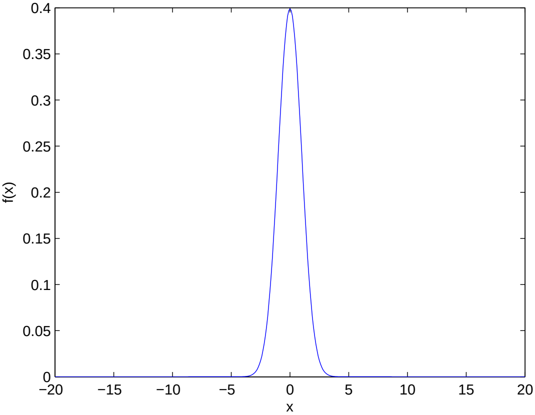
Outline

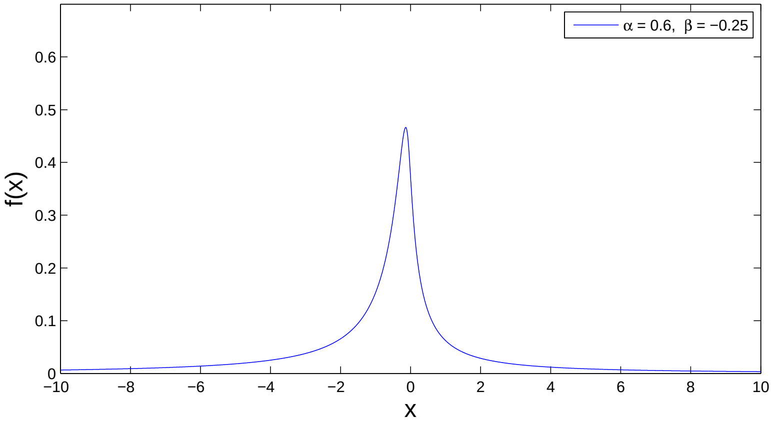
Normal distribution

Normal distribution (or Gaussian distribution):
$X \sim \mathcal{N}(0,1)$
Probability density function for a Gaussian random variable



Non-Normal distribution

There is only one normal distribution.
All others: Non-normal (i.e., anomalous) distributions



Normal vs. non-normal distributions

Gaussian vs. non-Gaussian random variables

Light vs. heavy tails

Local vs. nonlocal diffusions ?

Local vs. nonlocal Laplacians ?

Brownian motion: Einstein's theory

Brownian motion: Particles randomly moves in a liquid

Einstein 1905: Macroscopic theory (probability density p for particles)

Assumptions:
Particles spreading area grows linearly in time
(i.e., variance grows linearly in time)

Particle paths on non-overlapping time intervals are independent

L. C. Evans: An Intro to Stochastic Diff Eqns, 2013

Brownian motion: Einstein's theory

A particle randomly walks on 1D lattice: Space step Δx, time step Δt, location (m, n)

$$p(m, n+1) = \frac{1}{2}[p(m-1, n) + p(m+1, n)]$$

Rewrite:

$$p(m, n+1) - p(m, n) = \frac{1}{2}[p(m-1, n) - 2p(m, n) + p(m+1, n)]$$

Assumption: Particles spreading area growing linearly in time

$$\frac{(\Delta x)^2}{\Delta t} = D$$

$$\frac{p(m, n+1) - p(m, n)}{\Delta t} = \frac{D}{2}\frac{p(m-1, n) - 2p(m, n) + p(m+1, n)}{(\Delta x)^2}$$

Letting $\Delta x \to 0$ and $\Delta t \to 0$: $p_t = \frac{D}{2} p_{xx}$

Diffusion equation (Fokker-Planck eqn for Brownian motion):

$$p_t = \frac{D}{2} p_{xx}$$

Local Laplacian: $\Delta = \partial_{xx}$

For $p(x,0) = \delta(0)$ and $D = 1$:

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Estimate: $0 < p(x,t) \leq \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}}$
Diffusion!

Particle paths: Normal distribution $\mathcal{N}(0,t)$
Guess: Brownian motion $B_t \sim \mathcal{N}(0,t)$

Related works around the time

Macroscopic equations for microscopic motions

Bachelier 1900

Smoluchowski 1906

G. I. Taylor 1921

Uhlenbeck-Ornstein 1930

Brownian motion B_t: Wiener's theory

Wiener's theory 1923: Microscopic theory (Paths of a particle)

- Independent increments: $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ independent
- Stationary increments with $B_t - B_s \sim \mathcal{N}(0, t - s)$
- Continuous sample paths (but nowhere differentiable in time)

Remarks:
$B_t \sim \mathcal{N}(0, t)$
$\mathrm{Var}(B_t) = t$
Variance linear in time; spreading area linear in time

I. Karatzas and S. E. Shreve,
Brownian Motion and Stochastic Calculus

White noise

White noise: $\frac{dB_t}{dt}$

Generalized time derivative

Brownian particles in a moving liquid with velocity “b"

Brownian motion with an ambient or underlying velocity field “b":
$\frac{dX_t}{dt} = b + \frac{dB_t}{dt}$ or

$$dX_t = b\ dt + dB_t$$

$$X_t = b\,t + B_t \sim \mathcal{N}(bt, t)$$

For $p(x, 0) = \delta(0)$ and $D = 1$:

$$p(x,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-bt)^2}{2t}}$$

This satisfies: Diffusion-Advection equation

$$p_t = \frac{D}{2} p_{xx} - b p_x$$

b: Drift, or convection
Fokker-Planck eqn for Brownian motion with a drift “b"

For Brownian motion with a constant drift “b"

$$dX_t = b\,dt + dB_t,$$

the Fokker-Planck eqn is:

$$p_t = \frac{D}{2} p_{xx} - b p_x$$

Guess: For Brownian motion with a state-dependent drift “b", the Fokker-Planck eqn is:

$$p_t = \frac{D}{2} p_{xx} - (bp)_x$$

Related works

Fokker 1914

Planck 1918

Smoluchowski 1915

Kolmogorov forward equation 1931

So, local Laplacian Δ:
Macroscopic description of Brownian particles

In fact, it is also the **generator** for Brownian motion

Generator for Brownian motion B_t

Brownian motion starting at x: $X_t = x + B_t$

Generator: Time derivative of ‘mean observation of a stochastic process’

$Af(x) \triangleq \frac{d}{dt}|_{t=0}\mathbb{E}f(X_t)$

It is a linear operator.
Connecting **stochastics** with **deterministics**.

Generator A carries info about process X_t

Generator for Brownian motion: Local Laplacian!
$\Delta = \partial_{xx}$

Let us verifying this

For $X_t = x + B_t$,

$$\begin{aligned} \mathbb{E} f(X_t) &= \frac{1}{\sqrt{2\pi t}} \int f(y)\, e^{-\frac{(y-x)^2}{2t}} dy \\ &= \frac{1}{\sqrt{2\pi}} \int f(x + z\sqrt{t})\, e^{-\frac{z^2}{2}} dz, \end{aligned}$$

where we have changed variables via $z = \frac{y-x}{\sqrt{t}}$.

$$
\begin{aligned}
\frac{\mathbb{E}f(X_t)-f(x)}{t} & \\
&= \frac{1}{\sqrt{2\pi}}\int \frac{z\sqrt{t}f'(x)+\frac{1}{2}z^2tf''(x+\theta z\sqrt{t})}{t}\, e^{-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\frac{f'(x)}{t}\int z\, e^{-\frac{z^2}{2}}dz \\
&+ \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int z^2f''(x+\theta z\sqrt{t})\, e^{-\frac{z^2}{2}}dz \\
&= 0+\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int z^2f''(x+\theta z\sqrt{t})\, e^{-\frac{z^2}{2}}dz.
\end{aligned}
$$

Finally: the generator A is local Laplacian —

$$
\begin{aligned}
Af(x) &= \frac{d}{dt}|_{t=0}\mathbb{E}f(X_t)=\lim_{t\downarrow 0}\frac{\mathbb{E}f(X_t)-f(x)}{t} \\
&= \frac{1}{2}f''(x),
\end{aligned}
$$

So, local Laplacian Δ:
Macroscopic description of Brownian motion
Generator for Brownian motion

Same in the context of Fokker-Planck eqns

How about the **nonlocal Laplacian**: $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$?

Nonlocal Laplacian: Macroscopic description of Lévy motion

Nonlocal Laplacian: $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$:

Macroscopic description or generator for symmetric $\alpha-$stable motion L_t^α

Central Limit Theorem

$X_1, X_2, \cdots, X_n$ are independent, identically distributed (iid) random variables (i.e., ‘measurements’) and then “averaging":

Central Limit Theorem

A stable random variable X comes from “averaging the measurements":

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n - b_n}{a_n} = X \quad \text{in distribution}$$

for some constants a_n, b_n ($a_n \neq 0$)

Notation: $X \sim S_\alpha, \quad 0 < \alpha \leq 2$

$\alpha-$stable random variable
α: Non-Gaussianity index

A special case: $\alpha = 2$

Well-known normal random variable emerges when $\alpha = 2$

$\mathbb{E}X_i = \mu, \quad \mathrm{Var}(X_i) = \sigma^2$

Central limit theorem: A normal random variable comes from “averaging the measurements"

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = X \ \sim \mathcal{N}(0,1) \quad \text{in distribution}$$

Namely,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

Gaussian vs. Non-Gaussian random variables

Gaussian: Normal random variable $X \sim \mathcal{N}(0,1)$

Probability density function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

Non-Gaussian: $\alpha-$stable random variable $X \sim S_\alpha, \; 0 < \alpha < 2$

Probability density function $f_\alpha(x)$

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f_\alpha(x) \, dx$$

Prob density function for a Gaussian random variable

$X \sim \mathcal{N}(0, 1)$

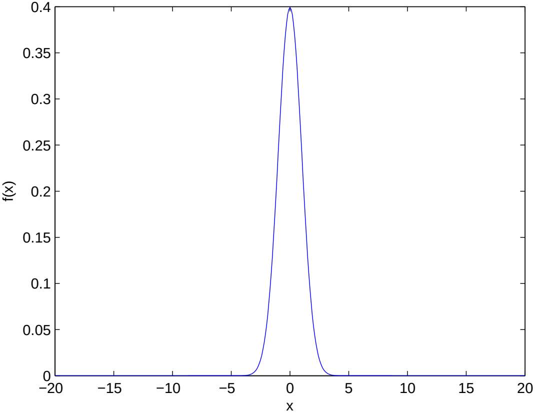


Figure : "Bell curve": Exponential decay, light tail

Prob density function for a non-Gaussian, $\alpha-$stable random variable

$X \sim S_\alpha$

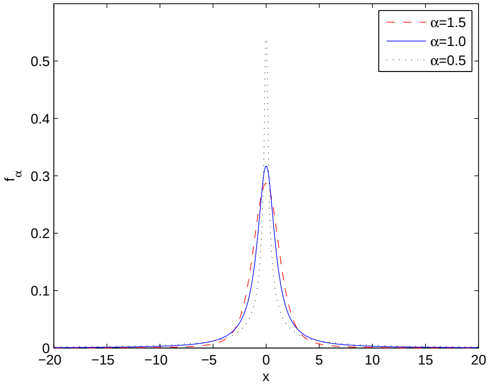


Figure : Polynomial power decay, heavy tail

Some references:

P. E. Protter: **Stochastic Integration and Diff Equations**, 1990

D. Applebaum: **Lévy Processes and Stochastic Calculus**, 2009

Lévy Motion L_t^α

Definition: Lévy motion L_t^α with $0 < \alpha \leq 2$:

(1) Stationary increments $L_t^\alpha - L_s^\alpha \sim S_\alpha(|t-s|^{\frac{1}{\alpha}}, 0, 0)$
(2) Independent increments
(3) Stochastically continuous sample paths (continuous in probability):
$\mathbb{P}(|L_t - L_s| > \delta) \to 0$, as $t \to s$, for all $\delta > 0$

Note: Paths are stochastically continuous (i.e., right continuous with left limit; countable jumps): $L_t^\alpha \to L_s^\alpha$ in probability as $t \to s$

Countable jumps in time!

Lévy-Khintchine Theorem:
Countable jumps in time:

Jump measure: a Borel measure
$\nu_\alpha(dy) = C_\alpha \frac{dy}{|y|^{1+\alpha}}$, for $0 < \alpha < 2$

$\nu_\alpha(a, b) = C_\alpha \int_a^b \frac{dy}{|y|^{1+\alpha}}$:
Mean number of jumps of “size" (a, b) per unit time!

Brownian motion B_t: A **Gaussian** process
(Brownian noise: $\frac{dB_t}{dt}$)

Lévy motion L_t^α: A **non-Gaussian** process
(Lévy noise: $\frac{dL_t^\alpha}{dt}$)

Heavy tail for $0 < \alpha < 2$: Power law

$$\mathbb{P}(|L_t^\alpha| > u) \sim \frac{1}{u^\alpha}$$

Light tail for $\alpha = 2$: Exponential law

$$\mathbb{P}(|B_t| > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi}u}$$

Generator for Lévy Motion: a Nonlocal operator

Lévy-Khintchine Theorem:
Specifies Fourier transform (i.e., characteristic function) of L_t^α: $g(k, \alpha)$
Thus: $L_t^\alpha = \mathbb{F}^{-1} g(k, \alpha)$

$$
\begin{aligned}
Au &= \frac{d}{dt}\Big|_{t=0} \mathbb{E} u(x + L_t^\alpha) \\
&= \int_{\mathbb{R}^n \setminus \{0\}} [u(x+y) - u(x)]\, \nu_\alpha(\mathrm{d}y) \\
&\triangleq -K_\alpha\, (-\Delta)^{\frac{\alpha}{2}}
\end{aligned}
$$

$\nu_\alpha(dy) = C_\alpha \frac{dy}{|y|^{n+\alpha}}$: Jump measure for L_t^α
C_α, K_α: Positive constants depending on n, α
Nonlocal Laplacian: $(-\Delta)^{\frac{\alpha}{2}}$

Generator for Lévy Motion: a Nonlocal operator

Justify the notation for $(-\Delta)^{\frac{\alpha}{2}}$:

$$\int_{\mathbb{R}^n \setminus \{0\}} [u(x+y) - u(x)] \, \nu_\alpha(\mathrm{d}y) \triangleq -K_\alpha \, (-\Delta)^{\frac{\alpha}{2}}$$

$$\mathbb{F}(\text{left hand side}) = |k|^\alpha \mathbb{F}(u)$$

Clearly, this notation is inspired by the fact that

$$\mathbb{F}(-\Delta u(x)) = |k|^2 \mathbb{F}(u)$$

Applebaum: **Lévy Processes and Stochastic Calculus**

Nonlocal diffusion equation (Fokker-Planck eqn for Lévy motion):

$$p_t = -K_\alpha \, (-\Delta)^{\frac{\alpha}{2}} p$$

Nonlocal Laplacian: $(-\Delta)^{\frac{\alpha}{2}}$

It is the **Generator** for Lévy motion

Two Laplacians

Local Laplacian: Δ

Nonlocal Laplacian: $(-\Delta)^{\frac{\alpha}{2}}, \quad$ for $0 < \alpha < 2$

Macroscopic manifestation of corresponding microscopic descriptions:
Brownian motion and $\alpha-$stable Lévy motion

Brownian motion vs. $\alpha-$stable Lévy motion

Brownian Motion ($\alpha = 2$)	$\alpha-$stable Levy Motion ($0 < \alpha < 2$)
Gaussian process	Non-Gaussian process
Independent increments	Independent increments
Stationary increments	Stationary increments
Continuous sample paths	Stoch continuous paths ("jumps")
Light tail	Heavy tail
Jump measure: 0	Jump measure: ν_α

Fokker-Planck eqn for system with Brownian motion

For a stochastic system with Brownian motion:

$$dX_t = b(X_t)dt + dB_t, \; X_0 = x$$

Fokker-Planck eqn for probability density evolution $p(x, t)$:

$$p_t = \Delta p - \nabla \cdot (b(x)p)$$

When the vector field (drift) $b(x)$ is divergence-free:

$$p_t = \Delta p - b(x) \cdot \nabla p$$

Fokker-Planck eqn for system with Lévy motion

For a stochastic system with Lévy motion:

$$dX_t = b(X_t)dt + dL_t^{\alpha}, \ X_0 = x$$

Fokker-Planck eqn for probability density evolution $p(x,t)$:

$$p_t = -K_{\alpha} \, (-\Delta)^{\frac{\alpha}{2}} p - \nabla \cdot (b(x)p)$$

When the vector field (drift) $b(x)$ is divergence-free:

$$p_t = -K_{\alpha} \, (-\Delta)^{\frac{\alpha}{2}} p - b(x) \cdot \nabla p$$

for $0 < \alpha < 2$

Fokker-Planck eqn: Nonlinear, as well as nonlocal

When the vector field b depends on the distribution of system state, then we have a nonlinear, nonlocal PDE:

$$p_t = \Delta p - \nabla \cdot (\tilde{b}(p)p)$$

$$p_t = -K_\alpha \, (-\Delta)^{\frac{\alpha}{2}} p - \nabla \cdot (\tilde{b}(p)p)$$

for $0 < \alpha < 2$

Wellposedness & regularity of solutions? Useful for designing numerical schemes.

Effects of Nonlocal Laplacian $(-\Delta)^{\frac{\alpha}{2}}$:

- in some partial differential equations?
- in some dynamical phenomena?

Eigenvalues of Two Laplacians on bounded domain

Local Laplacian: Δ
One-dim, zero Dirichlet BC: $\lambda_n \sim -n^2$

Nonlocal Laplacian: $-(-\Delta)^{\frac{\alpha}{2}}, \quad$ for $0 < \alpha < 2$
One-dim, zero external Dirichlet BC: $\lambda_n \sim -(n - \frac{2-\alpha}{4})^\alpha + O(\frac{1}{n})$
Kwasnicki 2010
Reducing the “diffusion power" by the “amount" $2 - \alpha$!

Effects of Nonlocal Laplacian in the Burgers eqn:

$$u_t = -uu_x - (-\Delta)^{\frac{\alpha}{2}} u$$

Kiselev, Nazarov & Shterenberg 2008
Under periodic boundary condition:
Blowup in finite time for $0 < \alpha < 1$, but global solution for $1 \le \alpha < 2$.

Biler, Funaki & Woyczynski 1998
In the whole space: Global solution for $1.5 < \alpha < 2$ in $H^1(R)$

Question:

Motion of particles under the influence of Lévy motion:

$$dX_t = b(X_t)dt + dL_t^{\alpha}, \; X_0 = x$$

- **Examine** quantities that carry dynamical information:

Escape probability

Likelihood of transition between different dynamical regimes!

Escape probability: Carrying dynamical information

- **Contaminant transport:** likelihood for contaminant to reach a specific region
- **Climate:** likelihood for temperature to be within a range
- **Tumor cell density:** likelihood for tumor density to decrease (becoming cancer-free)

How to quantify escape probability?

Escape probability from a domain D

Consider a SDE

$$dX_t = b(X_t)dt + dL_t^{\alpha}, \quad X_0 = x \in D$$

Escape probability $p(x)$:
Likelihood that a “particle **x**" first escapes D and lands in U

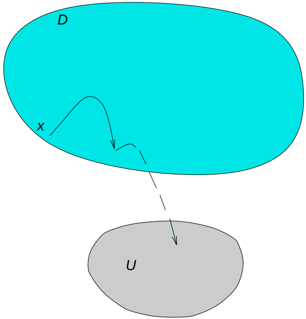


Figure : Domain D, with a target domain U in D^c

A surprising connection between escape probability and harmonic functions!

What is a harmonic function?

Recall: What is a harmonic function?

It is a solution of the Laplace equation:

$$\Delta h(x) = 0$$

But Δ is the generator of Brownian motion B_t

So we say:
$h(x)$ is a harmonic function with respect to Brownian motion

An analogy:

Harmonic function with respect to Lévy motion L_t^α:

$$(-\Delta)^{\frac{\alpha}{2}} h(x) = 0$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the generator of L_t^α

Note: Feedback of Probability Theory to Analysis!

A further analogy:

Consider a stochastic system

$$dX_t = b(X_t)dt + dL_t^{\alpha}$$

Generator for solution process X_t:

$$A_\alpha h(x) = b(x) \cdot \nabla h(x) - K_\alpha \, (-\Delta)^{\frac{\alpha}{2}} h(x)$$

Harmonic function with respect to X_t: $A_\alpha h(x) = 0$

Nonlocal deterministic partial differential equation

What is the connection between escape probability & harmonic functions?

Escape probability from a domain D

Escape probability $p(x)$:
Likelihood that a “particle **x**" first escapes D and lands in U

Exit time: $\tau_{D^c}(x)$ is the first time for X_t to escape D

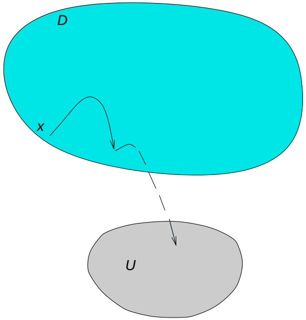


Figure : Domain D, with a target domain U in D^c

Connection: Escape probability & harmonic function

$$dX_t = b(X_t)dt + dL_t^{\alpha}, \quad X_0 = x \in D$$

For

$$\varphi(x) = \begin{cases} 1, & x \in U, \\ 0, & x \in D^c \setminus U, \end{cases}$$

$$\begin{aligned} \mathbb{E}[\varphi(X_{\tau_{D^c}(x)})] &= \int_{\{\omega : X_{\tau_{D^c}} \in U\}} \varphi(X_{\tau_{D^c}}) d\mathbb{P}(\omega) \\ &\quad + \int_{\{\omega : X_{\tau_{D^c}} \in D^c \setminus U\}} \varphi(X_{\tau_{D^c}}) d\mathbb{P}(\omega) \\ &= \mathbb{P}\{\omega : X_{\tau_{D^c}} \in U\} \\ &= p(x) \end{aligned}$$

But, left hand side is a harmonic function with respect to X_t

Liao 1989

Escape probability from a domain D

$$dX_t = b(X_t)dt + dL_t^\alpha, \quad X_0 = x \in D$$

Escape probability $p(x)$: Likelihood that a “particle **x**" first escapes D and lands in U

Theorem

Escape probability p is solution of Balayage-Dirichlet problem

$$\begin{cases} A_\alpha p = 0, \\ p|_U = 1, \\ p|_{D^c \setminus U} = 0, \end{cases}$$

with $A_\alpha = b(x) \cdot \nabla - K_\alpha(-\Delta)^{\frac{\alpha}{2}}$.

Qiao-Kan-Duan 2013

Escape probability to the right: under Brownian motion, no drift

$\Delta p = 0$, escape from $D = (-2, 2)$ to $U = (2, +\infty)$:

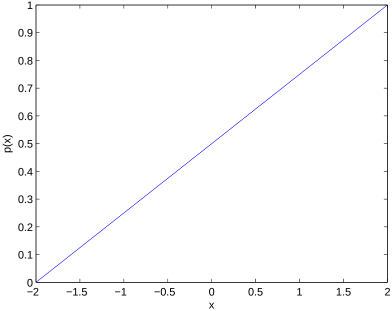


Figure : Escape probability: The case of Brownian motion

Escape probability to the right: under Lévy motion, no drift

$(-\Delta)^{\frac{\alpha}{2}} p = 0$, escape from $D = (-2, 2)$ to $U = (2, +\infty)$:

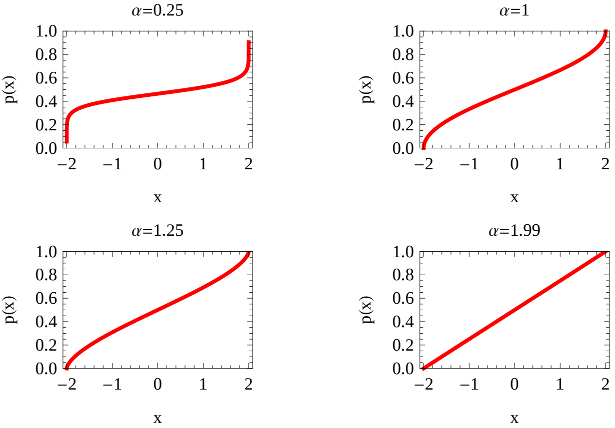


Figure : Escape probability: The case of Lévy motion

Impact of local & nonlocal diffusions

Under Brownian fluctuations (i.e., local diffusion):
— Escape probability $p(x)$ is linear in location

Under Lévy fluctuations (i.e., nonlocal diffusion):
— Escape probability $p(x)$ is nonlinear in location

When velocity field (drift) is present:

Escape probability under interactions between nonlinearity and fluctuations

Gao-Duan-Li-Song 2014

Fokker-Planck eqn:

Numerical simulations

Wang-Duan-Li-Lou 2014

Wellposedness under realistic conditions?
Behavior of solutions?
Impact of nonlocal Laplacian?

Summary

$$\Delta \text{ and } (-\Delta)^{\frac{\alpha}{2}}$$

- Microscopic origins of two Laplacians:

  Macroscopic descriptions of Brownian & Lévy motions

- Comparing Local & Nonlocal Diffusions:

  Escape probability: Quantifying particle dynamics under non-Gaussian fluctuations

  Fokker-Planck eqn: Quantifying probability density evolution