Global well-posedness of master equations for deterministic displacement convex potential mean field games

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(based on a joint work with W. Gangbo)

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Content of the talk

→ Mean field games.

→ Formal derivation of master equations.

→ Potential mean filed games linked to optimal control problems in infinite dimensions.

→ Vectorial vs. scalar master equations.

→ Displacement convexity, regularity estimates and well-posedness of master equations.

On Mean Field Games

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$$\tilde{u}(t, x) := \inf_{Q; Q_t = x} \left\{ \int_0^t L(Q_s, \dot{Q}_s) + f(Q_s, \rho_s) \, \mathrm{d}s + u_0(Q_0, \rho_0) \right\}, \tag{1}$$

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- → Data: → $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ Lagrangian function; → $T_0 > 0$: time horizon;

→ $f, u_0 : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ running- and the initial costs of the agents;
→ $\mu \in \mathscr{P}_2(\mathbb{R}^d)$: distribution of the agents at time T_0.

- → Notations:
  → $\mathscr{P}_2(\mathbb{R}^d) := \{\mu$ Borel probability measure on $\mathbb{R}^d : \int_{\mathbb{R}^d} |x|^2\, \mathrm{d}\mu(x) < +\infty\}$;
  → $\mathcal{B}_r := \{\mu \in \mathscr{P}_2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2\, \mathrm{d}\mu(x) \le r^2\}$.

The MFG system

- $\rightarrow$ The value function formally solves a Hamilton-Jacobi-Bellman equation.
- $\rightarrow$ The density of the population is transported by the velocity field given by the optimal control $\alpha^* := D_pH(\cdot, D\tilde{u})$ in the above problem.
- $\rightarrow$ One arrives to the coupled system

$$\begin{cases} \partial_t \tilde{u} + H(x, D\tilde{u}) & = f(x, \rho) & \text{in } (0, T_0) \times \mathbb{R}^d \\ \partial_t \rho + \nabla \cdot (\rho D_p H(x, D\tilde{u})) & = 0 & \text{in } (0, T_0) \times \mathbb{R}^d \\ \tilde{u}(0, x) = u_0(x, \rho_0), & \rho(T_0, \cdot) = \mu & \text{in } \mathbb{R}^d. \end{cases} \tag{MFG}$$

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- $\rightarrow$ The Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined as $H(x,\cdot) = L^*(x,\cdot)$.
- $\rightarrow$ A solution $(\tilde{u}, \rho)$ of the above system characterizes equilibrium situations.

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Its derivation

- $\rightarrow$ Let $T > 0$ be given. Define $u : [0, T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ as

$$u(T_0, x, \mu) := \tilde{u}(T_0, x), \ \forall (T_0, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d),$$

where $(\tilde{u}, \rho)$ is the solution of (MFG) with time horizon T_0 and final agent distribution μ.

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- → We must have $u(t,x,\rho_t)=\tilde{u}(t,x),\ \forall (t,x)\in[0,T_0]\times\mathbb{R}^d$.
- → Question: which equation does u satisfy? Answer:

$$\partial_t(u(t,x,\rho_t))=\partial_t\tilde{u}(t,x)=f(x,\rho_t)-H(x,D\tilde{u})=f(x,\rho_t)-H(x,D_xu).$$

- → Need to identify $\partial_t(u(t,x,\rho_t))$! We use the theory of optimal transport.

OT toolbox

→ For $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ we define the 2-Wasserstein distance W_2 as

$$W_2^2(\mu,\nu) := \inf\left\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \,\mathrm{d}\gamma(x,y) :\ \gamma \in \Pi(\mu,\nu) \right\}$$
$$= \inf\left\{ \int_\Omega |X(\omega)-Y(\omega)|^2 \,\mathrm{d}\omega :\ X, Y \in \mathbb{H},\ X_\sharp \mathscr{L}^d \llcorner \Omega = \mu,\ Y_\sharp \mathscr{L}^d \llcorner \Omega = \nu \right\}.$$

Notations:

→ $\Pi(\mu,\nu) := \left\{ \gamma \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d),\ (\pi^x)_\sharp \gamma = \mu,\ (\pi^y)_\sharp \gamma = \nu \right\}$,
$\pi^x, \pi^y : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are the canonical projections.

→ $\Pi_o(\mu,\nu) \subseteq \Pi(\mu,\nu)$: set of optimal plans.

→ For $T : \mathcal{X} \to \mathcal{Y}$ Borel function $T_\sharp \rho_0 = \rho_1$ means that $\rho_1(A) = \rho_0(T^{-1}(A))$ for any $A \subseteq \mathcal{Y}$ Borel set.

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→ Revolutionary result from [Otto, CPDE, 2001] and [Ambrosio-Gigli-Savaré, Birkhäuser, Springer, 2005]): $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ has a differential geometric structure.

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→ $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) = \overline{\nabla C_c^\infty(\mathbb{R}^d)}^{L^2_\mu}$ and $\mathcal{T}\mathscr{P}_2(\mathbb{R}^d) = \bigcup_{\mu \in \mathscr{P}_2(\mathbb{R}^d)} \{\mu\} \times \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Wasserstein gradients

Definition (Ambrosio-Gigli-Savaré, 2005; Gangbo-Tudorascu, JMPE, 2019)

Let $\mathcal{U} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. We say that $\xi \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ belongs to the subdifferential of $\mathcal{U}$ at μ, and we denote $\xi \in \partial^- \mathcal{U}(\mu)$, if for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$

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Chain rule:

- → If $(\sigma_t)_{t\in(0,1)}$ is a geodesic curve (i.e. $\partial_t\sigma + \nabla\cdot(\mathbf{v}\sigma) = 0$ with $\int_0^1\int_{\mathbb{R}^d}|\mathbf{v}|^2\,\mathrm{d}\sigma_t\,\mathrm{d}t < +\infty$ and with $\|\mathbf{v}_t\|_{\sigma_t}$ minimal for a.e. t) along which $\mathcal{U}$ is differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(\sigma_t) = \int_{\mathbb{R}^d}\nabla_w\mathcal{U}(\sigma_t)(x)\cdot\mathbf{v}_t(x)\,\mathrm{d}\sigma_t(x),\ \mathscr{L}^1 - \text{a.e. } t \in (0,1).$$

Examples

→ Let $\mathcal{U} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be defined as

$$\mathcal{U}(\mu) = \int_{\mathbb{R}^d} \varphi_0(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

where for $i = 0, 1$, $\varphi_i \in C^1(\mathbb{R}^d)$, has at most quadratic growth at infinity, with a gradient which has at most linear growth at infinity. Then $\mathcal{U}$ is differentiable at any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Let φ be even. We have

$$\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x) + (D\varphi_1 * \mu)(x).$$

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→ In the previous example, if $\varphi_1 \equiv 0$, and if φ_0 is differentiable on a bounded open set $B \subset \mathbb{R}^d$ and elsewhere not, $\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x)$, provided $\mathrm{spt}(\mu) \subseteq B$. Clearly, this object makes sense only for $x \in B$. If $\mathrm{spt}(\mu) \setminus B \neq \emptyset$, $\mathcal{U}$ is not differentiable at μ.

Back to the deterministic master equation

→ Now we can formally derive

$$\partial_t(u(t,x,\rho_t)) = \partial_t u(t,x,\rho_t) + \int_{\mathbb{R}^d} \nabla_w u(t,x,\rho_t)(z) \cdot D_p H(z, D_x u(t,z,\rho_t))\,\mathrm{d}\rho_t(z)$$

→ The master equation reads as

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→ An important application of master equations: serve as a tool to show the convergence of Nash equilibria of N–player differential game to solutions of MFGs, as $N \to +\infty$. (cf. [Cardaliaguet-Delarue-Lasry-Lions, 2019]).

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Our objective:

→ Describe a class of data u_0, f, H for which one can find a classical solution $u : [0,T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to (Master) for arbitrary large $T > 0$ (independent of the data)!

Literature on the problem

Deterministic case:

→ [Gangbo-Swiech, JDE, 2015]: potential games (to be described in a moment); f, g smooth $H(x,p) = \frac{1}{2}|p|^2 \rightarrow$ short time existence (T depends on the data, and cannot be arbitrary large).

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→ [Mou-Zhang, arXiv, 2019]: monotonicity assumption on f, g; two notions of weak solutions; global well-posedness in that sense.

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Potential game setting

- → Suppose that $\exists \mathcal{F}, \mathcal{U}_0 : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ of class C^1 such that

$$D_x f(x,\mu) = \nabla_w \mathcal{F}(\mu)(x) \text{ and } D_x u_0(x,\mu) = \nabla_w \mathcal{U}_0(\mu)(x), \ \forall (x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d).$$

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$$\mathcal{H}(\mu,\xi) := \int_{\mathbb{R}^d} H(x,\xi(x))\, \mathrm{d}\mu(x) \text{ and } \mathcal{L}(\mu,\zeta) = \int_{\mathbb{R}^d} L(x,\zeta(x))\, \mathrm{d}\mu(x).$$

The optimal control problem

→ Solve the optimal control problem

$$\mathcal{U}(t,\mu) := \inf\left\{\mathcal{U}_0(\sigma_0) + \int_0^t \mathcal{L}(\sigma_s, v_s) + \mathcal{F}(\sigma_s)\,\mathrm{d}s : \partial_s\sigma + \nabla\cdot(v\sigma) = 0,\ \sigma_t = \mu\right\} \quad \text{(HL-}\mathscr{P}_2\text{)}$$

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→ Question: how do we get further regularity? Since we are aiming for u to be differentiable w.r.t. μ, it is necessary to have $\mathcal{U}$ twice differentiable w.r.t. μ.

Master equations from (HL-$\mathscr{P}_2$) – the scalar case

Candidate for the solution to (Master):

- → Suppose that $(\sigma_s)_{s\in[0,t]}$ with $\sigma_t = \mu$ is the unique minimizer in (HL-$\mathscr{P}_2$).
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- → Thus, if $x \in \mathrm{spt}(\mu)$, the strict convexity of $H(y, \cdot)$ yields

$$D_xu(s, \gamma_s, \sigma_s) = \nabla_w\mathcal{U}(s, \sigma_s)(\gamma_s).$$

Master equations from (HL-$\mathscr{P}_2$) – the scalar case

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- → Suppose that $(\sigma_s)_{s\in[0,t]}$ with $\sigma_t = \mu$ is the unique minimizer in (HL-$\mathscr{P}_2$).
- → Define $u : [0,T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$ as

$$u(t,x,\mu) := \inf\left\{u_0(\gamma_0,\sigma_0) + \int_0^t L(\gamma_s,\dot\gamma_s) + f(\gamma_s,\sigma_s)\,\mathrm{d}s :\ \gamma \in W^{1,2}([0,t];\mathbb{R}^d),\ \gamma_t = x\right\}.$$

- → Notice that $u(t,\cdot,\mu)$ is defined for all $x \in \mathbb{R}^d$ (not only for $x \in \mathrm{spt}(\mu)$).
- → The regularity of $u(\cdot,\cdot,\mu)$ can be studied by classical methods.
- → Suppose that $u(s,\cdot,\sigma_s)$ is differentiable. Then $\dot\gamma_s = D_pH(\gamma_s, D_xu(s,\gamma_s,\sigma_s))$.
- → Suppose that $\mathcal{U}(s,\cdot)$ is differentiable. Then $\partial_s\sigma_s + \nabla\cdot(\sigma_s D_pH(\cdot,\nabla_w\mathcal{U}(s,\sigma_s)(\cdot))) = 0$.
- → Thus, if $x \in \mathrm{spt}(\mu)$, the strict convexity of $H(y,\cdot)$ yields

$$D_xu(s,\gamma_s,\sigma_s) = \nabla_w\mathcal{U}(s,\sigma_s)(\gamma_s).$$

- → So, one formally has $D_xu(s,\cdot,\mu) = \nabla_w\mathcal{U}(s,\mu)(\cdot)$ on $\mathrm{spt}(\mu)$.

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→ Therefore, $D_xu(s,\cdot,\mu)$ would produce a natural extension for $\nabla_w\mathcal{U}(s,\mu)(\cdot)$ to the whole $\mathbb{R}^d$!

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→ Take ∇_w of (HJB-$\mathscr{P}_2$) to get

$$\partial_s \nabla_w \mathcal{U}(s,\mu)(x) + D_x H(x, \nabla_w \mathcal{U}(s,\mu)(x)) + D_x \nabla_w \mathcal{U}(s,\mu)(x) D_p H(x, \nabla_w \mathcal{U}(s,\mu)(x)) \\ + \int_{\mathbb{R}^d} D^2_{ww} \mathcal{U}(s,\mu)(x,y) D_p H(y, \nabla_w \mathcal{U}(s,\mu)(y))\, \mathrm{d}\mu(y) = \nabla_w \mathcal{F}(\mu)(x).$$

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GOAL: obtain the necessary regularity on both $\mathcal{U}$ and u which let us justify the previous heuristic arguments!

Lift of HJB from $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ to $\mathbb{H}$

→ Recall, $\Omega = [0,1)^d$ and $\mathbb{H} = L^2(\Omega; \mathbb{R}^d)$. We define $\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0 : \mathbb{H} \to \mathbb{R}$ and $\tilde{\mathcal{H}}, \tilde{\mathcal{L}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ as

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→ In fact, we have $\mathscr{P}_2(\mathbb{R}^d) = \mathbb{H}/\sim$, where $X \sim Y$, if $X_\sharp \mathscr{L}^d \llcorner \Omega = Y_\sharp \mathscr{L}^d \llcorner \Omega$.

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$$\tilde{\mathcal{U}}(t,X) := \inf\left\{\tilde{\mathcal{U}}_0(X_0) + \int_0^t \tilde{\mathcal{L}}(X_s,\dot{X}_s) + \tilde{\mathcal{F}}(X_s)\,\mathrm{d}s : X_t = X\right\} \qquad \text{(HL-}\mathbb{H}\text{)}$$

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→ Remark: since the data $\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0, \tilde{\mathcal{L}}$ are rearrangement invariants, so is $\tilde{U}(t,\cdot)$ (this means $\tilde{\mathcal{F}}(X) = \tilde{\mathcal{F}}(Y)$, whenever $X \sim Y$).

Important links between the control problems and HJB equations

→ Under reasonable assumptions ($\tilde{\mathcal{L}}$ is convex in the second variable, regular enough, bounded from below, $\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0$ are bounded below and regular), the control problem (HL-$\mathbb{H}$) has a solution (at least for short time).

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- → A similar observation was made by Lions in his lectures.

Further regularity of $\tilde{\mathcal{U}}(t,\cdot)$ for arbitrary time horizon

→ Innocent observation: if in addition $\tilde{\mathcal{U}}_0$ and $\tilde{\mathcal{L}} + \tilde{\mathcal{F}}$ are convex, then so is $\tilde{\mathcal{U}}(t,\cdot)$ for all $t \in [0,T]$.

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→ $C^{1,1}(\mathbb{H})$: $\tilde{V}$ is Fréchet differentiable and there exists $\lambda > 0$ such that for all $X, Y \in \mathbb{H}$

$$\left| \tilde{V}(Y) - \tilde{V}(X) - \int_\Omega \nabla\tilde{V}(X)(\omega)\cdot(Y(\omega)-X(\omega))\,\mathrm{d}\omega \right| \le \frac{\lambda}{2}\|X-Y\|^2$$

or equivalently

$$\|\nabla\tilde{V}(X) - \nabla\tilde{V}(Y)\| \le \lambda\|X-Y\|.$$

→ And so, in this setting one can obtain $\tilde{\mathcal{U}}(t,\cdot) \in C^{1,1}_{\mathrm{loc}}(\mathbb{H})$.

Further regularity of $\mathcal{U}(t,\cdot)$

Theorem (Gangbo-M., 2020)

Let $V : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *and let* $\tilde{V} : \mathbb{H} \to \mathbb{R}$ *be its lift. Then* $\tilde{V} \in C^{1,1}_{\text{loc}}(\mathbb{H})$ *if and only if* $V \in C^{1,1}_{\text{loc}}(\mathscr{P}_2(\mathbb{R}^d))$.

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Here, we say that $V \in C^{1,1}(\mathscr{P}_2(\mathbb{R}^d))$, if V is differentiable and there exists $\lambda > 0$ such that

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$$\left| V(\nu) - V(\mu) - \int_{\mathbb{R}^{2d}} \nabla_w V(\mu)(x) \cdot (y-x)\, \mathrm{d}\gamma(x,y) \right| \le \frac{\lambda}{2} W_2^2(\mu,\nu),\ \forall \gamma \in \Pi_o(\mu,\nu)$$

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$\to$ So, in this convex setting, one can obtain that $\mathcal{U}(t,\cdot) \in C^{1,1}_{\text{loc}}(\mathscr{P}_2(\mathbb{R}^d))$.

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Here we say that V is displacement (or geodesically) semi-convex (see [McCann, 1997], [Ambrosio-Gigli-Savaré, 2005]) if there exists $\lambda \in \mathbb{R}$ such that for any $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $[0, 1] \ni s \mapsto \mu_s := ((1 - s)x + sy)_\sharp \gamma$, $(\gamma \in \Pi_o(\mu, \nu))$

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→ Typical examples of coupling functions in MFG:

$$\mathcal{F}(\mu) = \int_{\mathbb{R}^d} \varphi_0(x) \, \mathrm{d}\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x-y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y),$$

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→ If φ_0 and φ_1 are λ–convex, then $\mathcal{F}$ is displacement semi-convex.

Displacement convexity vs monotonicity à la Lasry-Lions

→ Previously, in all works on the global well-posedness of master equation in the literature, it was assumed the monotonicity condition

$$\int_{\mathbb{R}^d} [f(x,\mu) - f(x,\nu)] \, \mathrm{d}(\mu - \nu)(x) \geq 0 \text{ and } \int_{\mathbb{R}^d} [u_0(x,\mu) - u_0(x,\nu)] \, \mathrm{d}(\mu - \nu)(x) \geq 0.$$

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Let $\mathcal{F}(\mu)=\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi_1(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$ such that φ_1 is even. Clearly, if we set $f(x,\mu):=\delta_\mu\mathcal{F}(\mu)(x)=(\varphi_1*\mu)(x)$, we have that

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Lemma

Let $\varphi_1\in L^1(\mathbb{R}^d)$. f is monotone in the sense of Lasry-Lions if and only if the Fourier transform of φ_1 is nonnegative.

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Lemma

Let $\varphi_1\in L^1(\mathbb{R}^d)$. f is monotone in the sense of Lasry-Lions if and only if the Fourier transform of φ_1 is nonnegative. As a consequence, there are φ_1 such that $\mathcal{F}$ is displacement convex, but f fails to be monotone.

Higher regularity of $\mathcal{U}(t,\cdot)$?

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- However, this would require to impose $\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0$ to be of class C^2 (in the Fréchet sense) or better.
- Surprisingly, such regularity assumption might be too restrictive.

Hilbert space regularity is too restrictive for the study of MFG

Let $\Phi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{\Phi} \in C^2\left(\mathbb{H}\right)$ be its lift.

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$$\nabla^2 \tilde{\Phi}(X)(h, h_*) = \int_\Omega D_x\big(\nabla_w \Phi(\mu)\big) \circ X\; h \cdot h_* d\omega \qquad (2)$$
$$+ \int_{\Omega^2} \nabla^2_{ww} \Phi(\mu)\big(X(\omega), X(\omega_*)\big) h(\omega) \cdot h_*(\omega_*) d\omega d\omega_*$$

if $\xi, \xi_* \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ and $h = \xi \circ X$ and $h_* = \xi_* \circ X$.

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Lemma (Gangbo-M., 2020)

Let $\alpha \in (0, 1]$ and assume $\tilde{\Phi} \in C^{2,\alpha}_{\mathrm{loc}}\big(\mathbb{H}\big)$ is rearrangement invariant so that it is the lift of a function Φ. If (2) holds for all $h, h_* \in \mathbb{H}$ then $D_x\big(\nabla_w \Phi(\mu)(\cdot)\big)$ is a constant function on $\mathrm{spt}(\mu)$.

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Corollary: if $\tilde{\Phi}^{(k)}_g(X) := \int_{\Omega^k} g\big(X(\omega_1), \cdots, X(\omega_k)\big) d\omega_1 \cdots d\omega_k\ \forall X \in \mathbb{H}$, then $\tilde{\Phi}^{(k)}_g \in C^{2,\alpha}_{\mathrm{loc}}\big(\mathbb{H}\big)$ if and only if $g^{(k)}$ is a polynomial of degree at most 2.

Further consequences

→ Actually, the previous corollary for locally representable functions $\Phi_g^{(k)}$ holds even for the class $C^2(\mathbb{H})$ (instead of $(C^{2,\alpha}_{\mathrm{loc}}(\mathbb{H}))$.

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Theorem (Gangbo-M., 2020)

Let f, u_0 and $\mathcal{F}, \mathcal{U}_0$ are such that $D_x f = \nabla_w \mathcal{F}$ and $D_x u_0 = \nabla_w \mathcal{U}_0$. Let moreover $\mathcal{F}, \mathcal{U}_0$ be of class $C^{2,1,w}_{\mathrm{loc}}$, $\mathcal{U}_0$ and $\mathcal{L} + \mathcal{F}$ displacement convex and let $L, H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be C^3. Then there exists a unique, global in time classical solution $\mathcal{U}$ to the equation (HJB-$\mathscr{P}_2$) which is such that $\mathcal{U}(t,\cdot) \in C^{2,1,w}_{\mathrm{loc}}$. Moreover, there exists a unique global in time classical solution $u \in C^{1,1}_{\mathrm{loc}}([0,+\infty) \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$ to (Master) and $D_x u(t,\cdot,\mu)(\cdot) = \nabla_w \mathcal{U}(t,\mu)(\cdot)$ on $\mathrm{spt}(\mu)$.

$C^{2,1,w}_{\text{loc}}$ functions on $(\mathscr{P}_2(\mathbb{R}^d), W_2)$

We propose the following (inspired by [Chow-Gangbo, JDE 2019])

Definition

Let $\mathcal{B} \subseteq \mathscr{P}_2(\mathbb{R}^d)$ be open and convex. We say that $\mathcal{U} \in C^{2,1,w}(\mathcal{B})$, if $\mathcal{U} \in C^1(\mathcal{B})$, and if there exist $\Lambda_0 : \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}^{d\times d}$ and $\Lambda_1 : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}^{d\times d}$ such that $\Lambda_0 \in L^\infty(\mathbb{R}^d; \mu)$, $\Lambda_1 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mu \otimes \mu)$ and there exists a constant $C > 0$ such that
(1)

$$\left|\nabla_w \mathcal{U}(\nu)(y) - \nabla_w \mathcal{U}(\mu)(x) - \Lambda_0(x,\mu)(y-x) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Lambda_1(x,a,\mu)(b-a)\,\mathrm{d}\gamma(a,b)\right| \leq C\left(|x-y|^2 + W_2^2(\mu,\nu)\right),$$

for all $\mu, \nu \in \mathcal{B}$, $\gamma \in \Pi_o(\mu,\nu)$ and $(x,y) \in \mathrm{spt}(\mu) \times \mathrm{spt}(\nu)$.
(2) Λ_0 and Λ_1 are Lipschitz continuous, i.e. there exists $C > 0$ such that

$$|\Lambda_0(x,\mu) - \Lambda_0(y,\mu)|_\infty \leq C(|x-y| + W_2(\mu,\nu))$$

and $|\Lambda_1(x_1,x_2,\mu) - \Lambda_1(y_1,y_2,\nu)|_\infty \leq C(|x_1-y_1| + |x_2-y_2| + W_2(\mu,\nu))$, for any $\mu, \nu \in \mathcal{B}$ and $(x,y), (x_1,y_1), (x_2,y_2) \in \mathrm{spt}(\mu) \times \mathrm{spt}(\nu)$.

Examples

$\rightarrow$ In the previous definition $C^{2,1,w}_{\mathrm{loc}}(\mathscr{P}_2(\mathbb{R}^d))$ means that it is satisfied on each $\mathcal{B}_r$!

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for $\varphi_0, \varphi_0 : \mathbb{R}^d \to \mathbb{R}$ of class $C^{2,1}_{\mathrm{loc}}(\mathbb{R}^d)$ such that both of them have at most quadratic growth at infinity and bounded second order derivatives. Let moreover φ_1 be even. Then, $\mathcal{U} \in C^{2,1,w}_{\mathrm{loc}}(\mathscr{P}_2(\mathbb{R}^d))$.

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$$\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x) + (D\varphi_1 * \mu)(x)$$

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$$D_x \nabla_w \mathcal{U}(\mu)(x) = D^2\varphi_0(x) + (D^2\varphi_1 * \mu)(x); \;\; D^2_{ww}\mathcal{U}(\mu)(x,y) = D^2\varphi_1(x-y).$$

Notice that in this case $\Lambda_0(x,\mu) = D_x \nabla_w \mathcal{U}(\mu)(x)$ and $\Lambda_1(x,y,\mu) = D^2_{ww}\mathcal{U}(\mu)(x,y)$.

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and

$$D^2_{x_i x_j} U^{(m)}(x) = \begin{cases} \frac{1}{m^2} D^2_{ww}\mathcal{U}(\mu_x^{(m)})(x_i, x_j), & i \neq j, \\ \frac{1}{m} D_x \nabla_w \mathcal{U}(\mu_x^{(m)})(x_i) + \frac{1}{m^2} D^2_{ww}\mathcal{U}(\mu_x^{(m)})(x_i, x_i), & i = j. \end{cases}$$

From (HJB-$\mathscr{P}_2$) to (HJ-$(\mathbb{R}^d)^m$)

Lemma

Let the data be as in our theorem and let $\mathcal{U} \in C^{1,1}_{\text{loc}}([0,T] \times \mathscr{P}_2(\mathbb{R}^d))$ *be a classical solution to* (HJB-$\mathscr{P}_2$)*. Let* $m \in \mathbb{N}$ *and define* $U^{(m)} : [0,T] \times (\mathbb{R}^d)^m \to \mathbb{R}$ *be defined as* $U^{(m)}(t,x) = \mathcal{U}(t, \mu_x^{(m)})$*. Then* $U^{(m)}$ *is of class* $C^{1,1}_{\text{loc}}$ *and the unique classical solution of*

$$\begin{cases} \partial_t U^{(m)}(t,x) + H^{(m)}(x, D_x U^{(m)}(t,x)) = F^{(m)}(x), & \text{in } (0,T) \times (\mathbb{R}^d)^m, \\ U^{(m)}(0,x) = U_0^{(m)}(x), & \text{in } (\mathbb{R}^d)^m, \end{cases} \tag{HJ-$(\mathbb{R}^d)^m$}$$

where $F^{(m)}(x) := \mathcal{F}(\mu_x^{(m)})$, $U_0^{(m)}(x) := \mathcal{U}_0(\mu_x^{(m)})$ *and*

$$H^{(m)}(x_1, \ldots, x_m, p_1, \ldots, p_m) := \frac{1}{m} \sum_{i=1}^{m} H(x_i, m p_i).$$

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Corollary

As a consequence of the classical theory, one has also that $U^{(m)} \in C^{2,1}_{\rm loc}([0,T] \times (\mathbb{R}^d)^m)$!

How to deduce the desired $C^{2,1,w}_{\mathrm{loc}}$ properties of $\mathcal{U}$?

We need fine quantitative estimates on derivatives of $U^{(m)}$!

Theorem

Under the assumptions of our main theorem, we have that the solution $U^{(m)}$ of (HJ-$(\mathbb{R}^d)^m$) satisfies the following. For all $t \in [0,T]$ and $r > 0$ there exists $C = C(t,r) > 0$ such that for all $x \in \mathbb{B}^m_r$ we have

$$|D^2_{x_i x_j} U^{(m)}(t,x)|_\infty \leq \begin{cases} \dfrac{C}{m}, & i = j; \\ \dfrac{C}{m^2}, & i \neq j; \end{cases} \tag{3}$$

and

$$|D^3_{x_i x_j x_k} U^{(m)}(t,x)|_\infty \leq \begin{cases} \dfrac{C}{m}, & i = j = k; \\ \dfrac{C}{m^2}, & i = j \neq k,\ i \neq j = k,\ i = k \neq j; \\ \dfrac{C}{m^3}, & i \neq j \neq k, \end{cases} \tag{4}$$

where $\mathbb{B}^m_r := \{x \in (\mathbb{R}^d)^m : \frac{1}{m}\sum_{i=1}^m |x_i|^2 \leq r^2\}$.

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→ We derive the regularity estimates on the associated finite dimensional Hamiltonian flow.

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Since $U^{(m)}$ is of class $C^{1,1}_{\text{loc}}$, we have

→ $P_i(s,x) = D_{x_i}U^{(m)}(s, Q_1(s,x),\dots,Q_m(s,x))$.

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→ Therefore, regularity estimates on derivatives of $P(s,\cdot)$ and $Q^{-1}(s,\cdot)$ will give the required estimates on $U^{(m)}(t,\cdot)$.

From (HJ-$(\mathbb{R}^d)^m$) to (HJB-$\mathscr{P}_2$) and to (Master)

Theorem

Suppose that the solution $U^{(m)}$ to (HJ-$(\mathbb{R}^d)^m$) satisfies the fine quantitative derivative estimates up to order 3. Then $\mathcal{U}$, the solution to (HJB-$\mathscr{P}_2$), is of such that
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- $\to$ This regularity is enough to deduce the existence of a global classical solution to the vectorial master equation.
- $\to$ The regularity of $u(t,x,\cdot)$ does not follow immediately from these arguments! Need to have the regularity of $\mu \mapsto \sigma_s$ $(s \in (0,t))$, where $\sigma_t = \mu$ and $\partial_s \sigma_s + \nabla \cdot (\sigma_s \nabla_w \mathcal{U}(s,\sigma_s)(\cdot)) = 0$.
- $\to$ One needs to perform a 'new' discretization argument to show this remaining regularity!
- $\to$ As a conclusion, we obtain $u \in C^{1,1}_{\mathrm{loc}}([0,T] \times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d))$ is a classical solution to (Master). The uniqueness follows from the (strict) displacement convexity of the data!

Thank you for your attention!