On the approximation of first order mean field games

Francisco J. Silva (XLIM, Université de Limoges)

Workshop III: Mean Field Games and Applications
IPAM

May 5, 2020

Plan

Introduction

On the approximation of the HJB equation

Approximation of the MFG system

Convergence of the approximation

Solving the finite MFG problem

Extensions

Plan

Motivation

The main focus of this talk is illustrate the approximation of MFGs of the form

$$\left.\begin{array}{rcl} -\partial_t v + \frac{1}{2}\left|\nabla v\right|^2 & = & f(x, m(t)) \quad \text{in } [0,T]\times\mathbb{R}^d \\ v(T,\cdot) & = & g(\cdot, m(T)) \quad \text{in } \mathbb{R}^d \\ \partial_t m - \mathrm{div}\big(\nabla v m\big) & = & 0 \quad \text{in } [0,T]\times\mathbb{R}^d \\ m(0,\cdot) & = & m_0 \quad \text{in } \mathbb{R}^d. \end{array}\right\} \qquad (MFG)$$

- A semi-Lagrangian scheme to solve (MFG) has been proposed in Carlini-S. '14. Full-convergence result when $d = 1$.
- Fourier methods to treat (MFG) have been proposed recently in Nuberkyan-Saúde'19 and Li-Jacobs-Li-Nuberkyan-Osher '20.

- The discretization in Carlini-S. '14 is mainly based on the representation formulae

$$v(t,x) = \inf_{\alpha \in L^2\left([t,T];\mathbb{R}^d\right)} \int_t^T \left[\tfrac{1}{2}|\alpha(s)|^2 + f(\gamma(s), m(s))\right] \mathrm{d}s + g(\gamma(T), m(T))$$

$$\text{s.t.} \qquad \dot{\gamma}(s) = \alpha(s) \quad \text{in } (t,T), \quad \gamma(t) = x$$

$$= \inf_{\gamma \in H^1_x\left([t,T];\mathbb{R}^d\right)} \int_t^T \left[\tfrac{1}{2}|\dot{\gamma}(s)|^2 + f(\gamma(s), m(s))\right] \mathrm{d}s + g(\gamma(T), m(T)),$$

for the value function v, and

$$m(t) = \Phi(\cdot, t)\sharp m_0,$$

where, for all $x \in \mathbb{R}^d$, $\Phi(x,t)$ is the solution to

$$\dot{\gamma}(s) = -\nabla v(s, \gamma(s)) \quad \text{in } (0,T), \quad \gamma(0) = x,$$

at time t.

Plan

On the approximation of the HJB equation

Consider the HJB equation

$$\begin{aligned} -\partial_t v + \tfrac{1}{2}\left|\nabla v\right|^2 &= f \quad \text{in } [0,T]\times\mathbb{R}^d \\ v(T,\cdot) &= g \quad \text{in } \mathbb{R}^d \end{aligned}$$

- Given $\Delta t > 0$ consider the time grid $\mathcal{T}_{\Delta t} := \{0, \Delta t, \ldots, T\}$ and set $t_k = k(\Delta t)$ $(k = 0, \ldots, n)$. A standard semi-discrete scheme to approximate the HJB equation is

$$\begin{aligned} v_{\Delta t}(t_k, x) &= \inf_{\alpha\in\mathbb{R}^d} \left\{ \Delta t \left[\frac{|\alpha|^2}{2} + f(x) \right] + v_{\Delta t}(t_{k+1}, x + \Delta t \alpha) \right\}, \\ v_{\Delta t}(T, x) &= g(x), \end{aligned}$$

which follows by discretizing the DP of the continuous-time optimal control problem.

- Given a space-step $\Delta x > 0$, with associated grid $\mathcal{G}_{\Delta x} = \{x_i = i(\Delta x) \mid i \in \mathbb{Z}^d\}$, the fully-discrete SL scheme is

$$\begin{aligned} v_{\Delta t,\Delta x}(t_k, x_i) &= \inf_{\alpha \in \mathbb{R}^d} \left\{ \Delta t \left[\frac{|\alpha|^2}{2} + f(x_i) \right] + I[v_{\Delta t,\Delta x}](t_{k+1}, x_i + \Delta t \alpha) \right\}, \\ v_{\Delta t,\Delta x}(T, x_i) &= g(x_i), \end{aligned}$$

  where $I[\cdot]$ is an interpolation operator associated to a triangulation with vertices in $\mathcal{G}_{\Delta x}$.

- Given the particular structure of the dynamics, we can avoid an infinite grid and, more importantly, interpolation by choosing controls such that

$$x_i + \Delta t \alpha \quad \text{is a grid point.}$$

  Namely, controls having the form

$$\alpha = \frac{x_j - x_i}{\Delta t}, \qquad x_j \in \mathcal{G}^f_{\Delta x} \ \text{(finite grid).}$$

- In this case, the scheme becomes

$$\begin{aligned} v_{\Delta t,\Delta x}(t_k,x_i) &= \inf_{x_j\in\mathcal{G}^f_{\Delta x}} \left\{ \Delta t\left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2 + f(x_i)\right] + v_{\Delta t,\Delta x}(t_{k+1},x_j)\right\}, \\ v_{\Delta t,\Delta x}(T,x_i) &= g(x_i), \end{aligned}$$

or, equivalently,

$$\begin{aligned} v_{\Delta t,\Delta x}(t_k,x_i) &= \inf_{p\in\mathcal{P}\left(\mathcal{G}^f_{\Delta x}\right)} \left\{ \sum_{x_j\in\mathcal{G}^f_{\Delta x}} p_j\left[\Delta t\left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2 + f(x_i)\right] + v_{\Delta t,\Delta x}(t_{k+1},x_j)\right]\right\}, \\ v_{\Delta t,\Delta x}(T,x_i) &= g(x_i), \end{aligned}$$

- For each $(t_k,\, x_i)$ the problem defined by $v_{\Delta t,\Delta x}(t_k,x_i)$ can have several solutions. In order to get uniqueness, for $\varepsilon>0$, consider the entropy penalized scheme

$$
\begin{aligned}
v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i) &= \inf_{p\in\mathcal{P}\left(\mathcal{G}^f_{\Delta x}\right)} \left\{ \sum_{x_j\in\mathcal{G}^f_{\Delta x}} p_j \left[\Delta t \left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2 + f(x_i) \right] \right.\right. \\
&\qquad\qquad \left.\left. + v^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j) + \varepsilon \log p_j \right] \right\}, \\
v^{\varepsilon}_{\Delta t,\Delta x}(T,x_i) &= g(x_i).
\end{aligned}
$$

- For every t_k, x_i and $\varepsilon > 0$, there exists a unique minimizer $p^{\varepsilon}_{t_k,x_i}$ of the above problem.
- If $(\Delta t_n, \Delta x_n, \varepsilon_n) \to 0$, $\Delta x_n/\Delta t_n \to 0$, and $\varepsilon_n|\log(\Delta x_n)|/\Delta t_n \to 0$, then for every compact set $K \subseteq \mathbb{R}^d$ we have

$$
\sup_{(t,x)\in\mathcal{T}_n\times(\mathcal{G}^f_{\Delta x_n}\cap K)} \left| v^{\varepsilon_n}_{\Delta t_n,\Delta x_n}(t,x) - v(t,x) \right| \underset{n\to\infty}{\to} 0.
$$

- The previous scheme and the convergence result can be extended to several interesting contexts.

Plan

Approximation of the MFG system

Given the approximation of the HJB equation described before, it is natural to consider the following approximation (MFG_f) of (MFG)

$$\left.\begin{aligned}
&v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i) = \inf_{p\in\mathcal{P}(\mathcal{G}^f_{\Delta x})} \sum_{x_j\in\mathcal{G}^f_{\Delta x}} p_j\left[c_{i,j}(p_j, m^{\varepsilon}_{\Delta t,\Delta x}(t_k,\cdot)) + v^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j)\right]\\
&v^{\varepsilon}_{\Delta t,\Delta x}(T,x_i) = g\left(x_i, m^{\varepsilon}_{\Delta t,\Delta x}(T,\cdot)\right)\\
&m^{\varepsilon}_{\Delta t,\Delta x}(t_{k+1},x_j) = \sum_{x_i\in\mathcal{G}^f_{\Delta x}} p_{\mathrm{opt}}(x_i,x_j,t_k) m^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i)\\
&m^{\varepsilon}_{\Delta t,\Delta x}(0,x_i) = M_0(x_i) \qquad \forall\, x_i\in\mathcal{G}^f_{\Delta x},
\end{aligned}\right\}$$

where M_0 is a discretization of the initial distribution,

$$c_{i,j}(p_j,m) = \Delta t\left[\frac{1}{2}\left|\frac{x_j-x_i}{\Delta t}\right|^2 + f(x_i,m)\right] + \varepsilon\log(p_j),$$

$p_{\mathrm{opt}}(x_i,\cdot,t_k)\in\mathcal{P}(\mathcal{G}^f_{\Delta x})$ is the minimizer of $v^{\varepsilon}_{\Delta t,\Delta x}(t_k,x_i)$.

- System (MFG_f) is a particular instance of discrete time, finite state space MFGs introduced in Gomes-Mohr-Souza'10.

- Existence of a solution $(v^{\varepsilon}_{\Delta t,\Delta x}, m^{\varepsilon}_{\Delta t,\Delta x})$ follows from a fixed-point argument (see Gomes-Mohr-Souza'10).

- If f and g are monotone, i.e. for $h = f$, g,

$$\sum_{x_j \in \mathcal{G}^f_{\Delta x}} \left[h(x_j, m_1) - h(x_j, m_2)\right](m_1 - m_2) \geq 0 \quad \forall\, m_1, m_2 \in \mathcal{P}(\mathcal{G}^f_{\Delta x}),$$

  it is possible to show that the solution is unique.

- Under this monotonicity assumption we will show later how to compute the equilibrium $(v^{\varepsilon}_{\Delta t,\Delta x}, m^{\varepsilon}_{\Delta t,\Delta x})$ of (MFG_f).

Plan

Convergence of the approximation

- Consider a sequence $(\Delta t_n, \Delta x_n, \varepsilon_n) \to 0$. Our aim now is to study the asymptotic behaviour of $(v_n, m_n) := (v^{\varepsilon_n}_{\Delta t_n, \Delta x_n}, m^{\varepsilon_n}_{\Delta t_n, \Delta x_n})$ as $n \to \infty$.
- Set $\Gamma = C([0,T]; \mathbb{R}^d)$, set

$$\mathcal{P}_{m_0}(\Gamma) = \{\eta \in \mathcal{P}(\Gamma) \mid e_0 \sharp \eta = m_0\}, \text{ where } e_t(\gamma) = \gamma(t) \quad \forall\, \gamma \in \Gamma,\ t \in [0,T].$$

Let us comment on the relation between MFG equilibria and $\mathcal{P}(\Gamma)$.

- Continuous case: For $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ set

$$J_m(\gamma) := \int_0^T \left[\frac{1}{2} |\dot{\gamma}(t)|^2 + f(\gamma(t), m(t)) \right] \mathrm{d}t + g(\gamma(T), m(T)).$$

Then

(v, m) solves $(MFG) \Leftrightarrow\ \exists \xi^* \in \mathcal{P}_{m_0}(\Gamma)$, for ξ^*-a.e. γ we have

$$\gamma \in \text{argmin}\, \{J_{m^*}(\gamma') \mid \gamma'(0) = \gamma(0)\}$$

- Discrete case: Set

$$\mathcal{G}_n := \mathcal{G}^f_{\Delta x_n},$$
$$\mathcal{K}^n := \{P : \mathcal{G}_n \times \mathcal{G}_n \times \mathcal{T}_n \to [0,1] \mid P(x_i, \cdot, t_k) \in \mathcal{P}(\mathcal{G}_n) \;\; \forall\, t_k, x_i\}\,.$$

Clearly, any $P \in \mathcal{K}^n$ induces a probability measure $\xi_P \in \mathcal{P}(\Gamma)$.

Given $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ define $J^n_m : \mathcal{K}^n \to \mathbb{R}$ by

$$J^n_m(P) := \mathbb{E}_{\mathbb{P}^{M_0}_P}\left(\sum_{k=0}^{N_n - 1} c_{\gamma(t_k), \gamma(t_{k+1})}(P(\gamma(t_k), \cdot, t_k), m(t_k)) + g(\gamma(T), m(T)) \right)$$

Then,

(v^n, m^n) solves $(MFG_f) \Leftrightarrow\; \exists\, P^{n,*} \in \mathcal{K}^n$, such that

$$P^{n,*} \in \text{argmin}\,\{J^n_{m^{n,*}}(\gamma') \;|P \in \mathcal{K}^n\}$$

where $m^{n,*}(t) = e_t \sharp \xi_{P^{n,*}} \;\; \forall\, t \in [0,T]$.

- Given a solution (v^n, m^n) of (MFG_f), let us denote by $\xi^{n,*} \in \mathcal{P}(\Gamma)$ the probability on Γ induced by $P^{n,*}$.

- Our aim is to show that, up to some subsequence, $\xi^{n,*} \to \xi^*$, where ξ^* is a MFG equilibrium.

- Assume that $\varepsilon_n |\log(\Delta x_n)|/\Delta t_n$ is bounded. From the form of the cost functional, there exists a constant $C > 0$, independent of n, such that
$$\mathbb{E}_{\xi^{n,*}}\left(\int_0^T |\dot{\gamma}(t)|^2 \mathrm{d}t\right) \leq C.$$

- Since, for every $C > 0$, the set
$$\left\{\gamma \in H^1([0,T];\mathbb{R}^d) \mid |\gamma(0)| \leq C, \ \mid \int_0^T |\dot{\gamma}(t)|^2 \mathrm{d}t \leq C\right\}$$
is compact in Γ, the sequence $(\xi^{n,*})_{n\in\mathbb{N}}$ is tight, and, hence, there exists ξ^* such that, up to some subsequence, $\xi^{n,*} \to \xi^*$ in $\mathcal{P}(\Gamma)$.

- $\forall t \in [0,T]$, set $m^n(t) := e_t \sharp \xi^{n,*}$. We have $m^n(t_k) = m^{\varepsilon_n}_{\Delta t_n, \Delta x_n}(t_k, \cdot)$ and, hence, the notation is consistent.
- We have $m^n \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$. Moreover, there exists a constant $C > 0$ (independent of n) such that

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} |x|^2 \mathrm{d}m^n(t)(x) \leq C, \quad d_1\left(m^n(t), m^n(s)\right) \leq C|t-s|^{\frac{1}{2}}.$$

- Arzelá-Ascoli theorem yields the existence of cluster points in $C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$.
- Altogether, up to some subsequence, we have

$$\xi^{n,*} \to \xi^* \quad \text{in } \mathcal{P}(\Gamma),$$

$$m^n \to m := e_{(\cdot)} \sharp \xi^* \quad \text{in } C([0,T]; \mathcal{P}_1(\mathbb{R}^d)),$$

$$\sup_{(t,x) \in \mathcal{T}_n \times (\mathcal{G}^f_{\Delta x_n} \cap K)} |v^n(t,x) - v_m(t,x)| \underset{n \to \infty}{\to} 0.$$

- Then we can prove the

Theorem: [Hadikhanloo-S'19] Assume that $\Delta x_n/\Delta t_n \to 0$, and that $\varepsilon_n|\log(\Delta x_n)|/\Delta t_n \to 0$.Then

(i) There exists at least one cluster point $\xi^* \in \mathcal{P}(\Gamma)$ such that, up to some subsequence, $\xi^n \to \xi^*$ in the narrow topology. Moreover, every such ξ^* is a MFG equilibrium.

(ii) Up to some subsequence, there exists a solution (v,m) of (MFG) such that

$$\sup_{(t,x)\in\mathcal{T}_n\times(\mathcal{G}^f_{\Delta x_n}\cap K)} |v^n(t,x)-v(t,x)| \underset{n\to\infty}{\to} 0,$$

$$\sup_{t\in[0,T]} d_1\left(m^n(t),m(t)\right) \to 0.$$

In particular, the previous result hold for the entire sequence if (MFG) has a unique solution.

Plan

Solving the finite MFG problem

We consider the following "fictitious play" procedure to solve (MFG^f)

- Consider an arbitrary initial sequence of time marginals $M^1 = (M_0^1, \dots, M_N^1)$ and let $\bar{M}^1 = M^1$.
- For $\ell \geq 1$ compute

$$\begin{aligned} V_k^\ell &= \mathrm{HJB}(V_{k+1}^\ell, \bar{M}_k^\ell), \quad V_N^\ell = g(\bar{M}_N^\ell) \\ \text{and then} \qquad M_{k+1}^{\ell+1} &= \mathrm{FP}(M_k^\ell, V_{k+1}^\ell), \quad M_0^{\ell+1} = m_0. \end{aligned}$$

  Set

$$\bar{M}^{\ell+1} := \frac{1}{\ell+1} \sum_{\ell'=1}^{\ell} M^{\ell'}.$$

- In terms of the best response (BR), the method can be written as

$$M^{\ell+1} = BR(\bar{M}^\ell).$$

Theorem: [Hadikhanloo-S'19] If f and g are monotone and Lipschitz w.r.t. to the second argument, then $(V^\ell, M^\ell, \bar{M}^\ell) \to (v^n, m^n, m^n)$.

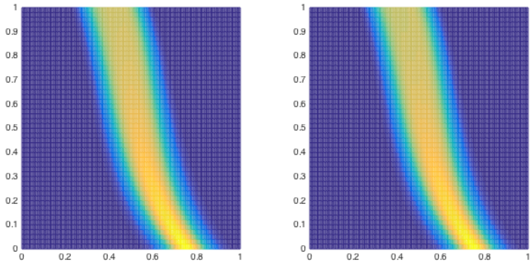
Example:

- Set $d=1$, $T=1$, $\rho_\sigma(z) = e^{-z^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, with $\sigma = 0.25$, and

$$\begin{aligned} f(x,m) &= 2(x-0.5)^2 + (\rho_\sigma * m) * \rho_\sigma(x) \\ g(x,m) &= 2(x-0.2)^2 + (\rho_\sigma * m) * \rho_\sigma(x) \\ m_0(x) &= \frac{h(x)}{\int_0^1 h(x')\mathrm{d}x'} \mathbb{I}_{[0,1]}(x), \quad \text{with } h(x) := e^{-\frac{(x-0.75)^2}{0.02}} \end{aligned}$$

- Discretization parameters: $\Delta x = 0.005$, $\Delta t = 0.02$ and $\varepsilon = 0.002$.

- We apply the fictitious play procedure to (MFG^f).



$\overline{M}^{\ell}$ (left) versus its best response $M^{\ell+1}$ (right), at step $\ell = 1000$.

- We have also tested the intuitive procedure $M^{\ell+1} = BR(M^{\ell})$. Convergence fails in general. Indeed, there are configurations M such that $M = BR(BR(M))$ and $M \neq BR(M)$.

Plan

Extensions (ongoing work with J. Gianatti)

- The previous techniques can be adapted to dynamics having the form

$$\dot{\gamma}(t) = A\gamma(t) + B\alpha(t).$$

(Cannarsa-Mendico'19 and Achdou-Mannucci-Marchi-Tchou'19).

- We can also consider state constraints

$$\gamma(t) \in K \quad \forall\, t \in [0, T]$$

(Cannarsa-Capuani'18).

- Other boundary conditions?