Finite dimensional approximations of Hamilton-Jacobi-Bellman equations in spaces of probability measures

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(joint work with W. Gangbo and S. Mayorga)

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Approximations of HJB equations

- HJB equation (of some type) in the space of probability measures $\Rightarrow$ appropriately interpreted viscosity solution $\mathcal{U}$
- Finite dimensional HJB equations corresponding to problems for n-particle systems $\Rightarrow$ solutions u_n

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Questions:

- Can we prove that "$u_n \to \mathcal{U}$"?
- Can we interpret the limit? For instance, is $\mathcal{U}$ the value function of variational/optimal control/... problem in spaces of probability measures?

Approximations of HJB equations

Some results in these directions are mentioned around without proofs or in a formal way (for instance lectures of P.L. Lions, notes of Cardaliaguet), some may be considered to be part of folklore of the theory. The goal is to investigate the problem rigorously from the point of view of viscosity solutions.

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Main points:

- Work with viscosity solutions. No regularity besides continuity. Proper form of the equation and the Hamiltonian.
- Viscosity solution $\mathcal{U}$ is interpreted as the viscosity solution of the lifted HJB equation in a Hilbert space $\Rightarrow$ L-viscosity solution.
- Obtain uniform estimates for u_n which give some compactness in $\mathcal{P}_2(\mathbb{R}^d)$.
- Prove that the functions u_n when converted to functions of empirical measures converge to the unique L-viscosity solution of the HJB equation.

Known and related results

• Convergence problems in in the context of master equations of mean field games (P.L. Lions, Cardaliaguet, Cardaliaguet-Delarue-Lasry-Lions, Delarue-Lacker-Ramanan, Gangbo-Mésáros, Mou-Zhang,...). In particular it was proved in Cardaliaguet-Delarue-Lasry -Lions that classical solutions of finite dimensional second order Nash systems converge, in a suitable sense, to classical solutions of the corresponding master equations.

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• Regularity estimates and convergence for f. dimensional approximations of first order HJB equations in spaces of probability measures when solutions are regular (Gangbo-Mésáros, Gangbo-Ś, Mayorga,...).

Equations in abstract spaces

- HJB equations and master equations for mean field games or mean field control problems in spaces of probability measures have been studied a lot in recent years using various approaches: Bensoussan-Frehse-Yam, Bensoussan-Graber-Yam, Bensoussan-Yam, Bessi, Buckdahn-Li-Peng-Rainer, Cardaliaguet, Cardaliaguet-Cirant-Porretta, Cardaliaguet-Delarue-Lasry-Lions, Cardaliaguet-Porretta, Carmona-Delarue, Chassagneux-Crisan-Delarue, Chow-Gangbo, Delarue-Lacker-Ramanan, Gangbo-Mésáros, Gangbo-Nguyen-Tudorascu, Gangbo-Ś., Gangbo-Tudorascu, Hynd-Kim, P.L. Lions, Mayorga, Mou-Zhang, Pham-Wei.

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• Equations related to control problems with partial observation: Bandini-Cosso-Fuhrman-Pham.

• Equations related to differential games: Cosso-Pham, Jimenez-Quincampoix.

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• Huge literature on first and second order HJB equations in Hilbert spaces, beginning with works of Crandall-Lions for first order equations and P.L. Lions and later Ś. for second order equations. Recent book: G. Fabbri, F. Gozzi and A. Święch, *Stochastic Optimal Control in Infinite Dimension: Dynamic Programming and HJB Equations*, with a contribution by M. Fuhrman and G. Tessitore, Probability Theory and Stochastic Modelling 82, Springer, 2017.

HJB equations

We will be concerned with first and second order degenerate HJB equations of the form

$$\begin{cases} \partial_t \mathcal{U} - \kappa \Delta_w \mathcal{U} + \mathcal{H}(\mu, \mu, \nabla_\mu \mathcal{U}) + \mathcal{F}(\mu) = 0 & \text{in } (0,T) \times \mathcal{P}_2(\mathbb{R}^d) \\ \mathcal{U}(0,\mu) = \mathcal{U}_0(\mu) \quad \text{on } \mathcal{P}_2(\mathbb{R}^d), \end{cases} \tag{1}$$

where $\Delta_w \mathcal{U}$ is the partial Laplacian of $\mathcal{U}$, $T > 0, \kappa \geq 0, \mathcal{P}_2(\mathbb{R}^d)$ is the Wasserstein space of probability measures on $\mathbb{R}^d$ with bounded second moments, $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ and, for $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\xi \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$, the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(\mu, \nu, \xi) = \int_{\mathbb{R}^d} H(x, \nu, \xi(x)) \mu(dx)$$

for some function $H : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$.

Approximating HJB equations

Goal: Justify that the solution $\mathcal{U}$ of (1) can is obtained as the limit of the approximating finite dimensional problems

$$\begin{cases} \partial_t u_n - \kappa \mathrm{Tr}(A_n D^2 u_n) + \frac{1}{n}\sum_{i=1}^n H(x_i, \frac{1}{n-1}\sum_{j\neq i}^n \delta_{x_j}, nD_{x_i}u_n) \\ \qquad + \mathcal{F}(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}) = 0 \quad \text{in } (0,T)\times(\mathbb{R}^d)^n, \\ u_n(0,x_1,\cdots,x_n) = \mathcal{U}_0(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}) \quad \text{on } (\mathbb{R}^d)^n, \end{cases} \tag{2}$$

where for $n\in\mathbb{N}$, A_n is the $nd\times nd$ matrix composed of n^2 block matrices I_d.

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We define the functions $U_0, F : E \to \mathbb{R}$ by

$$U_0(X) = \mathcal{U}_0(X_\sharp \mathcal{L}_1), \quad F(X) = \mathcal{F}(X_\sharp \mathcal{L}_1),$$

where $_\sharp$ denotes pushforward. Thus $X_\sharp \mathcal{L}_1$ is the law of the random vector X and is an element of $\mathcal{P}_2(\mathbb{R}^d)$.

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Partial Laplacian: Partial trace of Wasserstein Hessian. Take canonical basis $\{e_1, \cdots, e_k\}$ of $\mathbb{R}^d$. Consider its elements as constant functions in E.

$$\Delta_w \mathcal{U}(\mu) := \sum_{k=1}^{d} \nabla_w^2 \mathcal{U}(\mu)(e_k, e_k)$$

If $U : E \to \mathbb{R}$ is twice differentiable and such that $\mathcal{U}(\mu) = U(X)$ when μ is the law of X then

$$\Delta_w \mathcal{U}(\mu) \circ X = \sum^{d} \langle D^2 U(X) e_k, e_k \rangle.$$

L-viscosity solutions

For $X, P \in E$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, we define

$$\tilde{H}(X, \mu, P) := \int_\Omega H(X(\omega), \mu, P(\omega)) d\omega.$$

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"Lifted" HJB equation in the Hilbert space E:

$$\begin{cases} \partial_t U - \kappa \sum_{k=1}^{d} \langle D^2 U e_k, e_k \rangle + \tilde{H}(X, X_\sharp \mathcal{L}_1, DU) + F(X) = 0 & \text{in } (0,T) \times E \\ U(0, X) = U_0(X) & \text{on } E. \end{cases} \tag{3}$$

DU, D^2U are the Fréchet derivatives of U with respect to the X variable.

L-viscosity solutions

Definition

Let $\mathcal{U} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and define $U : [0,T) \times E \to \mathbb{R}$ by $U(t,X) = \mathcal{U}(t, X_\sharp \mathcal{L}_1)$.

(i) We say that $\mathcal{U}$ is an L-viscosity subsolution of (1) on the Wasserstein space if U is a viscosity subsolution of (3) on $[0,T) \times E$.

(ii) We say that $\mathcal{U}$ is an L-viscosity supersolution of (1) on the Wasserstein space if U is a viscosity supersolution of (3) on $[0,T) \times E$.

(ii) When $\mathcal{U}$ is both an L-viscosity subsolution and an L-viscosity supersolution of (1) on the Wasserstein space, we say that it is an L-viscosity solution of (1) on the Wasserstein space.

Assumptions about H

Assumptions:

- $1 < r < 2$.
- The function $H : \mathbb{R}^d \times \mathcal{P}_r(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is such that for all $p, q, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_r(\mathbb{R}^d)$

$$|H(x,\nu,p) - H(x,\nu,q)| \le C(1 + |p| + |q|)|p - q|,$$

$$|H(x,\mu,p) - H(y,\nu,p)| \le \sigma\big((|x-y| + d_r(\mu,\nu))(1+|p|)\big)$$

for some modulus of continuity σ and

$$|H(x,\mu,p)| \le C(1 + |p|^2).$$

- The functions $\mathcal{U}_0, \mathcal{F}$ are bounded and uniformly continuous on $\mathcal{P}_r(\mathbb{R}^d)$.

Main result

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Theorem (Convergence Theorem)

Let $\kappa \geq 0$. Suppose that for $n \geq 1$ the functions $u_n : [0,T) \times (\mathbb{R}^d)^n \to \mathbb{R}$ are the viscosity solutions of (2) *Then, for every bounded set B in $\mathcal{P}_2(\mathbb{R}^d)$,*

$$\lim_{n\to\infty} \sup \Big\{ \Big| u_n(t, x_1, \cdots, x_n) - \mathcal{U}(t, \frac{1}{n}\sum_{i=1}^n \delta_{x_i}) \Big| : \\ (t, x_1, \cdots, x_n) \in [0,T) \times (\mathbb{R}^d)^n, \frac{1}{n}\sum_{i=1}^n \delta_{x_i} \in B \Big\} = 0,$$

where $\mathcal{U}$ is the unique L-viscosity solution of (1) on the Wasserstein space.

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2. We prove that there exists a modulus ρ such that for every n

$$|u_n(t, \mathbf{x}) - u_n(s, \mathbf{y})| \le \rho(|t - s| + |\mathbf{x} - \mathbf{y}|_r) \quad \forall\ t, s \in [0, T], \mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^n,$$

where if $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$,

$$|\mathbf{x} - \mathbf{y}|_r = \frac{1}{n^{1/r}}(\sum_{i=1}^{n} |x_i - y_i|^r)^{1/r}.$$

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The proof is based on the proof of comparison for (2) to get the modulus of continuity in the $\mathbf{x}$ variable and construction of proper subsolution and supersolution functions to obtain the modulus of continuity in the time variable.

Idea of the proof

The proof works in more general case when either $A_n = A_n(\mathbf{x})$, $n = 1, 2, ...,$ is the $nd \times nd$ matrix composed of n^2 block matrices $a(x_i)a^*(x_j)$, $i, j = 1, 2, ..., n$ such that the function $a : \mathbb{R}^d \to S(d)$ is bounded and Lipschitz continuous or when A_n is any sequence of $nd \times nd$ symmetric matrices with constant coefficients such that $A_n \geq 0$ and $\mathrm{Tr}(A_n) \leq Ln$ for some $L \geq 0$. Thus it works for particle systems with independent noises and particle systems with common noise with diffusion coefficient σ.

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- Extension to $[0, T] \times \mathcal{P}_r(\mathbb{R}^d)$: We define the functions

$$\mathcal{V}_n(t, \mu_{\mathbf{x}}) := u_n(t, \mathbf{x}), \quad \text{where} \quad \mu_{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.$$

(u_n are invariant with respect to permutations of variables in $\mathbf{x}$.) We have

$$|\mathcal{V}_n(t, \mu_{\mathbf{x}}) - \mathcal{V}_n(s, \mu_{\mathbf{y}})| \leq \rho(|t-s| + d_r(\mu_{\mathbf{x}}, \mu_{\mathbf{y}})) \text{ for all } t, s \in [0, T], \mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^n$$

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$$M_R = \{\mu \in \mathcal{P}_r(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) \le R\}$$

are relatively compact in $\mathcal{P}_r(\mathbb{R}^d)$, up to a subsequence, $\mathcal{V}_n$ converges uniformly on every set $[0,T] \times M_R$ to a function $\mathcal{V} : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ which satisfies the same continuity estimate.

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$$V : [0,T] \times E \longrightarrow \mathbb{R}, \quad V(t,X) := \mathcal{V}(t, \mathrm{law}(X)).$$

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$$V:[0,T]\times E\longrightarrow\mathbb{R},\quad V(t,X):=\mathcal{V}(t,\mathrm{law}(X)).$$

If we can show that V is a viscosity solution of (3), since equation (3) has a unique bounded viscosity solution U, we can then conclude that $V=U$ and then the whole sequence $\mathcal{V}_n$ converges to $\mathcal{V}$.

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Then $u_n - \varphi_n$ must have local maxima at points $(t_n, \mathbf{x}^n) = (t_n, x_1^n, \cdots, x_n^n)$ such that

$$t_n \to t_0 \quad \text{and} \quad X^n = \sum_{i=1}^n x_i^n \mathbf{1}_{A_i^n} \to X_0.$$

Idea of the proof

Compute

$$D_{x_i}\varphi_n(t_n,\mathbf{x}^n)=\int_{A_i^n} D\varphi(t_n,\sum_{i=1}^n x_i^n \mathbf{1}_{A_i^n})$$

$$\frac{\partial^2\varphi_n}{\partial x_{ik}\partial x_{jk}}(t_n,\mathbf{x}^n)=\int_{A_i^n}\left(D^2\varphi(t_n,\sum_{i=1}^n x_i^n \mathbf{1}_{A_i^n})\mathbf{1}_{A_j^n}e_k\right)\cdot e_k.$$

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$$\mathrm{Tr}(A_n D^2\varphi_n) = \sum_{k=1}^d \langle D^2\varphi(t_n,\sum_{i=1}^n x_i^n \mathbf{1}_{A_i^n})e_k, e_k\rangle.$$

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Plug into the equations and pass to the limit (technical).

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$$\begin{cases} dX_i(s) = \frac{1}{n-1}\sum_{j\neq i} G(X_i(s) - X_j(s))ds + \sqrt{2\kappa}dW(s) \quad t \leq s \leq T, \\ X_i(t) = x_i. \end{cases}$$

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$$\begin{cases} dX_i(s) = \frac{1}{n-1} \sum_{j \neq i} G(X_i(s) - X_j(s))ds + \sqrt{2\kappa} dW(s) \quad t \le s \le T, \\ X_i(t) = x_i. \end{cases}$$

Define

$$u_n(t, \mathbf{x}) = \mathbb{E}\Big[- \int_t^T \mathcal{F}\Big(\frac{1}{n} \sum_{i=1}^n \delta_{X_i(s)}\Big) ds + \mathcal{U}_0\Big(\frac{1}{n} \sum_{i=1}^n \delta_{X_i(T)}\Big)\Big].$$

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Define

$$u_n(t, \mathbf{x}) = \mathbb{E}\Big[-\int_t^T \mathcal{F}\Big(\frac{1}{n}\sum_{i=1}^n \delta_{X_i(s)}\Big)ds + \mathcal{U}_0\Big(\frac{1}{n}\sum_{i=1}^n \delta_{X_i(T)}\Big)\Big].$$

Then the function u_n is the viscosity solution of the terminal value problem

$$\begin{cases} -\partial_t u_n - \kappa \mathrm{Tr}(A_n D^2 u_n) - \frac{1}{n-1}\sum_{i=1}^n \sum_{j\neq i} G(x_i - x_j)\cdot D_{x_i} u_n \\ \qquad\qquad + \mathcal{F}(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}) = 0 \qquad \text{in } (0,T)\times(\mathbb{R}^d)^n, \\ u_n(T, x_1, \cdots, x_n) = \mathcal{U}_0(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}) \qquad \text{on } (\mathbb{R}^d)^n, \end{cases}$$

where A_n is as in (2).

Example

In this example the Hamiltonian H is defined by

$$H(x,\nu,p) = -p \cdot \int_{\mathbb{R}^d} G(x-y)\nu(dy)$$

and it satisfies all assumptions of the convergence theorem.

Viscosity solutions on the Wasserstein space

Can we define viscosity solutions of (1) on the Wasserstein space without referring to the lifted equation?

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$\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Wasserstein second differential of $\mathcal{U}$ at μ:

$$\nabla_w^2 \mathcal{U}(\mu)(\xi) = A_1(\mu)\xi + \int_{\mathbb{R}^d} A_2(\mu)(q,\cdot)\xi(q)\mu(dq) \qquad \forall \xi \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d).$$

$A_2(\mu) \in L^\infty_{\mu\otimes\mu}(\mathbb{R}^{2d}, \mathbb{R}^{d\times d})$, $A_2(\mu)^T(q,x) = A_2(\mu)(x,q)$.

$A_1(\mu) \in L^\infty_\mu(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is symmetric and coincides almost everywhere with $\nabla_q\big(\nabla_\mu \mathcal{U}(\mu)\big)$.

Viscosity solutions on the Wasserstein space

We propose definitions of parabolic second order superjet $\mathcal{P}^{2,+}\mathcal{U}(t,\mu)$ and subjet $\mathcal{P}^{2,+}\mathcal{U}(t,\mu)$ of $\mathcal{U}$ at (t,μ).

Ideologically the definition is based on "restricting elements of jets" to derivatives of functions of the form

$$\mathcal{V}^{\mu}_{(\phi,\psi)}(\nu) := \int_{\mathbb{R}^d} \phi(q)(\nu-\mu)(dq) + \frac{1}{2}\int_{\mathbb{R}^{2d}} \psi(q_1,q_2)(\nu-\mu)(dq_1)(\nu-\mu)(dq_2)$$

where $\psi \in C_b^3(\mathbb{R}^{2d})$ which is symmetric and $\phi \in C_b^3(\mathbb{R}^d)$.

Viscosity solutions on the Wasserstein space

Definition

Suppose $\kappa \neq 0$. An upper semicontinuous function $\mathcal{U} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is an intrinsic-viscosity subsolution of (1) on the Wasserstein space if $\mathcal{U}(0,\cdot) \leq \mathcal{U}_0$ on $\mathcal{P}_2(\mathbb{R}^d)$ and

$$a - \kappa\Big(\int_{\mathbb{R}^d} \mathrm{Tr}(A_1(q))\mu(dq) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}(A_2(q_1,q_2))\mu(dq_1)\mu(dq_2) \Big) + \mathcal{H}(\mu,\mu,\xi_0) + \mathcal{F}(\mu) \leq 0$$

for all $(t,\mu) \in (0,T) \times \mathcal{P}_2(\mathbb{R}^d)$ and $(a,\xi_0,A_1,A_2) \in \mathcal{P}^{2,+}\mathcal{U}(t,\mu)$.

Intrinsic-viscosity supersolution is defined similarly.

Viscosity solutions on the Wasserstein space

Theorem

Let $\mathcal{U} : [0, T) \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. If $\mathcal{U}$ is an L-viscosity subsolution/supersolution/solution of (1) *on the Wasserstein space then it is an intrinsic-viscosity subsolution/supersolution/solution of* (1).

When $\kappa = 0$ a theorem like this was proved by Gangbo-Tudorascu (2019) with the definition of solution used there.

First order convex HJB equations

First order convex HJB equations:

Assume now that $\kappa = 0$, $H = H(x,p)$ (i.e. H does not depend on μ) and is convex in the gradient variable p, and

$$H(x,p) \geq C_1 + C_2|p|^2 \quad \text{for all } x,p$$

for some constants C_1, C_2, where $C_2 > 0$.

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Question: Can we characterize the solution $\mathcal{U}$ of (1)?

Define L to be the Legendre transform of H, that is

$$L(x,v) := \sup_{p \in \mathbb{R}^d} (p \cdot v - H(x,p)), \quad x, v \in \mathbb{R}^d.$$

First order convex HJB equations

Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\xi \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$, we define

$$\mathcal{L}(\mu,\xi) := \int_{\mathbb{R}^d} L(x,\xi(x))\mu(dx) - \mathcal{F}(\mu).$$

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Candidate for solution: Value function

$$\bar{\mathcal{U}}(t,\mu) := \inf_{(\sigma,v)} \Big\{ \int_0^t \mathcal{L}(\sigma_\tau, v_\tau)d\tau + \mathcal{U}_0(\sigma_0) \;\Big|\; \sigma_t = \mu \Big\},$$

with the infimum taken over all pairs (σ, v), where $\sigma = \sigma_\tau \in AC^2(0,t;\mathcal{P}_2(\mathbb{R}^d))$, $v = v_\tau$ is a velocity vector field for σ_τ and $\sigma_t = \mu$.

First order convex HJB equations

In this case we have

$$u_n(t,\mathbf{x}) = \inf_{\xi}\Bigg\{\int_0^t \Bigg(-\mathcal{F}\Bigg(\frac{1}{n}\sum_{j=1}^n \delta_{\xi_j(\tau)}\Bigg) + \frac{1}{n}\sum_{j=1}^n L(\xi_j(\tau),\dot{\xi}(\tau))\Bigg)d\tau$$

$$+\,\mathcal{U}_0\Big(\frac{1}{n}\sum_{j=1}^n \delta_{\xi(0)_j}\Big) \;\Big|\; \xi(\cdot)\in\mathcal{C}_n(t,\mathbf{x})\Bigg\},$$

$$\mathcal{C}_n(t,\mathbf{x}) := \{\xi(\cdot)\in AC^2(0,t;(\mathbb{R}^d)^n) \;\big|\; \xi(t)=\mathbf{x}\}.$$

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$$\lim_{n\to\infty} u_n(t, \mathbf{x}(n)) = \bar{\mathcal{U}}(t, \mu).$$

First order convex HJB equations

One proves:

- The function $\bar{\mathcal{U}}$ is continuous.
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Theorem

The value function $\bar{\mathcal{U}}$ is the unique L-viscosity solution of (1).

THANK YOU!