Convergence of some Mean Field Games to aggregation and flocking models

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Plan

1. Connecting MFGs to kinetic models ?
   - Mean Field Games and their system of PDEs
   - Agent-based models in population biology and crowd dynamics
   - Some previous results: Bertucci-Lasry-Lions and Degond-Herty-Liu
2. Convergence of MFGs to nonlocal continuity equations
   - a) MFG with controlled velocity and vanishing viscosity $\to$ aggregation equation
     - Outline of the proof: PDE approach.
   - b) deterministic MFG with controlled acceleration $\to$ kinetic Cucker-Smale model
     - Outline of the proof: variational approach.

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1. Mean Field Games PDEs

(MFE) $$\begin{cases} -\frac{\partial u}{\partial t} - \nu\Delta u + H(\nabla u) = F(x, m) & \text{in } (0,T)\times \mathbf{R}^d \\ \frac{\partial m}{\partial t} - \nu\Delta m - div(m\nabla H(\nabla u)) = 0 & \text{in } (0,T)\times \mathbf{R}^d \\ u(T,x) = g(x), \quad m(0,x) = m_o(x), \end{cases}$$

$m(x,t) =$ equilibrium distribution of the agents at time t;

$u(x,t)$ = value function of the representative agent

Data: $\nu \geq 0$, $H = L^*$, e.g., $H(p) = \frac{|p|^2}{2}$,

$F : \mathbf{R}^d \times \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}$ = running cost , g = terminal cost,

$m_o \geq 0$ = initial distribution of the agents, $\int_{\mathbf{R}^d} m_o(x)dx = 1$.

1st equation is backward H-J-B, 2nd equation is forward K-F-P eq.

Control interpretation of the MFE

- $$u(x,t) = \inf E[\int_t^T L(\alpha(s)) + F(y(s), m(s))ds + g(y(T))]$$
  over controls α and trajectories of
  $$dy(s) = \alpha(s)ds + \sqrt{2\nu}dW(s), \quad y(t) = x$$
- $$dy(s) = -\nabla H(\nabla u(y(s), s))ds + \sqrt{2\nu}dW(s)$$
  = optimal trajectory of the representative agent
- $m(x,t) =$ distribution of particles moving along optimal trajectories
- In particular, for $\nu = 0$ and $H(p) = |p|^2/2$ the dynamics with optimal feedback is $\dot{y}(s) = -\nabla u(y(s), s)$

  and the KFP equation becomes
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Agent-based models

They typically are nonlocal continuity equations of the form

$$\partial_t m - div(m\, Q[m]) = 0\,, \qquad Q : \mathcal{P}_p(\mathbf{R}^d) \to C^1(\mathbf{R}^d, \mathbf{R}^d)\,.$$

- The aggregation equation is a phenomenological model of dynamics of animal populations :

$$Q[m](x,t) = \nabla k * m(\cdot,t)(y) = \nabla \int_{\mathbf{R}^d} k(x-y)m(y,t)dy$$

The kernels

- $k(x) = -|x|e^{-a|x|}, \qquad a > 0,$
- $k(x) = e^{-|x|} - Fe^{-|x|/L}, \qquad 0 < F < 1, \quad L > 1$

describe repulsion at short distance , then attraction , decreasing at large distance (cfr. Bertozzi, Carrillo, Laurent and many others).

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- Models of crowd dynamics (Cristiani-Piccoli-Tosin)
  - $\partial_t m - div(m(v + Q[m])) = 0, \quad v = v(x)$ given,

    $Q[m] = \nabla \int_{\mathbf{R}^d} k(x-y)dm(y)$

    $k = \phi(|x|)$ with compact support, ϕ decreasing for small $|x|$, then increasing
  - models with "social forces", or mesoscopic, or kinetic: state variables: position and velocity $(x, v) \in \mathbf{R}^{2d}$

    $\partial_t m + v \cdot D_x m - div_v(mQ[m]) = 0 \qquad \text{in } (0,T) \times \mathbf{R}^{2d}$

    $Q[m](x,v) = \nabla_v \int_{\mathbf{R}^{2d}} k(x-y, v-v_*)m(y, v_*, t)dydv_*$

- Flocking and swarming models: as the last one with different k:

  Cucker-Smale : $k(x,v) = \frac{|v|^2}{(\alpha+|x|^2)^\beta}, \quad \alpha > 0, \beta \geq 0$

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Question: connection among MFGs and ABMs?

For MFG with dynamics $\dot{y} = \alpha$ the equation for the density m is

$$\partial_t m - div(m \nabla H(\nabla u)) = 0$$

which is a continuity equation with $Q[m] = \nabla H(\nabla u)$ and u depends on m in a non-local way via the HJB equation, so the dependence is not explicit.

For MFG with dynamics $\ddot{y} = \alpha$ the density m solves

$$\partial_t m + v \cdot D_x m - div_v(m \nabla_v H(\nabla u)) = 0$$

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Q.: can one connect in a rigorous way the classical ABMs to some MFGs ?

Stochastic case: Bertucci-Lasry-Lions 2018

$$
(MF0)\quad \begin{cases} -\partial_t u_\lambda + \lambda u_\lambda - \Delta u_\lambda + Q[m_\lambda]\cdot Du_\lambda + \dfrac{|Du_\lambda|^2}{2} = F(x), \\ \partial_t m_\lambda - \Delta m_\lambda - div(m_\lambda(Du_\lambda + Q[m_\lambda])) = 0 \quad \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m_\lambda(0) = m_0, \qquad \text{in } \mathbf{R}^d \end{cases}
$$

$\lambda =$ the discount factor in the cost functional associated to the HJB equation = "inter temporal preference parameter that measures the weight of anticipation for a given agent",

the dynamics of an agent in the MFG is the McKean-Vlasov equation

$$
dy(s) = (\alpha(s) - Q[m_\lambda])ds + \sqrt{2\nu}dW(s), \quad y(t) = x
$$

The limit $\lambda \to \infty$

Theorem (Bertucci-Lasry-Lions)

$Q : \mathcal{P}_1(\mathbf{R}^d) \to Lip(\mathbf{R}^d), \quad \|Q[m]\|_\infty \le C, \ \forall m \quad \Longrightarrow$

any solution (u_λ, m_λ) *of (MF0) is bounded uniformly in* λ *and*

for any $\lambda_n \to \infty$ *such that* $m_{\lambda_n} \to m$ *the limit m is a solution of the continuity equation*

$$\partial_t m - \Delta m - div(m\, Q[m]) = 0.$$

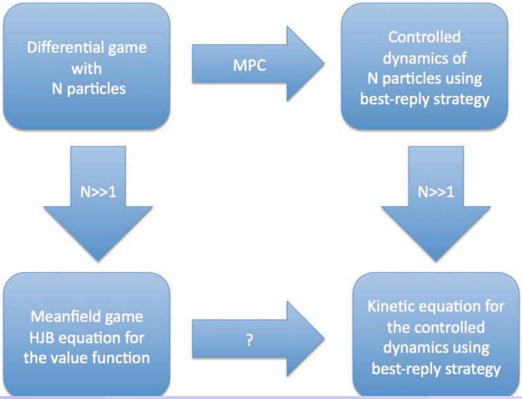
Idea: by estimates for uniformly parabolic equations $u_\lambda, \ Du_\lambda \to 0$.

So "any" ABM model with diffusion, driven by the vector field Q , has at least one solution that is the limit of the solution of a MFG.

The setting of Degond-Herty-Liu 2017

Finite horizon, deterministic problems. The arrow MPC (Model Predictive Control) is for short horizon and cheap control.

We address the horizontal ? $\Longrightarrow$? for infinite horizon problems.



The control problem for a single agent is

$$\dot{y} = v(y) + \alpha, \quad y(t) = x, \quad \inf_{\alpha(\cdot)} \int_t^T [\frac{|\alpha|^2}{2} + F(y(s), m(s))]ds$$

MPC approximation:

$$y(t+\Delta t) = x + \Delta t(v(x) + \alpha), \quad \min_{\alpha} \left[\Delta t \frac{|\alpha|^2}{2} + F(y(t+\Delta t), m(t)) \right]$$

Note that the scaling with Δt means that the control is cheap.
Taking the derivative w.r.t. α we get the optimal control $\bar{\alpha}$ if

$$\Delta t\, [\bar{\alpha} + DF(x, m(t))] = 0.$$

This suggests that, for short horizon T and cheap control, the optimal feedback should be approximated by the

steepest decent of the running cost $\bar{\alpha} \approx -DF(x, m(t))$.

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2 a). Convergence: the basic model

(MF1)

$$\begin{cases} -\partial_t u_\lambda - \nu_\lambda \Delta u_\lambda + \lambda u_\lambda - v(x)\cdot Du_\lambda + \frac{\lambda}{2}|Du_\lambda|^2 = F(x, m_\lambda(t)) \\ \partial_t m_\lambda - \nu_\lambda \Delta m_\lambda - div(m_\lambda(\lambda Du_\lambda - v(x))) = 0 \quad \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m_\lambda(0) = m_0, \qquad \text{in } \mathbf{R}^d \end{cases}$$

$\lambda > 0$, $\nu_\lambda \to 0+$, $v \in W^{2,\infty}$, $m_0 \in \mathcal{P}_2(\mathbf{R}^d)$ has bounded density

$F : \mathbf{R}^d \times \mathcal{P}_1(\mathbf{R}^d) \to \mathbf{R}$ and $D_x F$ continuous (on m with bdd. density)

$x \mapsto F(x, m)$ Lip and semiconcave uniformly in m.

Note: $H(p) = \lambda|p|^2/2 - v(x)\cdot p \implies DH(p) = \lambda p - v(x)$.

Representation of the solution

Note: no terminal condition for the HJB equation: existence via approximation on $\mathbf{R}^d \times [0, T]$ with terminal condition $u_\lambda^T(x, T) = 0$, $T \to +\infty$.

If (u_λ, m_λ) is a solution of (MF1) then

$$u_\lambda(x,t) = \inf E\left[\int_t^{+\infty} e^{-\lambda(s-t)}\left(\frac{1}{2\lambda}|\alpha(s)|^2 + F(y(s), m_\lambda(s))\right) ds\right],$$

for

$$dy(s) = [v(y(s)) + \alpha(s)]ds + \sqrt{2\nu_\lambda}dW(s), \quad s > t, \quad y(t) = x.$$

Rmks.:

1. Meaning of λ large: high discount factor (the near future counts much more than the far future, as in [B-L-L]) and cheap control (as in [DHL]).

2. $\nu_\lambda \to 0+$ means vanishing noise.

3. The limit as $\lambda \to \infty$ is not obvious to guess: $\lambda D u_\lambda \to$??

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Convergence Theorem for (MF1)

Under the previous assumptions, $\exists\ \lambda_n \to +\infty$ such that

$$m_{\lambda_n} \to m \quad \text{in } C([0,T],\mathcal{P}_1),\ \text{weak}^* \text{ in } L^\infty([0,T]\times \mathbf{R}^d),\ \forall\, T>0,$$

m solves (distribution sense) the continuity equation

$$\begin{cases} \partial_t m - div(m(DF(x,m) - v(x))) = 0 & \text{in } \mathbf{R}^d \times \mathbf{R}_+ \\ m(0) = m_0, \quad \text{in } \mathbf{R}^d, \end{cases}$$

$$\lambda_n u_{\lambda_n}(x,t) \to F(x,m(t)) \text{ loc. uniformly},$$

$$\lambda_n D u_{\lambda_n}(x,t) \to DF(x,m(t)) \ \text{ a.e.}$$

Interpretation : the optimal feedback $-\lambda D u_\lambda$ is close to the gradient descent of the running cost for the limit density m, $-DF(\cdot, m)$.

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Applications and variants

1. $F(x, m) = k * m(x)$ with $k \in W^{1,\infty}$ is ok for the Theorem.
2. The aggregation equation and model 1 of crowd dynamics are

$$\partial_t m - div(m(D\,k * m - v(x))) = 0$$

with $k = \phi(|x|)$ Lipschitz: not always C^1 in $x = 0$ but semiconcave if there is repulsion at short distance, $\Longrightarrow$ the Thm. applies.

3. Result remains true if $\nu_\lambda \to \nu_\infty > 0$, with $-\nu_\infty \Delta m$ in the limit PDE.
4. The deterministic case $\nu_\lambda = 0 \ \forall \lambda$ can be treated under the stronger assumption $\|F(\cdot, m)\|_{C^2} \leq C$, e.g. $F = k * m$, $k \in C^2 \cap W^{2,\infty}$.
5. More general Hamiltonians: H convex, with at most quadratic growth, $D_p H$, $D_{px} H$ growing at most linearly, H Lipschtz and semiconcave in x (with linear modulus in $|p|$).

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Uniqueness of the limit m

If the limit continuity equation has at most one solution m, then

$$m_\lambda \to m, \quad \lambda u_\lambda \to F(\cdot, m), \quad \lambda D u_\lambda \to D_x F(\cdot, m), \text{ as } \lambda \to \infty.$$

This occurs when the support of m_0 is bounded and

- $D_x F(\cdot, m) \in C^1, \quad |D_x F(x, m) - D_x F(y, m)| \le C_1 |x - y|,$
  and $\|D_x F(\cdot, m) - D_x F(\cdot, \bar{m})\|_\infty \le C_1 \mathbf{d}_1(m, \bar{m}),$ [Piccoli-Rossi '13];
- e.g., if $F = k * m, \quad k \in C^2 \cap W^{2,\infty}$;
- in the aggregation equation with the kernels $k = \phi(|x|)$ shown before by, e.g., Carrillo-Rosado 2010.

N.B.: MFG system (MF1) can have many solutions: the Lasry-Lions monotonicity condition on F is NOT satisfied if F models aggregation, but under the conditions above ALL of them converge to the same limit!

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Convergence: ingredients of the proof for (MF1)

Estimates for HJB, independent of ν_λ,

- L^∞ : $|u_\lambda| \le \frac{C}{\lambda}(1+|x|)$
- Lipschitz : $|Du_\lambda| \le \frac{C}{\lambda}$
- Semiconcavity : $D^2 u_\lambda \le \frac{C}{\lambda}$

Estimates on the KFP equation, independent of ν_λ,

- 2nd moment : $\sup_{t\in[0,T]} \int_{\mathbf{R}^d} |x|^2 m_\lambda(x,t)dx \le C_T$.
- L^∞ : $\sup_{t\in[0,T]} \|m_\lambda(t)\|_\infty \le C_T \|m_0\|_\infty$

Compactness of $\{m_\lambda\}$ in $C([0,T],\mathcal{P}_1) \Rightarrow F(x,m_\lambda) \to F(x,m)$ loc. unif.

Stability of viscosity solutions for HJB $\Longrightarrow \lambda u_\lambda(x,t) \to F(x,m(t))$,

Semiconcavity estimate for $u_\lambda \Longrightarrow \lambda Du_\lambda(x,t) \to DF(x,m)$ a.e.
$\Longrightarrow$ can pass to the limit in KFP.

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2. b) Convergence for controlled acceleration

First order MFG system

$$
\text{(MF2)} \quad \begin{cases} -\partial_t u_\lambda + \lambda u_\lambda - v \cdot D_x u_\lambda + \dfrac{\lambda}{2}|D_v u_\lambda|^2 = F(x, v, m_\lambda(t)) \\ \partial_t m_\lambda + v \cdot D_x m_\lambda - div_v(m_\lambda \lambda D_v u_\lambda) = 0 \quad \text{in } \mathbf{R}^{2d} \times (0, T) \\ m_\lambda(0) = m_0, \quad u_\lambda(x, v, T) = 0 \qquad \text{in } \mathbf{R}^{2d}. \end{cases}
$$

$\lambda > 0$, $m_0 \in \mathcal{P}_1(\mathbf{R}^{2d})$ with compact support,

$$
F(x, v, m(t)) = k * m(x, v, t) = \int_{\mathbf{R}^{2d}} k(x - y, v - v_*) m(y, v_*, t) dy dv_*,
$$

$$
k(x, v) = |v|^2 / g(x), \quad g(x) = (\alpha + |x|^2)^\beta, \quad \alpha > 0, \beta \geq 0,
$$

i.e., k is the potential associated to the Cucker-Smale ODEs.

More generally: $g(x) \geq c_o > 0$, even, smooth, with $|Dg|/g \leq C$.

The deterministic MFG associated to (MF2) has the cost functional

$$J(\gamma, m, t) = \int_t^T e^{-\lambda(s-t)} \left[\frac{1}{2\lambda} |\ddot{\gamma}(s)|^2 + F(\gamma(s), \dot{\gamma}(s), m(s)) \right] ds$$

A solution (u_λ, m_λ) of (MF2) is expected to satisfy

$$u_\lambda(x, v, t) = \inf J(\gamma, m_\lambda, t), \quad \gamma(t) = x, \ \dot{\gamma}(t) = v.$$

However deterministic MFG with controlled acceleration were studied only very recently by Cannarsa-Mendico (ArXiv 2019)
and Achdou-Mannucci-Marchi-Tchou (ArXiv 2019),
under assumption implying $F(x, v, m) \geq c_o|v|^2 - c_1$, $c_o > 0$,
whereas in our case $F(x, v, m) \sim |v|^2/g(x) \to 0$ as $|x| \to \infty$.

We first prove existence of a weak variational solution of (MF2).

More generally: $g(x) \geq c_o > 0$, even, smooth, with $|Dg|/g \leq C$.

The deterministic MFG associated to (MF2) has the cost functional

$$J(\gamma, m, t) = \int_t^T e^{-\lambda(s-t)} \left[\frac{1}{2\lambda} |\ddot{\gamma}(s)|^2 + F(\gamma(s), \dot{\gamma}(s), m(s)) \right] ds$$

A solution (u_λ, m_λ) of (MF2) is expected to satisfy

$$u_\lambda(x, v, t) = \inf J(\gamma, m_\lambda, t), \quad \gamma(t) = x, \ \dot{\gamma}(t) = v.$$

However deterministic MFG with controlled acceleration were studied only very recently by Cannarsa-Mendico (ArXiv 2019)

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under assumption implying $F(x, v, m) \geq c_o|v|^2 - c_1$, $c_o > 0$,

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The variational approach

It originates from Lasry-Lions, continued by Cardaliaguet, Benamou - Carlier - Santambrogio, and many others....

Take $\Gamma := C^1([0,T], \mathbf{R}^d)$ and for $\eta \in \mathcal{P}(\Gamma)$

$$\mathcal{J}_\lambda(\eta) := \int_\Gamma \int_0^T e^{-\lambda t} \frac{1}{2\lambda} |\ddot{\gamma}(t)|^2 dt\, \eta(d\gamma) + \int_0^T e^{-\lambda t} \mathcal{F}(m^\eta(t)) dt,$$

where $m^\eta(t) := e_t \sharp \eta$, $e_t : \Gamma \to \mathbf{R}^{2d}$, $e_t(\gamma) := (\gamma(t), \dot{\gamma}(t))$, and

$$\mathcal{F}(m) := \frac{1}{2} \int_{\mathbf{R}^{4d}} k(x - x_*, v - v_*) m(dx, dv) m(dx_*, dv_*).$$

is a potential of F.

We call variational solution of the MFG problem a minimizer of $\mathcal{J}_\lambda$.

Existence of a variational solution

Lemma

$\forall\ \lambda > 0$, there exists at least a minimizer $\bar\eta_\lambda$ of $\mathcal{J}_\lambda$ subject to

$e_0\sharp\bar\eta_\lambda = m_0$.

It is a weak solution or an equilibrium of the MFG problem in the sense that

for $\bar\eta_\lambda-$ a.e. $\bar\gamma$,

$$J(\bar\gamma, m^{\bar\eta_\lambda}, 0) = \inf J(\gamma, m^{\bar\eta_\lambda}, 0),\ \gamma \in H^2,\ \gamma(0) = \bar\gamma(0),\ \dot\gamma(0) = \dot{\bar\gamma}(0).$$

Link with the MFG system (MF2), at least formally: if

$$u_\lambda(x, v, t) := \inf J(\gamma, m^{\bar\eta_\lambda}, t),\ \gamma(t) = x,\ \dot\gamma(t) = v,$$

then the pair $(u_\lambda, m^{\bar\eta_\lambda})$ is a weak solution of (MF2), i.e., u_λ is a viscosity solution to the HJB equation and $m^{\bar\eta_\lambda}$ is a distribution solution the KFP equation.

The limit kinetic equation

In the limit we expect the kinetic equation describing the continuous version of Cucker-Smale model

$$\text{(K)} \quad \begin{cases} \partial_t m + v \cdot D_x m - div_v(m D_v F(x,v,m)) = 0 & in\ \mathbf{R}^{2d} \times \mathbf{R}_+, \\ m(0) = m_0, & in\ \mathbf{R}^{2d}. \end{cases}$$

Following Canizo-Carrillo-Rosado 2011, $m \in C^0([0,T], \mathcal{P}_2(\mathbf{R}^d))$ is a measure-valued solution to (K) if $m(t) = \Phi^{x,v}(t) \sharp m_0$, where $\Phi^{x,v} = (\Phi_1^{x,v}, \Phi_2^{x,v})$ solves the "ODE" driven by the vector field in (K)

$$\begin{cases} \frac{d}{dt} {\Phi_1}^{x,v}(t) = {\Phi_2}^{x,v}(t), \\ \frac{d}{dt} {\Phi_2}^{x,v}(t) = -D_v F({\Phi_1}^{x,v}(t), {\Phi_2}^{x,v}(t), m(t)), \\ \Phi^{x,v}(0) = (x,v). \end{cases}$$

By Canizo-Carrillio-Rosado 2011 there exists a unique measure - valued solution of (K).

Convergence Theorem for (MF2)

Let $\bar{\eta}_\lambda$ be a minimizer of $\mathcal{J}_\lambda$ subject to $e_0 \sharp \bar{\eta}_\lambda = m_0$.

Then $m^{\bar{\eta}_\lambda} = e_t \sharp \bar{\eta}_\lambda \to m$ in $C^0_{loc}([0, T), \mathcal{P}_2(\mathbf{R}^{2d}))$ as $\lambda \to +\infty$,

$m =$ the unique measure-valued solution to (K).

Remarks.

- We do not prove the convergence of all the equilibria of the MFG problem, but only of the minimizers of $\mathcal{J}_\lambda$.
- A somehow related limit for variational problems was studied by Rossi-Savarè-Segatti-Stefanelli 2019 and called Weighted Energy Dissipation, converging to a gradient flow (in metric spaces!).

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Convergence: main ingredients of the proof for (MF2)

Estimates on the value function

$$u_\lambda(x,v,s) := \inf J(\gamma, m^{\bar\eta_\lambda}, s),\ \gamma \in H^2,\ \gamma(s)=x,\ \dot\gamma(s)=v,$$

- $0 \le u_\lambda(x,v,s) \le \frac{C}{\lambda}(1+|v|^2 + M_{2,v}(m^{\bar\eta_\lambda}(s)))$,
  where $M_{2,v}(m^{\bar\eta_\lambda}(s)) := \int_{\mathbf{R}^{2d}} |v|^2 m^{\bar\eta_\lambda}(dx,dv,s)$
- $M_{2,v}(m^{\bar\eta_\lambda}(t)) \le C(1+\frac{e^{\lambda(t-s)}}{\lambda})M_{2,v}(m^{\bar\eta_\lambda}(s))$ for $0 \le s \le t \le T$.

Dynamic Programming Principle: set $\Gamma_s := C^1([s,T],\mathbf{R}^d)$ $\eta \in \mathcal{P}(\Gamma_s)$,

$$\mathcal{J}_{\lambda,s}(\eta) := \int_{\Gamma_s}\int_s^T e^{-\lambda(t-s)}\frac{1}{2\lambda}|\ddot\gamma(t)|^2 dt\,\eta(d\gamma) + \int_s^T e^{-\lambda(t-s)}\mathcal{F}(m^\eta(t))dt.$$

Then for any $s \in [0,T)$, the restriction $\bar\eta_{\lambda,s}$ of $\bar\eta_\lambda$ to $[s,T]$ is a minimizer of $\eta \to \mathcal{J}_{\lambda,s}(\eta)$ subject to $e_s\sharp\eta = e_s\sharp\bar\eta_\lambda$.

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Gradient estimate on u_λ: $\forall \varepsilon > 0$, for a.e. $(x, v, s) \in \mathbf{R}^{2d} \times [0, T - \varepsilon]$,

$$|D_v u_\lambda(x, v, s)| \le C \left(\frac{u_\lambda^{1/2}(x, v, s)}{\lambda^{1/2}} + \varepsilon u_\lambda(x, v, s) \right)$$

Bounds on the optimal trajectories: for $\varepsilon < \varepsilon_o$, $\lambda \ge 1/\varepsilon$, $t \in [0, T - \varepsilon]$

- $|\ddot{\bar\gamma}(t)| \le C$ for $\bar\eta_\lambda - a.e.\ \bar\gamma$,
- the support of $m^{\bar\eta_\lambda}(t)$ is contained in B_C .

The limit $\lambda \to \infty$:

- can extract $\bar\gamma_{\lambda_n} \to \gamma \in \Gamma$ and $m^{\bar\eta_{\lambda_n}} \to m \in C^0([0, T], \mathcal{P}_2(\mathbf{R}^{2d}))$
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Concluding remarks

- We established the first rigorous results on asymptotic connections between MFGs and phenomenological models of population dynamics (without diffusion);
- it shows a loss of rationality in the (singular) limit.
- Other sets of assumptions and variants are possible.
- The literature on aggregation and kinetic equations is large: it can inspire new results and models for MFG.
- Open question: MFG models assume high rationality of the agents, Agent-based models assume no choice is made by the agents:
  are there intermediate rationality models better describing social behavior, e.g., crowd motion?

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