How small a universal differentiability set can be?

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Background

We consider real-valued Lipschitz functions $f : X \to \mathbb{R}$.

1. If X is finite-dimensional, then Rademacher theorem implies f is differentiable almost everywhere.
    - If A has positive measure, then $\{x \in A : f \text{ is differentiable at } x\}$ is not empty.
    - What if A has measure 0?
2. For separable X, the dual X^* must be separable as otherwise there is an equivalent norm on X which is everywhere Fréchet non-differentiable.
3. If X^* is separable, then every Lipschitz function is differentiable on a dense subset of X [Preiss, 1990] and ...
4. ...moreover, points of differentiability can be found inside a fixed beforehand dense G_δ subset S of X satisfying the condition that S contains a dense set of lines.

Universal Differentiability Set (UDS)

A set $S \subseteq X$ is a UDS if for every Lipschitz function $f : X \to \mathbb{R}$ there is an $x \in S$ such that f is (Fréchet) differentiable at x.

Finite-dimensional case, Rademacher's Theorem

1. Every subset of $\mathbb{R}^n$ of positive measure is a UDS.
2. If $n \geq 2$ one can choose a G_δ set $S \subseteq \mathbb{R}^n$ to contain all rational lines and to have measure 0. Hence there are *null* universal differentiability subsets of $\mathbb{R}^n$, $n \geq 2$.
3. In $\mathbb{R}^1$, however, for every subset E of measure 0 one can find a Lipschitz function which fails to have a derivative inside E.

Sharpness of the result, $n \geq 2$

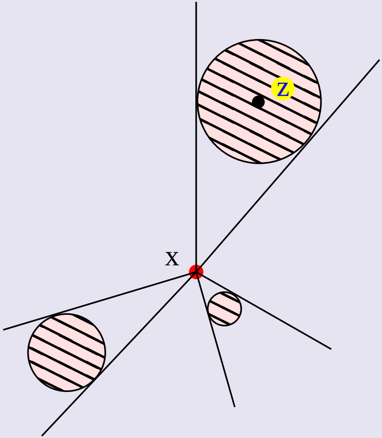
[Preiss, 1990] [Doré, M., 2010, 2011, 2012] [Dymond, M., 2013]
If $n \geq 2$, then $\mathbb{R}^n$ contains Lebesgue null universal differentiability subsets.

Examples of non-differentiability sets

Classical results

1. $E \subseteq X$ is porous.

Def. $E \subseteq X$ is porous at $x \in X$ if $\exists \lambda > 0$ s.t. for every $r > 0$ there is a $z \in B(x, r)$ such that $B(z, \lambda\|z - x\|) \cap E = \emptyset$.



E is porous at $x \in E \Rightarrow$
$f(y) = \mathrm{dist}(y, E)$ is 1-Lipschitz and is not differentiable at x.

$$\frac{f(z) - f(x)}{\|z - x\|} \geq \lambda$$

$E \subseteq X$ is porous if it is porous at each of its points.

Examples of non-differentiability sets of Lipschitz functions

Classical results

1. $E \subseteq X$ is porous, then $f(x) = \mathrm{dist}(x, E)$ is a 1-Lipschitz function and the set of points where f is not Fréchet differentiable contains E.

Thus porous sets are not UDS.

2. $E \subseteq X$ is σ-porous, i.e. a countable union of porous sets.

B. Kirchheim, D. Preiss, L. Zajíček (1980s):
There exists a Lipschitz function $f\colon X \to \mathbb{R}$ that is nowhere diff. on E.

Thus σ-porous sets are not UDS.

3. D. Preiss (1990):
If X^* is separable and the set $E \subseteq X$ is a G_δ set containing a dense set of lines, then every Lipschitz function $f\colon X \to \mathbb{R}$ is Fréchet differentiable at some point $x \in E$.

This set is a UDS.

UDS

Search for null or small universal differentiability sets

1. The set constructed by D. Preiss can be chosen to be Lebesgue null in every $X = \mathbb{R}^n$, $n \geq 2$,
however its closure is always equal to the whole space.

2. M. Doré, O.M. (2010 + 2011):
If $n \geq 2$, there exists a compact universal differentiability set $E \subseteq \mathbb{R}^n$ of Hausdorff dimension 1 (so it is Lebesgue null).

3. M. Doré, O.M. (2012):
If X^* is separable, then there exists a closed bounded totally disconnected universal differentiability set $E \subseteq X$ of Hausdorff dimension 1.

4. M. Dymond, O.M. (2013):
If $n \geq 2$, $\exists\ E \subseteq \mathbb{R}^n$ a compact universal differentiability set of the upper Minkowski (box counting) dimension 1 (and it is Hausdorff dim 1 too).

Hausdorff and Minkowski dimension

Let $A \subset \mathbb{R}^n$.

Hausdorff dimension

$$\mathcal{H}^p(A) = \lim_{\delta \downarrow 0} \inf \Big\{ \sum_i \mathrm{diam}(E_i)^p : A \subseteq \bigcup_i E_i, \mathrm{diam}(E_i) \leq \delta \Big\}.$$

is the p-dimensional Hausdorff measure of A.

Hausdorff dimension:

$$\underline{\mathrm{dim}_{\mathcal{H}}(A) = \inf\{p : \mathcal{H}^p(A) = 0\}}.$$

Minkowski (box counting) dimension

Now for each $\delta > 0$ let N_δ be the minimal possible number of balls of radius δ with which it is possible to cover A. Then

$$\underline{\overline{\dim}_{\mathcal{M}}(A) / \underline{\dim}_{\mathcal{M}}(A) = \inf\{p : \overline{\lim}_{\delta \downarrow 0} / \underline{\lim}_{\delta \downarrow 0} N_\delta \delta^p = 0\}}$$

is the upper (lower) Minkowski dimension of A.

Universal differentiability sets

$\overline{\dim}_{\mathcal{M}}(E) \geq \underline{\dim}_{\mathcal{M}}(E) \geq \dim_{\mathcal{H}}(E) \geq 1$:, E – UDS

Assume $\dim_{\mathcal{H}}(E) < 1$; let $e \in X$, $P \in X^*$ be s.t. $P(e) = 1$.

$\dim_{\mathcal{H}}(P(E)) < 1 \quad \Rightarrow \quad S = P(E) \subseteq \mathbb{R}$ is Lebesgue null.

$\exists\ g : \mathbb{R} \to \mathbb{R}$ Lipschitz, not differentiable everywhere on S,
thus $f := g \circ P : X \to \mathbb{R}$ is Lipschitz and
$\forall x \in E$, directional derivative $f'(x, e)$ does not exist
$\Rightarrow \forall x \in E$, f is not differentiable at x.

Finding a point of differentiability in a set

$E \subseteq X$, $f : X \to \mathbb{R}$ is Lipschitz
How to find a point $x^* \in E$ s.t. f is differentiable at x^*?

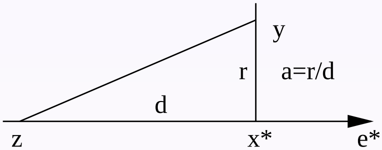
Step by step

We construct a sequence (x_k, e_k), $x_k \in E$ and $\|e_k\| = 1$ such that $f'(x_k, e_k)$ exists and is "almost maximal" among $f'(x, e)$ when $x \in E$, $\|x - x_k\|$ is small and e is arbitrary direction.

$x_k \to x^*$ by completeness,
$e_k \to e^*$ by adjusting f on each step: $f_n(x) \approx f_{n-1}(x) + \alpha_n \langle e_{n-1}, x \rangle$
and $f'(x^*, e^*)$ exists, is equal to $\lim f'(x_k, e_k)$ and is therefore "almost maximal" in every neighbourhood of x^*.

We then prove f is differentiable at x^* and $f'(x^*)(u) = f'(x^*, e^*)\langle e^*, u \rangle$.

Finding a point of differentiability

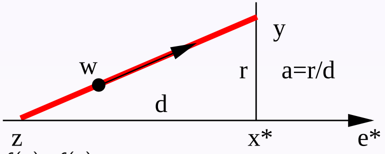


$$M = f'(x^*, e^*) \geq 0$$

$f(y) > f(x^*) + \varepsilon r$
$f(z) \approx f(x^*) - f'(x^*, e^*)d$

$$\frac{f(y) - f(z)}{\|y - z\|} \geq \frac{Md + \varepsilon r}{\sqrt{d^2 + r^2}} = \frac{M + \varepsilon a}{\sqrt{1 + a^2}} > M + \varepsilon a + O(a^2) > M + \tau$$

Finding a point of differentiability



$\frac{f(y)-f(z)}{\|y-z\|} > f'(x^*, e^*) + \tau$, $\tau > 0$ is fixed

Therefore there exists $w \in [y, z]$ such that $f'(w, \frac{y-z}{\|y-z\|}) > f'(x^*, e^*) + \tau$

If $[y, z] \subseteq E$, we get a contradiction.

Thus $f'(x^*, e^{*\perp}) = 0$.

Essential properties of a UDS

- The limit point x^* must not be a porosity point of the set E to be constructed
- Our argument works if around each limit point x^* the set E contains straight line segments in a dense set of directions

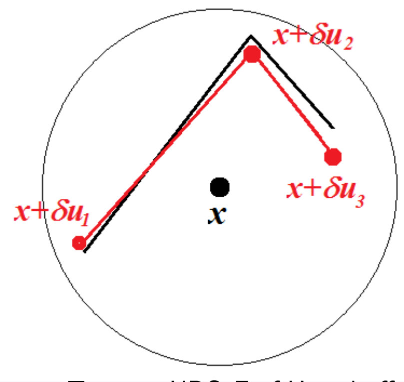
Key Geometric Lemma

If $n \geq 2$ and

$$E = \bigcup_{\lambda \in (0,1)} E_\lambda \subseteq \mathbb{R}^n,$$

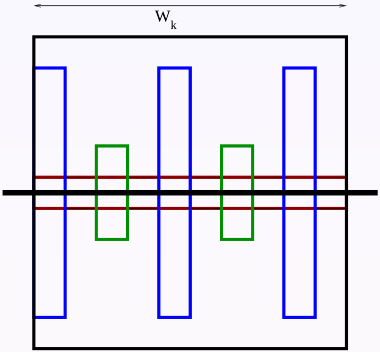
where (E_λ) is an increasing sequence of closed sets, and for all $0 < \lambda < \lambda' < 1$ and $\eta > 0$ there is a threshold $\delta^* = \delta^*(\lambda, \lambda', \eta)$ such that $x \in E_\lambda$, $\|v_i\| \leq 1$ $(i = 1, 2, 3)$, $0 < \delta < \delta^* \implies$ there exist $[x + \delta u_1, x + \delta u_2] \cup [x + \delta u_2, x + \delta u_3] \subseteq E_{\lambda'}$ with $\|u_i - v_i\| < \eta$,

then E is a universal differentiability set.



- To get a UDS E of Hausdorff dimension 1 we can choose a G_δ set G of Hausd. dim. 1 that contains **all** rational lines in it, and to construct $E \subseteq G$.
- This is not possible if we look for a UDS of Minkowski dimension 1: any set containing dense set of lines has maximal possible Minkowski dimension.

Construction



$R = R_{k+1} = Q^s,\ Q > 1,\ w_{k+1} = w_k / R$

Total number of cubes $w_{k+1} \times w_{k+1}$:

$$R + s \times Q^s + sQ \times Q^{s-1} + \cdots + \\ + sQ^{s-1} \times Q \sim s^2 Q^s = R(\log R)^2$$

Repeat for $\forall$ new tube $\implies R(\log R)^4$,

Again and again: $R(\log R)^{2m}$ cubes.

$N_{w_{k+1}} \leq N_{w_k} \times mR(\log R)^{2m}$

$\frac{N_{w_{k+1}} w_{k+1}^p}{N_{w_k} w_k^p} \leq (\log R)^{2m+1} R^{1-p} < 1,\ R$ large

For $\delta \in (w_{k+1}, w_k)$: $N_\delta \delta^p \leq N_{w_{k+1}} w_k^p = N_{w_{k+1}} w_{k+1}^p R^p$.

We show: $N_{w_{k+1}} w_{k+1}^p R_{k+1}^p \to 0$

Geometric measure theory

Equivalent definitions of a u.p.u. sets

Theorem. G. Alberti, M. Csörnyei, D. Preiss (2010): $S \subseteq \mathbb{R}^n$ The following two conditions are equivalent:

1. There exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\forall x \in S$ and $\forall \|e\| = 1$ the directional derivative $f'(x, e)$ does not exist
2. S is C-null for every cone C, i.e.
   for every $C = \{v : \|v - v_0\| < \alpha\}$ and for every $\varepsilon > 0$
   there exists an open set G_ε with $S \subseteq G_\varepsilon$ and
   $$\mathcal{H}^1(\gamma \cap G_\varepsilon) \le \varepsilon$$
   for every C^1-curve γ whose tangents lie in C.

u.p.u. $\Rightarrow$ p.u.

Each uniformly purely unrectifiable set is **purely unrectifiable**:
its intersection with any smooth curve has 1-dimensional measure 0.

Geometric measure theory: Open question

Does there exist a purely unrectifiable set which is NOT uniformly purely unrectifiable?

Question

Does there exist a purely unrectifiable UDS?

- In our original construction the final set contains many straight line intervals $\Rightarrow$ not p.u.
- However we know how to eliminate all straight line intervals from the UDS
  - Now eliminate the *measure* from these intervals (and smooth curves)
  - In the construction, replace straight segments by broken lines or curves with Lipschitz constants $\to \infty$
- If the measure of $E \cap I$ is zero for straight line intervals I then we cannot have a sequence of points $x_n \in E$ (nothing to start with!) so $x_n \in E_n$ for each $n \geq 1$
  - If $x_n \in E_n \setminus E_{n+1}$ then how to find $x_{n+1} \in E_{n+1}$ close to x_{n+1}?

Open questions

Conjecture 1

In $\mathbb{R}^d$, $d \geq 2$, every set of positive measure contains a (closed) universal differentiability subset of Lebesgue measure zero;

Conjecture 2

Every UDS contains a closed universal differentiability subset.