# A free boundary problem from Brownian bees in the infinite swarm limit in $\mathbb{R}^d$

**Jim Nolen**
**Mathematics Department**
**Duke University**

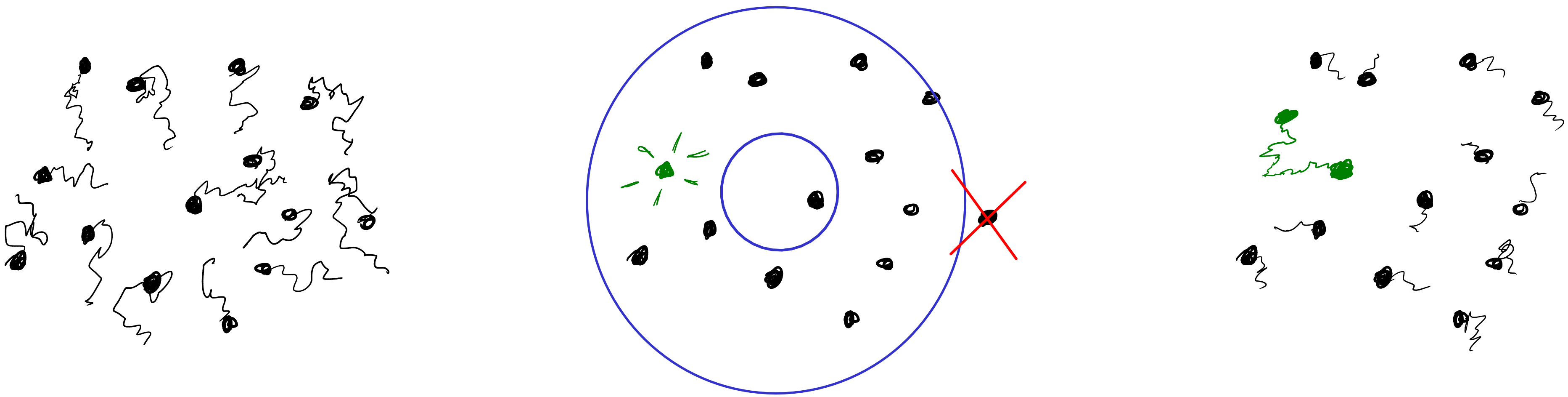
Including joint work with
Julien Berestycki (Oxford), Éric Brunet (ENS-Paris), Sarah Penington (Bath)
and with Erin Beckman (Concordia)

The **Brownian Bees Process** is a special case of Branching Brownian Motion with selection ($N$**-BBM**).

There are $N$ particles, moving as independent Brownian motions in $\mathbb{R}^d$.

Each particle branches independently, at rate one.

At each branch event, the particle farthest from the origin is removed.



General $N$-**BBM process** with spatial selection:

There are $N$ particles, moving as independent Brownian motions in $\mathbb{R}^d$.

Each particle branches independently, at rate one.

At each branch event, remove the particle with minimal "fitness", according to fitness function $V(x) : \mathbb{R}^d \to \mathbb{R}$.

*Thus, the total number of particles is always $N$.*

$$\mathbf{X}(t) = \{X_1(t), \dots, X_N(t)\} \subset \mathbb{R}^d$$

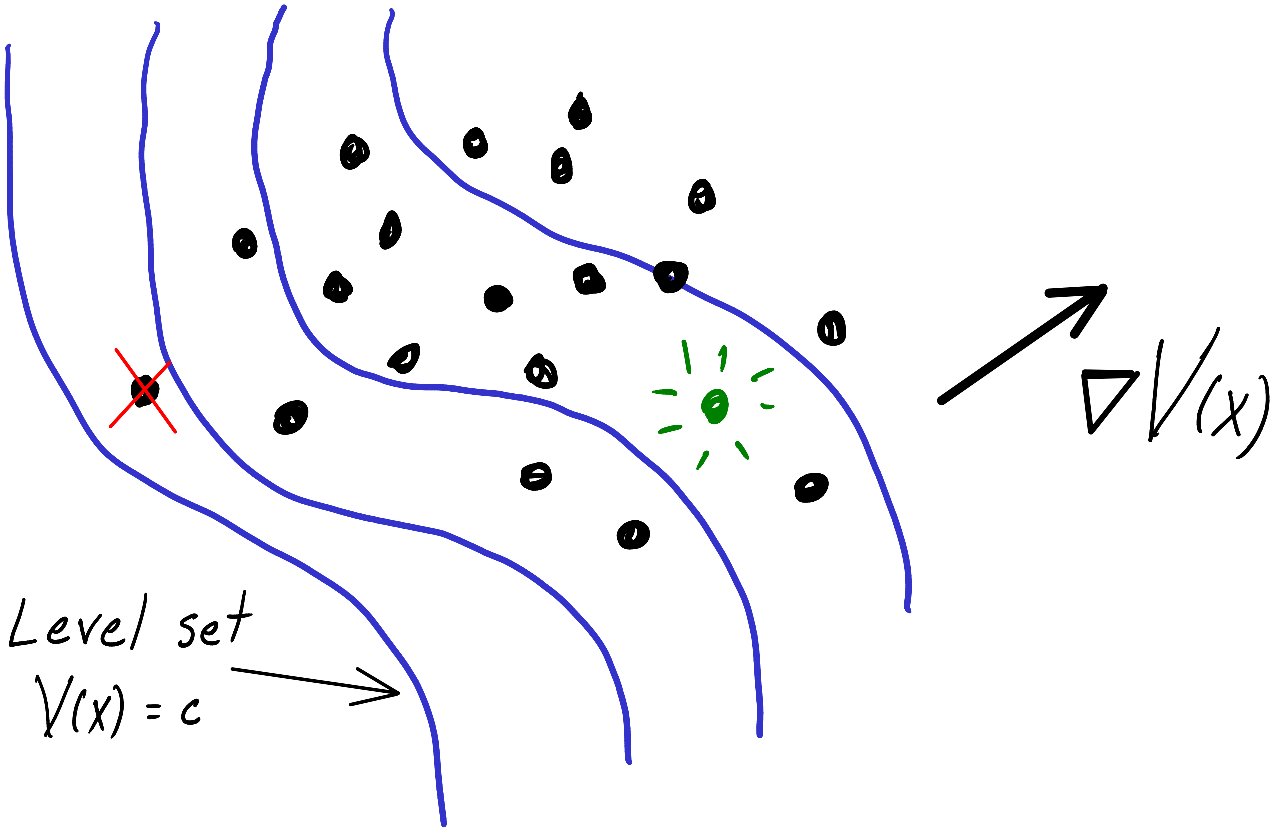
$$\text{Particle positions:} \quad X_1(t), \dots, X_N(t) \in \mathbb{R}^d$$

$$\text{Fitness values:} \quad V(X_1(t)), \dots, V(X_N(t))$$

Brunet, Derrida, Mueller, Munier (2006)

N. Berestycki, L. Zhao (2018).

Selection pressure pushes the ensemble in the direction of $\nabla V$.




The process is unchanged by replacing $V$ with $F \circ V$, $F$ increasing. For the **Brownian Bees** process, $V(x) = -|x|$; the selection has a confining effect on the population.

**Some interesting questions:**

How does the structure of $V(x)$ effect the evolution of the population?

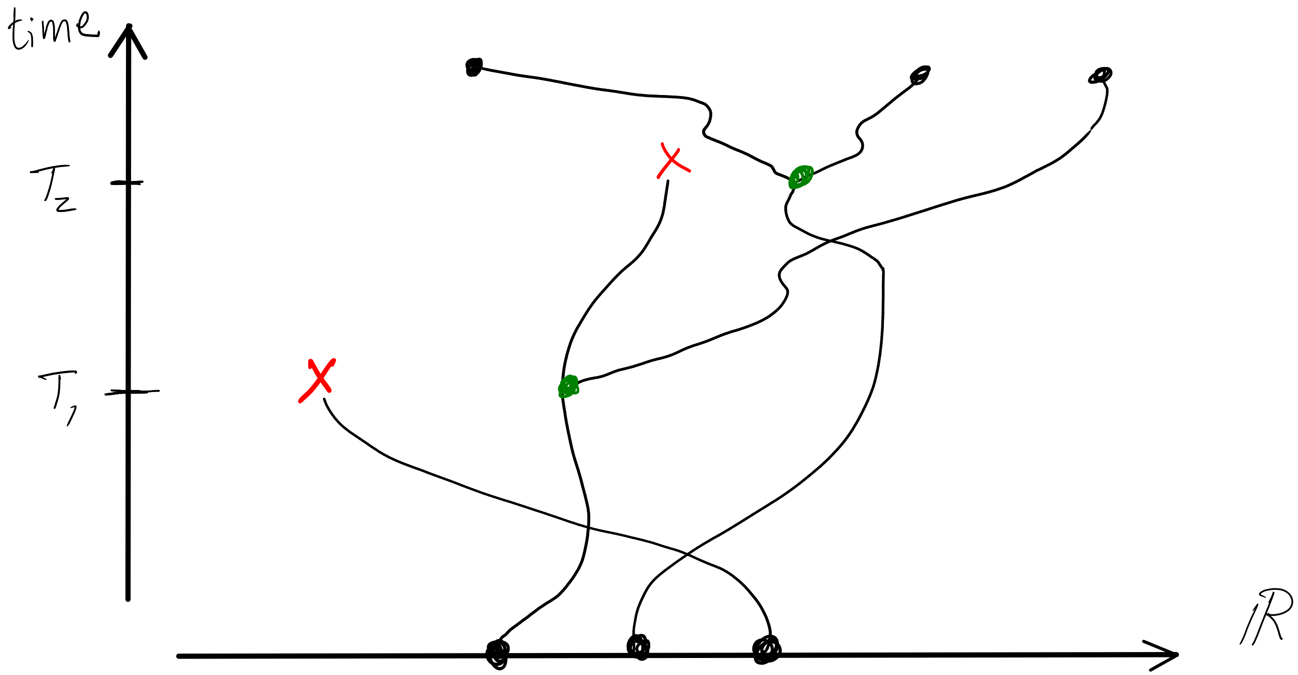
What is the behavior as $t \to \infty$?

Is there a hydrodynamic limit as $N \to \infty$?

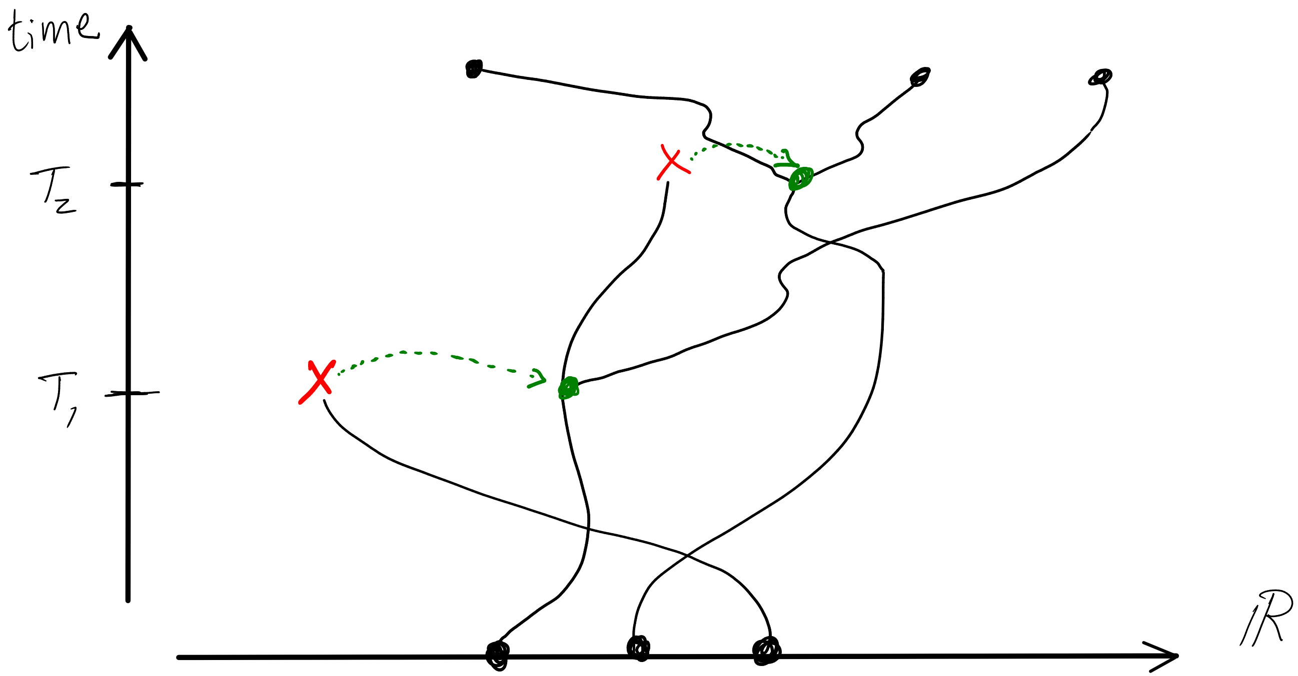
Do these limits commute?

How does $V$ effect the genealogy of the process?

There have been several works on the case $d = 1$ with monotone fitness: $V(x) = x$.
The selection pressure produces a “drift” to the right.



There have been several works on the case $d = 1$ with monotone fitness: $V(x) = x$. The selection pressure produces a "drift" to the right.




Alternatively, you might say that the left-most particle jumps to the location of another randomly chosen particle.

For each $N$, the selection pushes the ensemble to the right with positive speed $c_N$:

$$\lim_{t\to\infty} \max_k \frac{X_k(t)}{t} = \lim_{t\to\infty} \min_k \frac{X_k(t)}{t} = c_N > 0,$$

almost surely. Moreover,

$$c_N \sim 2 - \frac{\pi^2}{(\log N)^2}, \qquad \text{as } N \to \infty.$$

Bérard and Goúeré, Comm. Math. Phys, 2010.

N. Berestycki, L. Zhao, Ann. Appl. Prob. 2018.

There are some outstanding conjectures related to the **geneaology**:

It is expected that the ancestral process behaves like the Bolthausen-Sznitman coalescent on a time scale $(\log N)^3$.

This allows for multiple simultaneous mergers in the family ancestry, because a single particle may run far ahead and then quickly branch to make up a significant fraction of the population.

Brunet, Derrida, Mueller, Munier, Phys Rev E, 2006.

P. Maillard, PTRF 2016.

J. Berestycki, N. Berestycki, J. Schweinsberg, Ann. Probability 2013.

**Hydrodynamic limit, $d = 1$:**

For $d = 1$, with monotone fitness, De Masi, Ferrari, Presutti, Soprano-Lotto (2017) proved a hydrodynamic limit for the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k(t)}$$

As $N \to \infty$, $\mu_t^N \to v(t, x)$ which solves

$$\begin{aligned} &v_t = v_{xx} + v, \quad x \geq \gamma(t) \\ &v(t, \gamma(t)) = 0 \\ &\int_{\gamma(t)}^{\infty} v(t, x)\, dx = 1 \end{aligned}$$

$\gamma(t)$ defines a **free boundary**.

Global existence of PDE solution: J. Berestycki, É. Brunet, S. Penington 2018

This problem is equivalent to

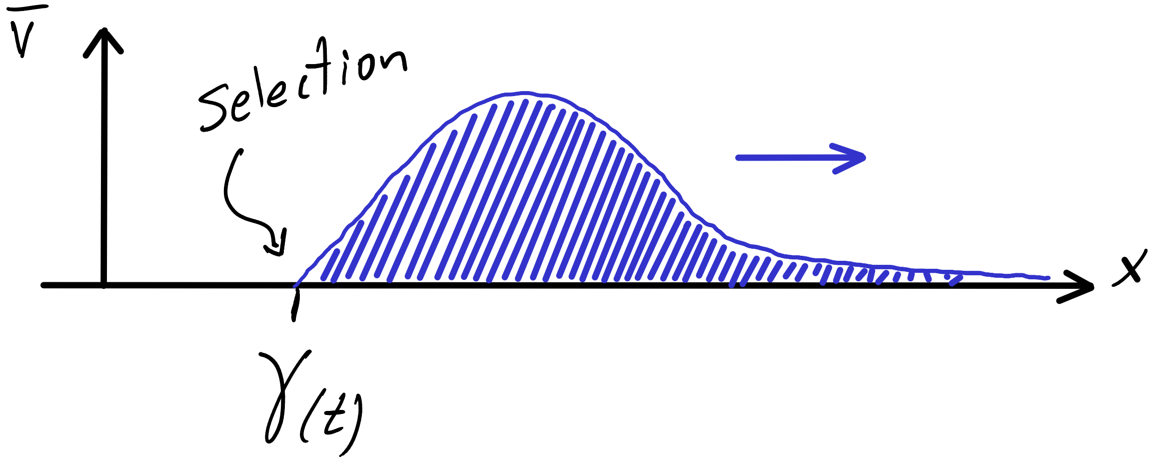
$$
\begin{aligned}
&v_t = v_{xx} + v, \qquad x \geq \gamma(t), \\
&v(t, \gamma(t)) = 0, \\
&\gamma'(t) = -v_{xx}(t, \gamma(t)) \\
&v_x(t, \gamma(t)) = 1
\end{aligned}
$$

Recall that the supercooled Stefan problem involves $\gamma'(t) = v_x(t, \gamma(t))$.

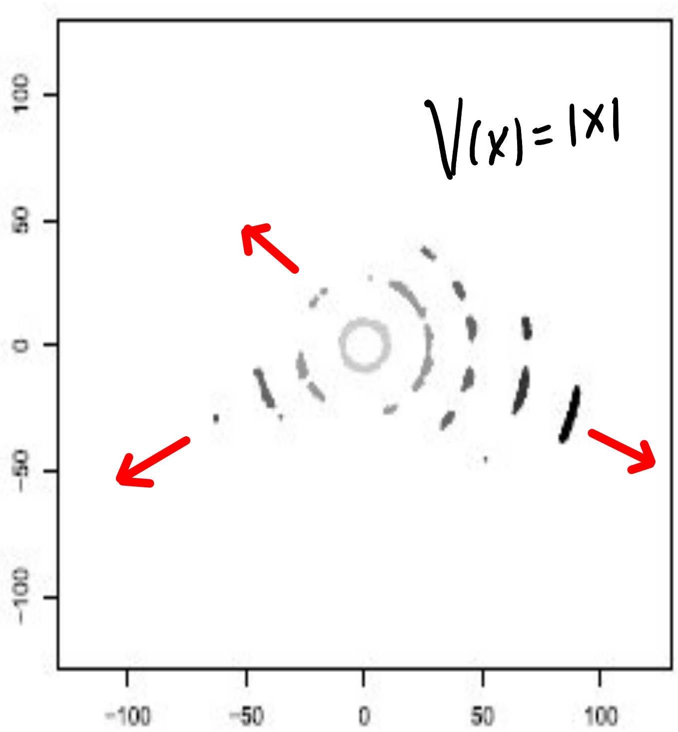
Chayes, Kim (2012), Delarue, Nadtochiy, Shkolnikov (2019)

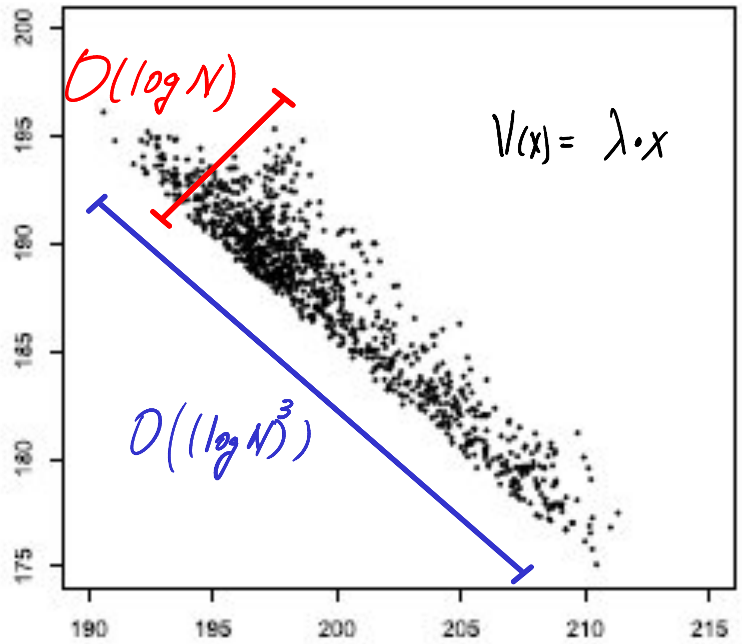
There is an explicit traveling wave solution

$$\bar{v}(t,x) = \phi(x - \gamma(t)), \qquad \phi(x) = xe^{-x}, \quad \gamma(t) = 2t.$$



For $d \geq 2$, or with non-monotone fitness, there are relatively few results.

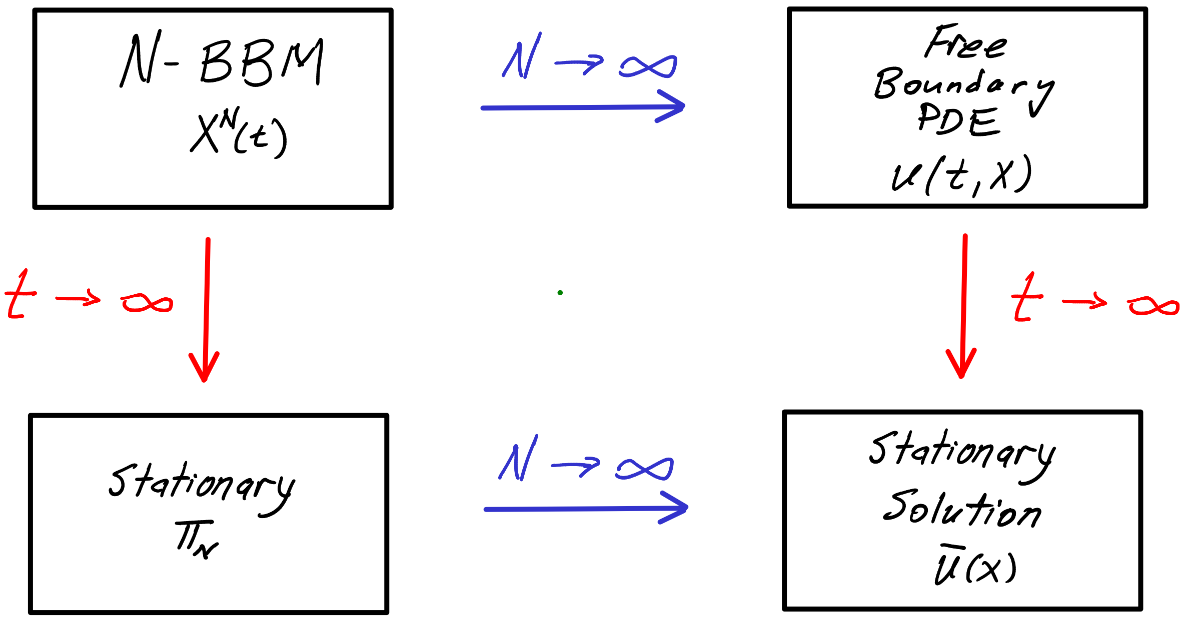



Images from N. Berestycki, L. Zhao, Ann. Appl. Prob. 2018.

**Our work:** N-BBM in dimension $d \geq 1$, with fitness $V(x) = -|x|$.

Super-level sets of $V$ are compact (Euclidean balls). So, the selection has a confining effect on the ensemble.




J. Berestycki, Brunet, Penington, N. 2020

**Hydrodynamic limit, $d \geq 1$:**

Let $\mu_t^N$ be the empirical measure for the N-BBM with fitness $V(x) = -|x|$ in $\mathbb{R}^d$, and define

$$M_t^N = \max_{k \in \{1,\ldots,N\}} |X_k(t)|.$$

Suppose $\{X_k(0)\}_{k=1}^N$ are i.i.d, $X_k(0) \sim \mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Then for every $t > 0$,

$$\begin{aligned} \mu_t^N(dx) &\to u(t,x)\,dx \;\; \text{weakly} \\ M_t^N &\to R_t \end{aligned}$$

almost surely, as $N \to \infty$, where $(u(t,x), R_t)$ solves a free-boundary problem with initial condition $\mu_0$....

**Free boundary problem** in $\mathbb{R}^d$ for the hydrodynamic limit: Given initial measure $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, find a pair $(u(x,t), R_t)$ such that

$$\begin{aligned}
&u_t = \Delta u + u, \qquad |x| < R_t, \;\; t > 0 \\
&u(t,x) = 0, \qquad |x| \geq R_t, \;\; t > 0 \\
&\int_{|x| \leq R_t} u(t,x)\, dx = 1 \\
&u(t,x) \to \mu_0(dx), \text{weakly as } t \to 0
\end{aligned}$$

**Theorem:** For any Borel probability measure $\mu_0$ on $\mathbb{R}^d$, there is a unique, global solution $(u, R_t)$ to this free boundary problem. Moreover, $R_t$ is finite and continuous for all $t > 0$, and $R_t \to R_0 := \inf\{r > 0 \mid \mu_0(\mathcal{B}(r)) = 1\} \in [0, \infty]$.

**Free boundary problem**

$$
\begin{aligned}
&u_t = \Delta u + u, \qquad |x| < R_t, \ \ t > 0 \\
&u(t,x) = 0, \qquad |x| \geq R_t, \ \ t > 0 \\
&\int_{|x|\leq R_t} u(t,x)\,dx = 1 \\
&u(t,x) \to \mu_0(dx), \text{weakly as } t \to 0
\end{aligned}
$$

$u(x,t)$ may not be spherically symmetric, even though the free boundary is a sphere: $\{|x| = R_t\}$.

If $\mu_0$ has an atom at the edge of its support, then $R_t$ very rapidly moves outward, $R_t \gg R_0 + \sqrt{t}$ as $t \to 0$. However, for any $\alpha \in (0, 1/2)$,

$$R_t - R_s \leq C_\alpha (t-s)^\alpha, \quad 0 \leq s < t.$$

**Large time limit** for the free boundary problem:

Let $R_\infty$ be chosen so that the principal Dirichlet eigenvalue of $(-\Delta)$ is $\lambda = 1$:

$$
\begin{aligned}
-\Delta \bar{u} = \bar{u}, \quad |x| \leq R_\infty, \\
\bar{u}(x) = 0, \quad |x| = R_\infty.
\end{aligned}
$$

Then,

$$\|u(t,\cdot) - \bar{u}(\cdot)\|_\infty \leq Ct^{-1}, \quad \text{and} \quad |R_t - R_\infty| \leq Ct^{-1}$$

In particular, for $d = 1$: $\quad R_\infty = \pi/2, \qquad \bar{u}(x) = \alpha \cos(x).$

**Stationary distribution:**

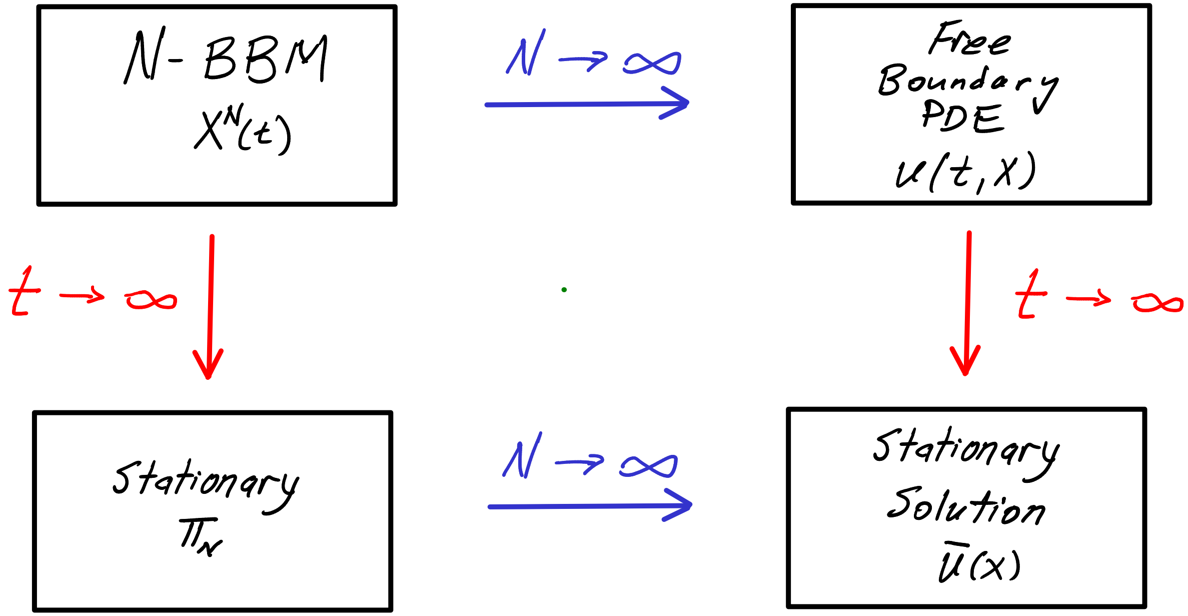
For each $N$, the N-BBM problem for $d \geq 1$ with $V(x) = -|x|$ has a stationary distribution $\pi^N$ on $(\mathbb{R}^d)^N$. ($\mathbf{X}(t) = (X_1(t), \ldots, X_N(t))$ is positive Harris recurrent)

The density of particles under $\pi^N$ converges to $\bar{u}$ as $N \to \infty$. For a configuration $\mathbf{X} \in (\mathbb{R}^d)^N$, let $\mu_{\mathbf{X}}^N$ be its empirical measure. For any $\epsilon > 0$, and any $A \subset \mathbb{R}^d$,

$$\pi^N \left( \mathbf{X} \in (\mathbb{R}^d)^N \mid \left| \mu_{\mathbf{X}}^N(A) - \int_A \bar{u}(x)\, dx \right| > \epsilon \right) \to 0$$

as $N \to \infty$. Also,

$$\pi^N \left( \mathbf{X} \in (\mathbb{R}^d)^N \mid \left| \max_{k=1,\ldots,N} |X_k| - R_\infty \right| > \epsilon \right) \to 0$$


N-BBM
$X^N(t)$
$N \to \infty$
Free Boundary PDE
$u(t,x)$
$t \to \infty$
$t \to \infty$
Stationary
$\pi_N$
$N \to \infty$
Stationary Solution
$\bar{u}(x)$

Observations about the free boundary problem

$$\begin{aligned}
&u_t = \Delta u + u, \qquad |x| < R_t, \;\; t > 0 \\
&u(t,x) = 0, \qquad |x| \geq R_t, \;\; t > 0 \\
&\int_{|x|\leq R_t} u(t,x)\, dx = 1 \\
&u(t,x) \to \mu_0(dx), \text{weakly as } t \to 0
\end{aligned}$$

Assuming $u$ is $C^2$ at the boundary, we formally derive

$$\int_{|x|=R_t} (\nu \cdot \nabla u) dS(x) = -1, \quad \text{and} \quad \frac{d}{dt} R_t = \int_{|x|=R_t} \Delta u(x,t)\, dS(x)$$

So, this has some similarities to a problem of Caffarelli and Vázquez (1992):

$$u_t = \Delta u, \quad u > 0, \quad x \in \Omega_t$$

with $u = 0$ and $|\nabla u| = c$ on the free boundary $\partial\Omega_t$.

Hence speed of free boundary is $\frac{u_t}{|\nabla u|} = c^{-1}\Delta u$.

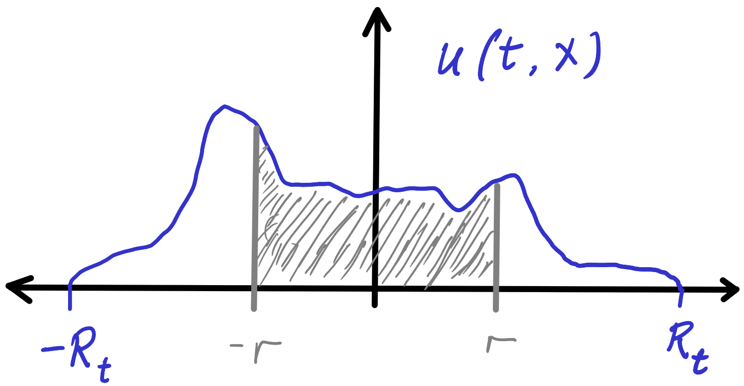
The free boundary problem is closely related to a **parabolic obstacle problem**.

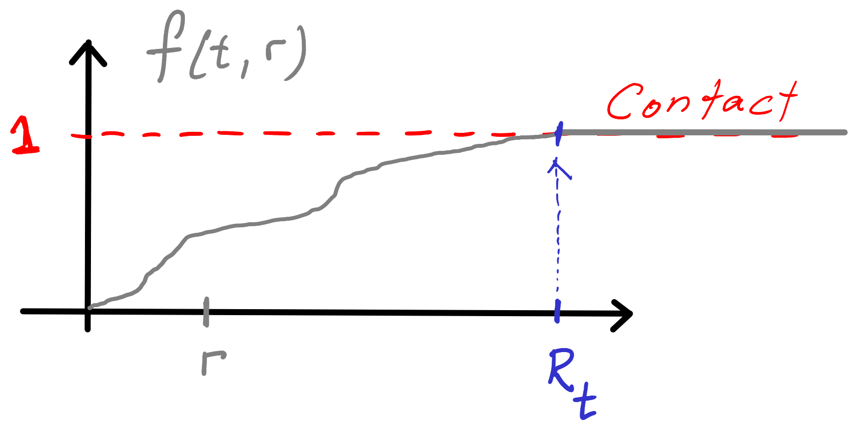
$$\text{The function} \quad f(t,r) = \int_{|x|\le r} u(t,x)\,dx, \quad r \ge 0$$

satisfies $f \le 1$ and

$$f_t = f_{rr} - \frac{d-1}{r} f_r + f, \quad \text{if } f(t,r) < 1$$
$$r \mapsto f_r(t,r) \ \text{ is continuous}$$

$R_t$ is the boundary of the contact set $\{f(t,\cdot) = 1\}$. At $R_t$, $f_r(t, R_t) = 0$.




Obstacle problems: see A. Friedman 1982.

General strategy for well-posedness of original free boundary problem:

(i) Well-posedness of the obstacle problem with initial condition $f_0(r) = \mu_0(\{|x| < r\})$. This defines $R_t = \inf\{r > 0 \mid f(r,t) = 1\}$.

(ii) $R_t$ is continuous for $t > 0$.

(iii) Given $R_t$, prove that

$$u(x,t) = e^t \int_{\mathbb{R}^d} \mu_0(dx) \rho^R(x,y,t)$$

solves the free boundary problem, where $y \mapsto \rho^R(x,y,t)$ is density of Brownian motion that avoids the boundary before time $t$:

$$\int_A \rho^R(x,y,t)\, dy = \mathbb{P}_x(B_t \in A, \quad |B_s| < R_s \ \ \forall\, s \in (0,t]).$$

The solution $f$ to the obstacle problem is approximated by $f = \lim_{n\to\infty} f_n$,

$$\partial_t f_n = \partial_r^2 f_n - \frac{d-1}{r}\partial_r f_n + f_n - (f_n)^n, \quad r > 0$$

Which means that $u$ is the limit of an approximating sequence $u_n(t,x)$ satisfying

$$\partial_t u_n = \Delta u_n + u_n - u_n A_n([u_n], x, t)$$

where the nonlinear, nonlocal term $A_n$ is

$$A_n([u_n], x, t) = n\left(\int_{B(|x|)} u_n(t,y)\,dy\right)^{n-1}$$

which strongly penalizes mass at points $|x| > R = R_{n,t}$ such that

$$\int_{B(R)} u_n(t,y)\,dy = 1,$$

but is negligible at points $|x| < R$.

The obstacle problem provides monotonicity that has an analogue for the stochastic system. The radial distribution function

$$f(t,r) = \int_{|x|\le r} u(t,x)\,dx, \quad r \ge 0$$

satisfies the obstacle problem: $f \le 1$ with

$$\begin{aligned} & f_t = f_{rr} - \frac{d-1}{r} f_r + f, \quad \text{where } \; f < 1 \\ & f(t,0) = 0, \quad t > 0, \\ & r \mapsto f_r(t,r) \; \text{ is continuous.} \end{aligned}$$

Given $f(t,r)$, the solution at time $t+\delta$ may be approximated by operator splitting

$$e^{\delta} G_\delta * \mathcal{C}_{e^{-\delta}} f(t,r) \le f(r, t+\delta) \le \mathcal{C}_1 (e^{\delta} G_\delta * f(t,r))$$

where operator $\mathcal{C}_m$ cuts mass to level $m$:

$$\mathcal{C}_m g(r) = \min(g(r), m)$$

For the stochastic particle system, this idea suggests a coupling of the N-BBM with unconstrained BBM.

Let $\mathbf{X}^{(N)}(t)$ be the N-BBM. Let $\mathbf{X}^{+}(t)$ be a regular BBM, unconstrained.

If the particles are labeled by *rank*: $|X_1^{(N)}| \leq |X_2^{(N)}| \leq \cdots \leq |X_N^{(N)}|$, then

$$|X_k^+| \preccurlyeq |X_k^{(N)}|, \qquad k = 1, \ldots, N.$$

That is, the CDF of $|\mathbf{X}^+|$ stochastically dominates the CDF of $|\mathbf{X}^N|$:

$$F^{(N)}(t,\cdot) \preccurlyeq F^+(t,\cdot), \qquad F^{(N)}(t,\cdot) \preccurlyeq \mathcal{C}_1 F^+(t,\cdot)$$

$$\text{where} \quad F^{(N)}(t,r) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{\{|X_k(t)| \leq r\}} \qquad F^+(t,r) = \frac{1}{N}\sum_{k=1}^{N_t^+} \mathbb{1}_{\{|X_k^+(t)| \leq r\}}.$$

$$\text{Compare this to:} \quad f(r, t+\delta) \leq \mathcal{C}_1(e^{\delta} G_\delta * f(t,r))$$

This coupling gives an upper bound on $F^{(N)}$.

A lower bound can be obtained by cutting first, then evolving the free BBM until $N_t^+ = N$.

Define $\mathbf{X}^+(0) \subset \mathbf{X}^{(N)}(0)$ by removing particles from $\mathbf{X}^{(N)}(0)$ with highest ranks, then

$$|X_k^{(N)}(t)| \preccurlyeq |X_k^+(t)|, \qquad k = 1, \ldots, N_t$$

as long as $N_t^+ \leq N$. Which implies

$$\mathbb{1}_{\{N_t^+ < N\}} F^+(t, \cdot) \preccurlyeq F^{(N)}(t, \cdot)$$

$$\text{Compare this to:} \quad e^\delta G_\delta * \mathcal{C}_{e^{-\delta}} f(t, r) \leq f(r, t + \delta)$$

The first version of this coupling comes from De Masi, Ferrari, Presutti, Soprano-Lotto (2017).

For **general fitness function** $V(x) : \mathbb{R}^d \to \mathbb{R}$, one expects the following limit problem:

For $\ell \in \mathbb{R}$, let $\Omega_\ell$ be the super-level set of fitness function $V$:

$$\Omega_\ell = \{x \in \mathbb{R}^d \mid V(x) > \ell\}$$

Find $(u(t,x), \ell_t)$ such that

$$u_t = \Delta u + u, \quad x \in \Omega_{\ell_t},$$
$$u(t,x) = 0, \quad x \in \partial\Omega_{\ell_t}$$
$$\int_{\Omega_{\ell_t}} u(t,x)\, dx = 1$$

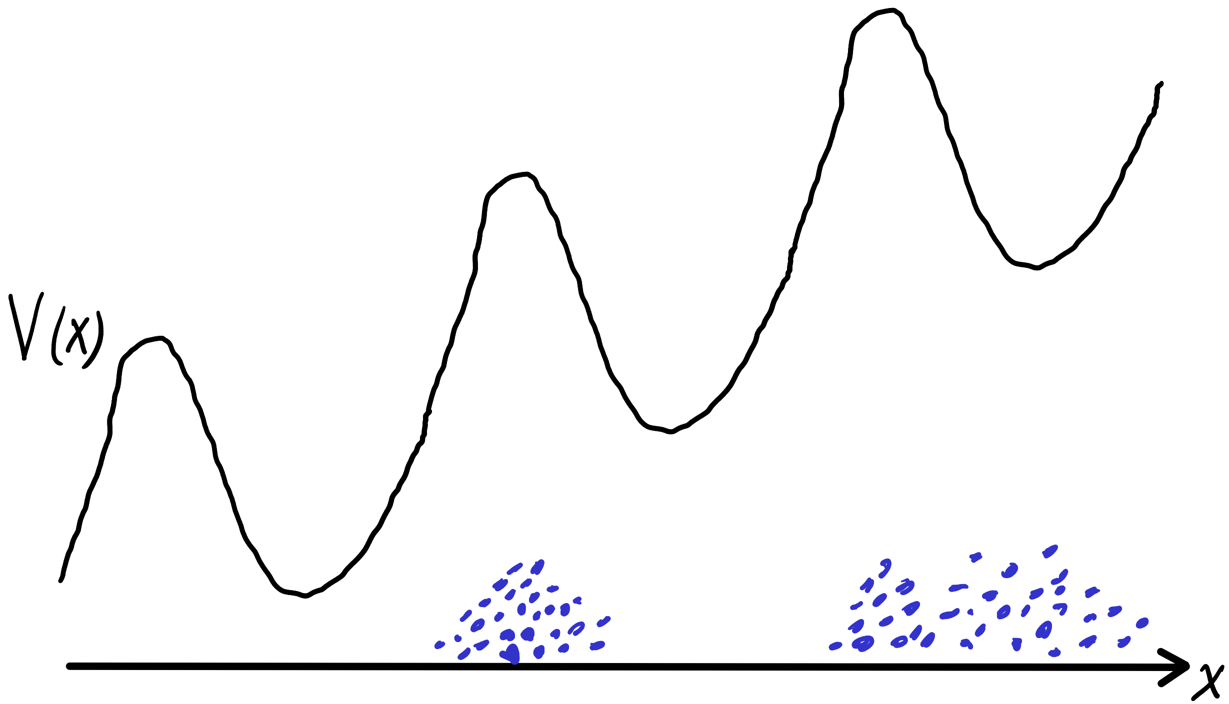
So, the free boundary is $\partial\Omega_{\ell_t}$, the $\ell_t$-level set of $V$.
$\ell_t$ is the fitness-level at which selection happens.
The topology of $\Omega_{\ell_t}$ may change as $\ell_t$ varies.

Back to N-BBM:

What is the effect of fitness barriers? Suppose $d = 1$, but non-monotone fitness.

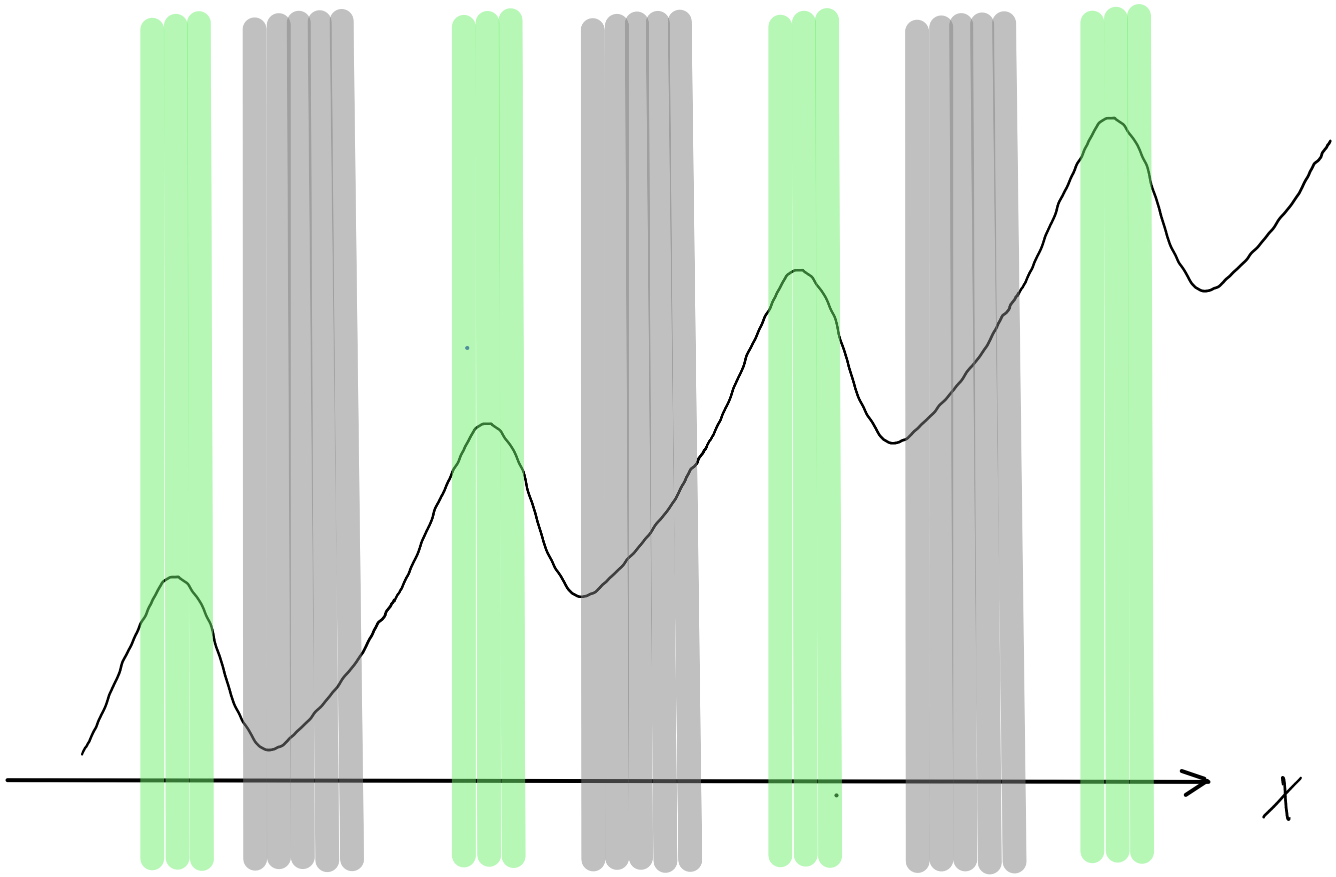
$$V(x) = x + \psi(x), \qquad \psi(x) \text{ is } L\text{-periodic.}$$

Particles must pass through a fitness valley before moving to better territory.



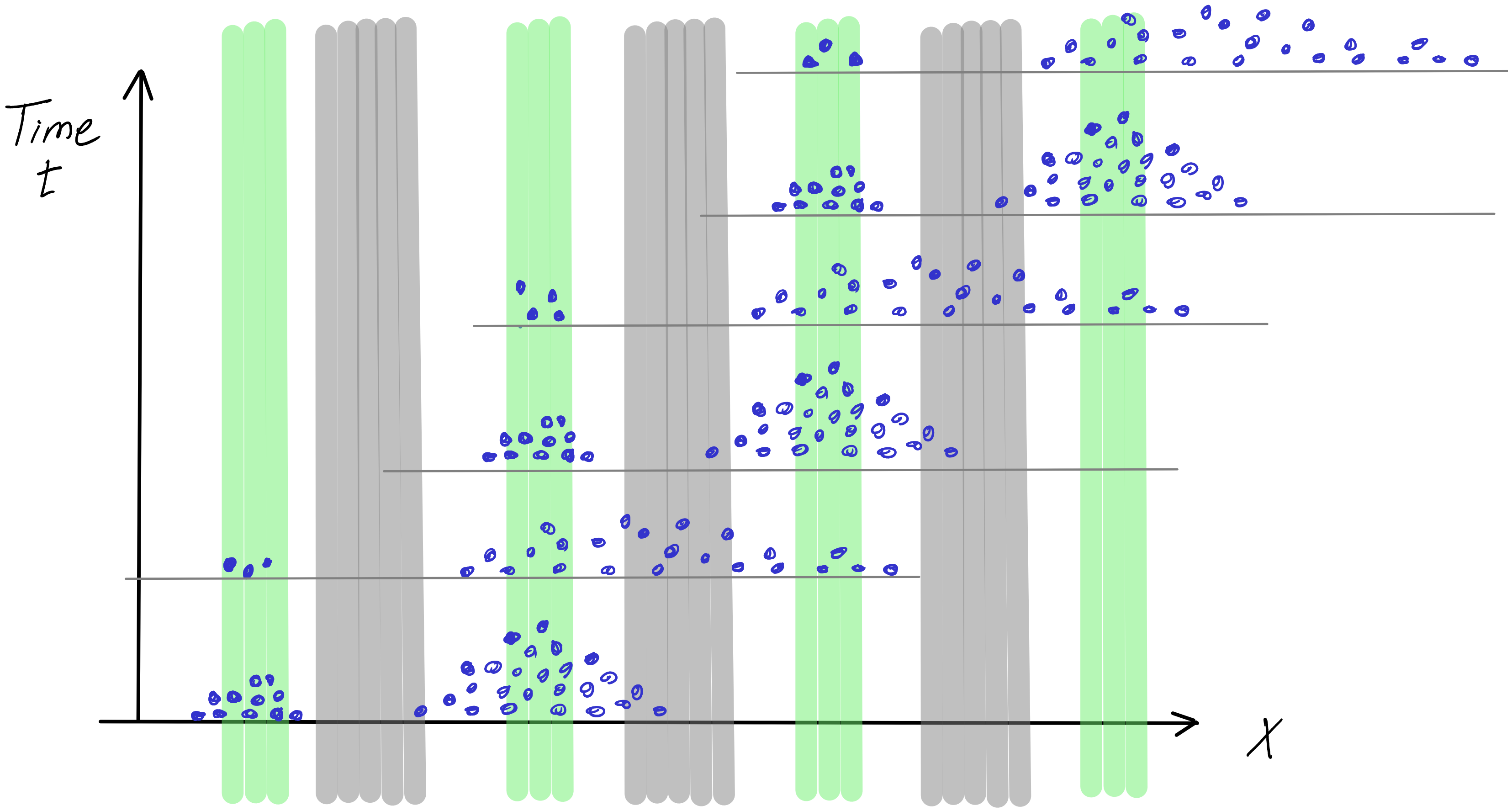
$$V(x) = x + \psi(x), \qquad \psi(x) \text{ is } L\text{-periodic.}$$

Ensemble behaves like a pulsating traveling wave.



$$V(x) = x + \psi(x), \qquad \psi(x) \text{ is } L\text{-periodic.}$$

Ensemble behaves like a pulsating traveling wave.



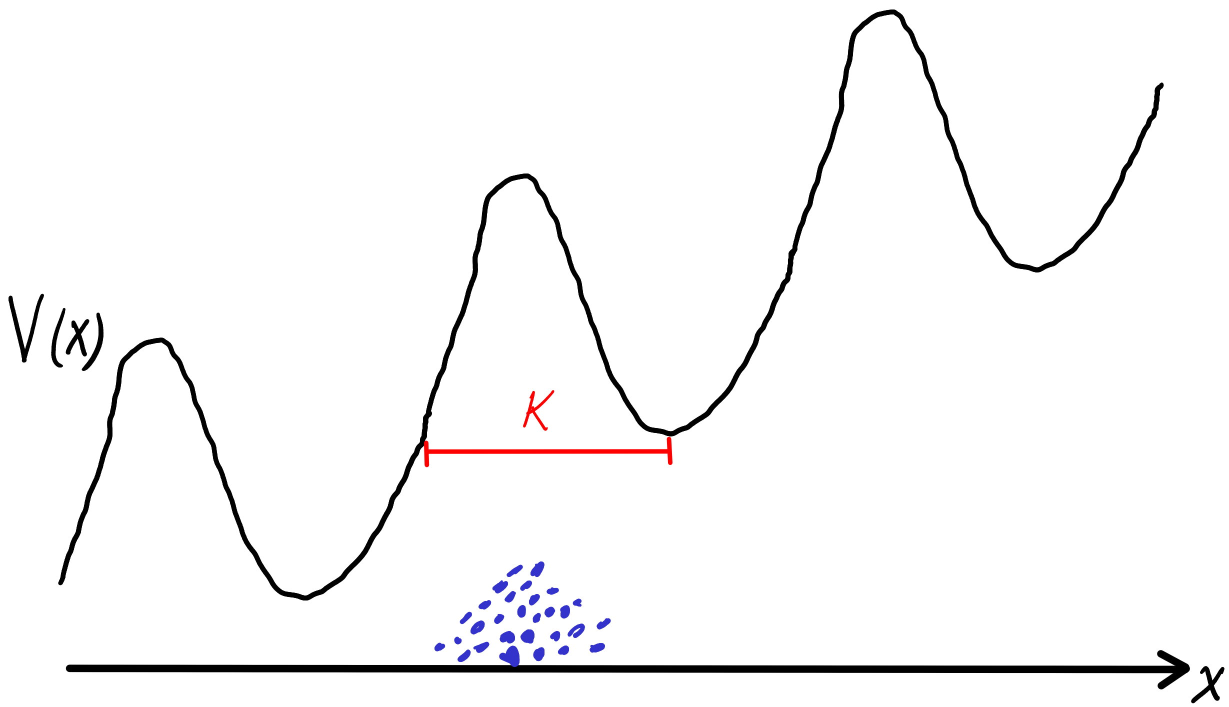
Theorem: for each $N$ there is a unique stationary distribution, within a moving frame. Moreover, there is $c_N$ such that

$$\lim_{t\to\infty} \max_k \frac{X_k(t)}{t} = \lim_{t\to\infty} \min_k \frac{X_k(t)}{t} = c_N$$

holds almost surely and in $L^1$.

Erin Beckman, N. (2019)

Unlike the case of monotone fitness, the hydrodynamic limit may be a standing wave (eigenfunction) inside a local fitness valley, if $K > 2R_\infty = \pi$.




Numerical simulations show a metastable behavior when $K > 2R_\infty$.

Is $c_N > 0$?

**Conclusion/summary:**

- What aspects of this extend to general fitness $V$?
- What is effect of $V$ on the population?
- Traveling waves? metastability? rate of adaptation?
- Strong selection principle?
- Variants on graphs?

**For references and further reading:**


- J. Berestycki, Brunet, N., Penington: *A free boundary problem arising from branching Brownian motion with selection*, arXiv:2005.09384
- J. Berestycki, Brunet, N., Penington: *Brownian bees in the infinite swarm limit*, to appear soon...