Existence Theory for Mean Field Games with Non-Separable Hamiltonian

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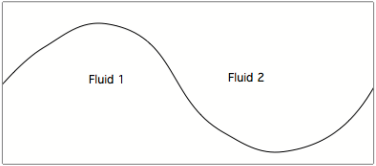
Introduction

- I will begin by discussing a problem in fluid dynamics.
- Next, we'll adapt the method from the fluids problem to get an existence theorem for mean field games.
- I will then give another existence theorem, using the implicit function theorem.
- These existence results are "in the small," but we have a few options for which feature of the problem can be taken small.
- These results also do not use much structure (separability, convexity, monotonicity) on the nonlinearity; mainly, just Lipschitz bounds and/or mapping properties.
- I will mention some current work at the end.

The Vortex Sheet Problem

- We consider two infinitely deep, horizontally periodic fluids, separated by a sharp interface.



- The fluid velocities are given by the irrotational, incompressible Euler equations:

$$\mathbf{v}_{i,t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = -\nabla p_i,$$

$$\mathrm{div}(\mathbf{v}_i) = 0,$$

$$\mathbf{v}_i = \nabla \phi_i.$$

- The fluids have densities ρ_1 and ρ_2. If $\rho_2 = 0$, then this is the water wave case.

The Vortex Sheet IVP Is Ill-Posed

- If $\rho_1\rho_2 > 0$, then the initial value problem is ill-posed.
- There have been several proofs of this over the years: Caflisch & Orellana 1989, Lebeau 2002, Lebeau & Kamotski 2005, Wu 2006. Ill-posedness is also implied by, but not explicitly discussed in, the work of Duchon & Robert 1988.
- The ill-posedness can be seen from linear theory: consider $y(\alpha, t) = \epsilon\eta(\alpha, t)$. Then, the linearized equation of motion is

$$\eta_{tt} = -\eta_{\alpha\alpha} + \tau H(\eta_{\alpha\alpha\alpha}),$$

  where H is the Hilbert transform and τ is the (non-negative) coefficient of surface tension.
- If $\tau = 0$, then the problem is elliptic in space-time, and has an ill-posed initial value problem.
- This ill-posedness is really the same thing as the Kelvin-Helmholtz instability (the problem is so unstable as to be ill-posed).

The Duchon-Robert Formulation

- In just about all studies of the vortex sheet, the irrotationality assumption is used to reduce the dimension by one; that is, only quantities on the interface need to be considered.
- Consider the interface to be a graph, $(x, y(x))$. Let $\Omega(x) = 1 + \omega(x)$ be the vortex sheet strength.
- Denote $v = y_x$. They write the evolution equations as

$$v_t - \Lambda\omega = F(v, \omega)_x, \qquad \omega_t - \Lambda v = G(v, \omega)_x,$$

  where F and G are nonlinear terms stemming from the Biot-Savart integral.
- Here, $\Lambda = \sqrt{-\partial_{xx}} = H\partial_x$, and thus $\hat{\Lambda}(\xi) = |\xi|$.
- Again, the linearization is elliptic in space-time, and we see the ill-posedness at the linear level.

The Duchon-Robert Result

- Specify half the data: $v(x,0) = v_0(x)$. If v_0 is sufficiently small in a certain function space (the Wiener algebra), then there exists a solution (v,ω) to the initial value problem for all time. This solution is analytic at all positive times.
- Method of proof: write a Duhamel formula which integrates forward in time from $t=0$ and backwards in time from $t=\infty$:

$$v = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G),$$

$$-\omega = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) + \frac{1}{2}I^-(F+G).$$

- Consider function spaces on space-time domain $\mathbb{R} \times [0,\infty)$; in such spaces, prove Lipschitz bounds on F and G, and prove that I^+, I^- are bounded linear operators.
- Put this all together using the contraction mapping theorem, to get existence of solutions in these function spaces.

The Mean Field Games System

- The following is the mean field games system of coupled PDEs:

$$u_t + \Delta u + \mathcal{H}(t, x, Du, m) = 0,$$

$$m_t - \Delta m + \mathrm{div}(m\mathcal{H}_p(t, x, Du, m)) = 0.$$

- We can take $x \in \mathbb{T}^n$ and $t \in [0, T]$, for some T.
- This is supplemented with boundary conditions. The *planning problem* specifies $m(0, x) = m_0(x)$, $u(T, x) = u_T(x)$.
- The *payoff problem* still specifies $m(0, x) = m_0(x)$, but now has $u(T, x) = G(x, m(T, x))$ for the second condition.
- In almost all works of which I am aware in the literature, $\mathcal{H}(t, x, p, m) = H(t, x, p) + F(t, x, m)$, and this H is taken to be convex (although works with congestion effects include a non-separable Hamiltonian). *These assumptions are unnecessary for our approach.*

Prior Results

- For most prior results, $\mathcal{H}$ is taken to be separable. The function F is known as the coupling.
- Existence of strong solutions in a few cases: when F is a nonlocal smoothing operator, or when $H(t, x, Du) = |Du|^2$. (Lasry-Lions, Cardaliaguet-Lasry-Lions)
- When the coupling F is local, proofs may be for weak solutions (Lasry-Lions, Porretta).
- The work of Gomes, Pimentel, and Sanchez Morgado shows that (still in the separable case) under a number of technical assumptions, strong solutions exist.
- Other results include stationary solutions or the limit as T goes to infinity.
- This list is not exhaustive, but gives the flavor of prior works.

Reformulating for the Duchon-Robert Method

- Project away the means: let $w = \mathbb{P}u$ and let $\mu = m - \bar{m}$.
- Let $\mathbb{P}\mathcal{H}(t,x,Dw,\mu) = \Xi(t,x,Dw,\mu) = \mathbb{P}(b(t,x)\mu + \Upsilon(t,x,Dw,\mu))$.
- Let $\Theta(t,x,Dw,\mu) = \mathcal{H}_p(t,x,Du,m)$.
- We get the following system:

$$w_t + \Delta w + \mathbb{P}(b\mu) + \mathbb{P}(\Upsilon(\cdot,\cdot,Dw,\mu)) = 0,$$

$$\mu_t - \Delta\mu + \mathrm{div}(\mu\Theta(\cdot,\cdot,Dw,\mu)) + \bar{m}\mathrm{div}(\Theta(\cdot,\cdot,Dw,\mu)) = 0.$$

- The reason for separating out a linear term in the w equation: say, for example, that $\mathcal{H} = m|Du|^4 + m^3$. Then, we have $m^3 = (\mu + \bar{m})^3$, which has a term linear in μ. Otherwise, Υ will be assumed to satisfy a nonlinear estimate.

The Duhamel Formulation

- Say we use "payoff" boundary conditions, $m(0,x) = m_0(x)$, and $u(T,x) = G(x, m(T,x))$.
- Introduce integral operators

$$I^+(f)(t) = \int_0^t e^{\Delta(t-s)} f(s,\cdot)\ ds$$

and

$$I^-(f)(t) = \int_t^T e^{\Delta(s-t)} f(s,\cdot)\ ds.$$

Also, let $I_T f = I^+(f)(T)$.

- We get the following Duhamel formula for the forward equation for μ :

$$\mu(t,\cdot) = e^{\Delta t}\mu_0 + I^+(\operatorname{div}(\mu\Theta(\cdot,\cdot,Dw,\mu)))(t,\cdot) \\ + \bar{m}(I^+(\operatorname{div}(\Theta(\cdot,\cdot,Dw,\mu))))(t,\cdot),$$

- Define $A(\mu, w) = \mu(T,\cdot)$; this involves I_T.

Continuing the Duhamel Formulation

- Then, using the payoff boundary condition, we integrate backward in time from time T, finding the following Duhamel formula for the backward equation for w :

$$w(t,\cdot) = e^{\Delta(T-t)}\tilde{G}(\cdot, A(\mu,w)) - I^{-}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu))(t) \\ - I^{-}(\mathbb{P}(be^{\Delta\cdot}\mu_0))(t) - I^{-}(\mathbb{P}(bI^{+}\mathrm{div}(\mu\Theta(\cdot,\cdot,Dw,\mu))(\cdot)))(t) \\ - \bar{m}I^{-}(\mathbb{P}(bI^{+}\mathrm{div}(\Theta(\cdot,\cdot,Dw,\mu))(\cdot)))(t).$$

- These equations for μ and w give a fixed point problem. We seek fixed points in some function space.

Function Spaces: The Wiener Algebra

- The original Duchon-Robert method uses a contraction mapping in spaces related to the Wiener algebra.
- The Wiener algebra is the space of functions with Fourier transform/series in L^1 or ℓ^1.
- This has a simple algebra property:

$$\|fg\| = \sum_k |\mathcal{F}(fg)(k)| \leq \sum_k \sum_j |\hat{f}(k-j)\hat{g}(j)| \leq \|f\|\|g\|.$$

- We use a space-time version of this, with some weights:

$$\|f\|_{\mathcal{B}_{\alpha,j}} = \sum_{k\in\mathbb{Z}^n} \sup_{t\in[0,T]} \left| |k|^j e^{\beta(t)|k|} \hat{f}(k) \right|.$$

- Here, $j \in \mathbb{N}$, and $\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & T \in [T/2, T]. \end{cases}$
- These spaces are related to the dissertation work of my student, Timur Milgrom. The spaces are still Banach algebras.

Bounds for the Linear Operators

- Our operators I^{+} and I^{-} are bounded linear operators between $\mathcal{B}_{\alpha,j}$ and $\mathcal{B}_{\alpha,j+2}$, for any j and for any $\alpha \in [0, T/2)$:

$$\|I^{\pm}\|_{\mathcal{B}_{\alpha,j}\to\mathcal{B}_{\alpha,j+2}} \leq \frac{2T}{T-2\alpha} + 2.$$

- The gain of two derivatives here comes from the presence of the Laplacian, and is a version of parabolic smoothing.

The Contraction Mapping Argument

- We find a fixed point of a mapping associated to the Duhamel formulation.
- This requires showing that the associated mapping is a local contraction.
- We make Lipschitz assumptions on G and on $\mathcal{H}$ via Υ and Θ. These Lipschitz assumptions are in spaces of Wiener algebra type.
- So, for example, we might assume

  $$\|\Upsilon(\cdot,\cdot,a,b)-\Upsilon(\cdot,\cdot,y,z)\|_{\mathcal{B}_{\alpha,j}} \leq M(a,b,y,z)\left[\|(a,b)-(y,z)\|_{(\mathcal{B}_{\alpha,j})^{n+1}}\right],$$

  where M is a function which is continuous and which satisfies $M(t,x,0,0)=0$.
- Together with estimates for I^{+} and I^{-}, we get a local contraction (local about the origin).

The Main Theorem

- Let $T > 0$ be given. Let $\alpha \in (0, T/2)$ be given. Let Υ, G, and Θ satisfy the appropriate Lipschitz properties.
- There exists $\delta > 0$ such that if $\|\mu_0\| < \delta$, then there exist w and μ in $\mathcal{B}_{\alpha,2}$ so that u and m solve the mean field games system.
- For all $t \in (0, T)$, the functions u and m are analytic.
- So, we get existence of smooth solutions, as long as the initial m is a small perturbation of the uniform distribution.

Examples

- In the Wiener algebra, we can get examples like

$$\mathcal{H}(t, x, p, m) = a(t, x)m^{k_1} p_i p_j p_\ell + m^{k_2},$$

  for k_1, $k_2 \in \mathbb{N}$ and $i, j, \ell \in \{1, 2, \ldots n\}$.
- Similarly, we could take

$$\mathcal{H}(t, x, Du, m) = a(t, x)m^{k_1} |Du|^4 + m^{k_2}.$$

- Generally speaking, except for polynomials, the hypotheses are a little difficult to check in the Wiener algebra.
- **Current work**: with some effort, the results can be carried over to Sobolev spaces, in which we can find more examples because of the availability of a composition estimate.

Another Approach: Large Data, Small Hamiltonian

- We previously considered only small data. There are works in the literature which take a small time horizon, such as the work of Gomes and Voskanyan on mean field games with congestion effects.
- We have another approach which allows for arbitrary data or time horizon, but requires a kind of smallness on $\mathcal{H}$.
- In particular, we replace $\mathcal{H}$ with $\varepsilon\mathcal{H}$.
- In the Duhamel formulas, all the nonlinear terms gain ε in front.
- We then use the implicit function theorem to get existence of solutions.
- So, we find existence of solutions for some mean field games with non-separable Hamiltonian with data of arbitrary size and for arbitrarily large time horizon, by restricting the size of ε.

Implicit Function Theorem Details

- Say we consider the planning problem.
- Consider the following mapping, F :

$$F\left(\begin{pmatrix} w \\ \mu \end{pmatrix}, \varepsilon\right) = \begin{pmatrix} w - e^{\Delta(T-\cdot)}w_T + \varepsilon I^-(\Xi(\cdot,\cdot,Dw,\mu)) \\ \mu - e^{\Delta\cdot}\mu_0 - \varepsilon I^+(\mathrm{div}((\mu+\bar{m})\Theta(\cdot,\cdot,Dw,\mu))) \end{pmatrix}.$$

- We know a solution when $\varepsilon = 0$; this is the linear solution, $w(t,\cdot) = e^{\Delta(T-t)}w_T$ and $\mu(t,\cdot) = e^{\Delta t}\mu_0$.
- We can take the derivative:

$$D_{(w,\mu)}F\Big|_{\varepsilon=0} = \mathrm{Id}.$$

- Since the identity is a bijection, and from the previously developed mapping properties of F (via the mapping properties of I^+ and I^-), the implicit function theorem applies. We get a solution for some interval of values of ε about $\varepsilon = 0$.

Summary

- Inspired by the work of Duchon-Robert, and the extension in Milgrom's thesis, we provide proof of existence of solutions for mean field games with non-separable Hamiltonian.
- This proof requires a small perturbation of the uniform distribution as data.
- We give another proof for larger data, but places a smallness condition on the Hamiltonian. This is still without assuming separability or other structure.
- Current work includes allowing Sobolev data, and formulating a smallness constraint which simultaneously considers the size of the data, the size of the time interval, and the size of the Hamiltonian.
- Of future interest: lowering regularity assumptions on the data, and perhaps studying nonexistence results via singularity formation.

Thanks for your attention.