Probabilistic max-plus schemes for solving Hamilton-Jacobi-Bellman equations

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Workshop 1: High Dimensional Hamilton-Jacobi Methods in Control and Differential Games

IPAM, virtual UCLA, March 30 - April 3, 2020

Based on joint works with Jean-Philippe Chancelier (CERMICS, ENPC), Benoît Tran and Eric Fodjo

A finite horizon diffusion control problem involving “discrete” and “continuum” controls

Maximize $J(t, x, \mu, U) := \mathbb{E}\left[\int_t^T \ell^{\mu_s}(\xi_s, U_s)ds + \psi(\xi_T) \mid \xi_t = x\right]$,

- $\xi_s \in \mathbb{R}^d$, the *state process*, satisfies the stochastic differential equation

$$d\xi_s = f^{\mu_s}(\xi_s, U_s)ds + \sigma^{\mu_s}(\xi_s, U_s)dW_s \ ,$$

- $\mu := (\mu_s)_{0 \leq s \leq T}$, and $U := (U_s)_{0 \leq s \leq T}$ are *admissible control processes*, $\mu_s \in \mathcal{M}$ a finite set and $U_s \in \mathcal{U} \subset \mathbb{R}^p$,
- $(W_s)_{s \geq 0}$ is a d-dimensional Brownian motion,

Compute the value function $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, $v(t, x) = \sup_{\mu, U} J(t, x, \mu, U)$ *and a* **feedback optimal control** $(t, x) \in [0, T] \times \mathbb{R}^d \mapsto (m(t, x), u(t, x)) \in \mathcal{M} \times \mathcal{U}$.

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The Hamilton-Jacobi-Bellman (HJB) equation

Theorem

Under suitable assumptions, the value function v is the unique (continuous) viscosity solution of the HJB equation

$$-\frac{\partial v}{\partial t} - \mathcal{H}(x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0, \quad x \in \mathbb{R}^d,\ t \in [0,T),$$
$$v(T,x) = \psi(x), \quad x \in \mathbb{R}^d,$$

satisfying also some growth condition at infinity (in space).

With the *Hamiltonian*:

$$\mathcal{H}(x,r,p,\Gamma) := \max_{m \in \mathcal{M}} \mathcal{H}^m(x,r,p,\Gamma) \ ,$$
$$\mathcal{H}^m(x,r,p,\Gamma) := \max_{u \in \mathcal{U}} \mathcal{H}^{m,u}(x,r,p,\Gamma) \ ,$$
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A discrete time (or time discretization of a) stochastic control problem / Multistage Stochastic Programming (MSP)

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- $\xi_s \in \mathbb{R}^d$, the *state process* satisfies

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- $\mu := (\mu_s)_{0 \le s \le T-1}$, and $U := (U_s)_{0 \le s \le T-1}$ are *admissible control processes* in the sense that $\sigma(U_s) \subset \sigma(W_1, \ldots, W_{s+1})$
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The Dynamic Programming or Bellman equation

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Theorem

The value function v is the unique solution of the Bellman equation

$$\begin{aligned} V_T &= \psi \\ \forall t \in [\![0, T-1]\!],\ V_t &= \mathcal{B}_t(V_{t+1}) \end{aligned} \quad .$$

Where the *Bellman operator* $\mathcal{B}_t$ from the space of functions $\mathbb{R}^d \to \bar{\mathbb{R}}$ to itself is given by

$$\begin{aligned} \tilde{\mathcal{B}}_t(\phi)(x, w) &= \min_{m,u} \left(c_t^m(x, u, w) + \alpha_t^m(x, u, w)\, \phi\left(g_t^m(x, u, w)\right)\right) \\ \mathcal{B}_t(\phi)(x) &= \mathbb{E}\left[\tilde{\mathcal{B}}_t(\phi)(x, W_{t+1})\right] \end{aligned} \quad .$$

Example of time discretization: a semilagrangian scheme

Consider the Euler discretization $\hat{\xi}$ of the process ξ:

$$\hat{\xi}(t+h) = \hat{\xi}(t) + f^{\mu_t}(\hat{\xi}(t), U_t)h + \sigma^{\mu_t}(\hat{\xi}(t), U_t)(W_{t+h} - W_t) \ .$$

Denoting $W_t^h = W_{t+h} - W_t$ the increments of the Brownian process, the discretisation can be written as a discrete time dynamics of a MSP:

$$\hat{\xi}(t+h) = g_t^{\mu_t}\left(\hat{\xi}(t), u, W_t^h\right) \ .$$

The following is a time discretization of HJB:

$$v_t^h = \mathcal{B}_{t,h}(v_{t+h}^h), \quad t \in \mathcal{T}_h = \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$\mathcal{B}_{t,h}(\phi)(x) = \sup_{m \in \mathcal{M},\ u \in \mathcal{U}} \left\{ \mathbb{E}\left[h\ell^m(x,u) + e^{-h\delta^m(x,u)} \phi(g_t^m\left(x, u, W_t^h\right)) \right] \right\} \ .$$

Under appropriate assumptions, v^h converges to the solution of HJB when h goes to zero.

Standard grid based space discretizations solving HJB equations suffer the curse of dimensionality malediction:
for an error of ϵ, the computing time of finite difference or finite element methods is at least in the order of $(1/\epsilon)^{d/2}$.

Possible curse of dimensionality-free methods:

- Idempotent/tropical methods: deterministic case: McEneaney (2007), Dower, Zhang, Zheng Qu (2014), stochastic case: McEneaney, Kaise and Han (2011), M.A. and Fodjo (2018).
- Sparse grids Garcke, Griebel, Bokanowski, Kang,...
- Special classes via Representation formula Osher, Darbon, Yegerov, Dower, McEneaney,...
- Tensor decompositions Dolgov, Kalise, Kunish (2019), Oster, Sallandt, Schneider (2019).
- Deep learning for deterministic problems: Nakamura-Zimmerer, Qi Gong, Wei Kang (2019).

Possible curse of dimensionality-free methods (cont):

- Probabilistic numerical methods based on a backward stochastic differential equation interpretation of the HJB equation, simulations and regressions:
  - Quantization Bally, Pagès (2003)
  - Introduction of a new process without control: Bouchard, Touzi (2004) for the semi-linear case; Cheridito, Soner, Touzi and Victoir (2007) and Fahim, Touzi and Warin (2011) in the fully-nonlinear case.
  - Control randomization: Kharroubi, Langrené, Pham (2013).
  - Fixed point iterations: Bender, Zhang (2008) for semilinear PDE (which are not HJB equations).
  - Neural Networks/Deep learning: Weinan E., Jiequn Han, Jentzen, Beck, Pham, Warin (2017-) , Pham, Warin (2019).
- Optimization along one or few optimal trajectories:
  - Deterministic case: Direct methods, Pontryagin principle,...
  - DP algorithm on a tree-structure Alla, Falcone, Saluzzi (2019)
  - Stochastic case: Stochastic Dual Dynamic Programming method (SDDP) Pereira and Pinto (1991), Shapiro (2011),...

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Overview

1. The curse of dimensionality-free idempotent method of McEneaney for deterministic problems
2. A finite horizon variant of the random idempotent method of Zheng Qu for deterministic problems (A., Chancelier, Tran, 2018)
3. A random idempotent method for stochastic control problems (A., Fodjo, 2018)
4. Comparison with the SDDP method (A., Chancelier, Tran, 2018)
5. A new probabilistic scheme for HJB equations (A., Fodjo, 2018)
6. A probabilistic idempotent method for stochastic control problems (A., Fodjo, 2018)

1. The curse of dimensionality-free idempotent method of McEneaney for deterministic problems

Recall: applying the semilagrangian scheme to the undiscounted deterministic control problem ($\delta^m \equiv 0$, $\sigma^m \equiv 0$), we obtain:

$$v_t^h = \mathcal{B}_{t,h}(v_{t+h}^h), \quad t \in \mathcal{T}_h = \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$\mathcal{B}_{t,h}(\phi)(x) = \sup_{m \in \mathcal{M},\ u \in \mathcal{U}} \left\{ h\ell^m(x,u) + \phi(g_t^m(x,u)) \right\} \ ,$$

and

$$g_t^m(x,u) = x + f^m(x,u)h \ .$$

The Bellman operators $\mathcal{B}_{t,h}$ are

- *monotone:* $\phi \leq \phi' \Rightarrow \mathcal{B}_{t,h}(\phi) \leq \mathcal{B}_{t,h}(\phi')$*;*
- *max additive:* $\mathcal{B}_{t,h}(\phi \vee \phi') = \mathcal{B}_{t,h}(\phi) \vee \mathcal{B}_{t,h}(\phi')$*;*
- *additively homogeneous and thus max-plus linear:* $\mathcal{B}_{t,h}(\lambda + \phi) = \lambda + \mathcal{B}_{t,h}(\phi)$*..*

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Similarly, the Lax-OLeinik semigroup associated to the HJ equation is max-plus linear: this is the superposition principle of Maslov.

Let q_i^{t+h} be “max-plus basic” functions, then

$$v_{t+h}^h(x) = \max_{i=1,\dots,N}(\lambda_i + q_i^{t+h}(x)) \implies v_t^h(x) = \max_{i=1,\dots,N}(\lambda_i + q_i^t(x)) \ ,$$

with $q_i^t = \mathcal{B}_{t,h}(q_i^{t+h})$ so

we only need to compute the effect of the composition of Bellman operators $\mathcal{B}_{t,h}$ on the basic functions $q_i^T, \ i = 1, \dots, N$.

- First type of max-plus methods: project the operator $T_{t,h}$ or the q_i^t on a fixed basis, see Fleming and McEneaney (2000) and A.,Gaubert,Lakoua (2008) $\Longrightarrow$ **same difficulty as grid based methods.**
- Second type of max-plus methods (McEneaney, 2007): Assume that the $\mathcal{H}^m$ correspond to LQ problems, then

$$\mathcal{B}_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} \mathcal{B}^m_{t,h}(\phi)(x)$$

with

q quadratic $\Longrightarrow \mathcal{B}^m_{t,h}(q)$ quadratic, given by a Riccati recurrence equation.

So

v^h_T finite sup of quadratic forms $\Longrightarrow v^h_t$ finite sup of quadratic forms.

- The number of quadratic forms for v^h_0 is exponential in the number of time step only. So the method is **curse of dimensionality-free** at the price of a curse of complexity.
- It can be reduced by pruning.

2. A finite horizon variant of the random idempotent method of Zheng Qu for deterministic problems (A., Chancelier, Tran, 2018)

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To simplify, assume that the $\mathcal{H}^m$ correspond to LQ homogeneous problems: $\ell^m(x, u)$ are pure quadratic functions (homogenous polynomials of degree 2) and $g_t^m(x, u)$ are linear.

Idea: Replace pruning by random sampling.

Compute at each step $k \geq 0$, the sets Φ_t^k, $t = 0, h, \ldots, T$, of quadratic forms representing the approximate value functions: $V_t^k(x) := \sup_{\phi \in \Phi_t^k} \phi(x)$, as follows:

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The algorithm proposed by Zheng Qu is similar but applied directly to the stationary equation $\mathcal{H}(x, v(t,x), Dv(t,x)) = 0$.

3. A random idempotent method for stochastic control problems

compare with (A., Fodjo, 2018)

Recall: applying the semilagrangian scheme to an undiscounted stochastic control problem ($\delta^m = 0$), we obtain:

$$v_t^h = \mathcal{B}_{t,h}(v_{t+h}^h), \quad t \in \mathcal{T}_h = \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$\mathcal{B}_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} \mathcal{B}_{t,h}^m(\phi)(x) \ ,$$

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and

$$g_t^m(x,u,w) = x + f^m(x,u)h + \sigma^m(x,u)w \ .$$

The $\mathcal{B}_{t,h}^m$ are not max-plus linear in general, but they are still additively homogeneous and monotone.

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Moreover, if σ^m is constant, f^m is affine, and ℓ^m is concave quadratic, then all $\mathcal{H}^m$ correspond to LQG problems, so

q quadratic $\Longrightarrow \mathcal{B}_{t,h}^m(q)$ quadratic, given by a Riccatti recurrence equation.

Theorem (McEneaney, Kaise and Han, 2011)

Assume $\delta^m = 0$, σ^m is constant, f^m is affine, ℓ^m is concave quadratic.

If ψ is the supremum of a finite number of concave quadratic forms. Then, for all $t \in \mathcal{T}_h$, there exists a set Z_t and a map $q_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ such that for all $z \in Z_t$, $q_t(\cdot, z)$ is a concave quadratic form and

$$v^h(t, x) = \sup_{z \in Z_t} q_t(x, z) \ .$$

Moreover, the sets Z_t satisfy

$$Z_t = \mathcal{U} \times \mathcal{M} \times \{\bar{z}_{t+h} : \mathcal{W} \to Z_{t+h} \mid \textit{Borel measurable}\} \ ,$$

where $\mathcal{W} = \mathbb{R}^d$ is the space of values of the Brownian process.

The proof uses the max-plus (infinite) distributivity property.

- In the deterministic case, the sets Z_t are finite, and their cardinality is exponential in time: $\#Z_t = M \times \#Z_{t+h} = \cdots = M^{N_t} \times \#Z_T$ with $M = \#\mathcal{M}$ and $N_t = (T-t)/h$.
- In the stochastic case, the sets Z_t are infinite as soon as $t < T$.
- If the Brownian process is discretized in space, then $\mathcal{W}$ can be replaced by the finite subset with fixed cardinality p, and if $\mathcal{U}$ is also replaced by a finite subset $\mathcal{U}_h$, then the sets Z_t become finite.
- Nevertheless, their cardinality increases doubly exponentially in time: $\#Z_t = M \times \#\mathcal{U}_h \times (\#Z_{t+h})^p = \cdots = (M \times \#\mathcal{U}_h)^{\frac{p^{N_t}-1}{p-1}} \times (\#Z_T)^{p^{N_t}}$ where $p \geq 2$ ($p = 2$ for the Bernouilli discretization).
- McEneaney, Kaise and Han proposed to apply a pruning method to reduce at each time step $t \in \mathcal{T}_h$ the cardinality of Z_t.
- **Here, we shall replace again pruning by a random sampling.**
- Given a sampling on $\mathbb{R}^d$, and an a priori cardinality of the set Φ_t of quadratic forms used to approximate V_t as $V_t(x) := \sup_{\phi \in \Phi_t} \phi(x)$, one may choose the parameters by minimizing the norm of the error $v_t^h - \mathcal{B}_{t,h}(v_{t+h}^h)$, restricted to a sample on $\mathbb{R}^d$. But this is not a convex program.
- **Here we find rather the maximal such a function V_t which is below the true solution. This a tropical projection.**

- Idea: use the formal proof of the previous theorem.
- If $V_{t+h}(x) := \sup_{\phi\in\Phi_{t+h}} \phi(x)$, for all $x \in \mathbb{R}^d$, with Φ_{t+h} a finite set of quadratic forms, then there exists a function $z : \mathbb{R}^d \to \Phi_{t+h}$, which chooses $z(x)$ as any element ϕ of Φ_{t+h} which maximizes $\phi(x)$.
- When m, x and u are fixed, denote $\tilde{Z} = z(g_t^m(x, u, W_t^h))$. This is a random element of Φ_{t+h}, so a random quadratic form.
- The map:

$$w(x', u') := \mathbb{E}\left[h\ell^m(x', u') + \tilde{Z}(g_t^m\left(x', u', W_t^h\right))\right]$$

is a quadratic form of (x', u') such that

$$w(x', u') \leq \mathbb{E}\left[h\ell^m(x', u') + \sup_{\phi\in\Phi_{t+h}} \phi(g_t^m\left(x', u', W_t^h\right))\right] \;,$$

with equality in (x, u).
- Then the map $\phi(x') = \sup_{u'\in\mathcal{U}} w(x', u')$ is a quadratic form such that $\phi \leq \mathcal{B}_{t,h}^m(V_{t+h})$, with equality in x, when u realizes the maximum in

$$\mathcal{B}_{t,h}^m(V_{t+h})(x) = \sup_{u'\in\mathcal{U}} \left\{\mathbb{E}\left[h\ell^m(x, u') + \sup_{\phi\in\Phi_{t+h}} \phi(g_t^m\left(x, u', W_t^h\right))\right]\right\} \;.$$

Adaptation of the random idempotent algorithm of (A., Fodjo (2018)).

Compute at each step $k \geq 0$, the sets Φ_t^k, $t = 0, h, \ldots, T$, of quadratic forms representing the approximate value functions: $V_t^k(x) := \sup_{\phi \in \Phi_t^k} \phi(x)$, as follows:

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1. Initialize the sets to $\Phi_t^0 = \emptyset$, $t = 0, h, \ldots, T$.
2. **Forward phase:** draw independently new points $\left(x_t^{k-1}, u_t^{k-1}\right)_t$ on $\mathbb{R}^d \times \mathcal{U}$.

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3. **Backward phase:** backward in time, draw a sample of fixed size m from W_t^h, compute the corresponding random sample of $\tilde{Z} \in \operatorname{Argmax}_{\phi \in \Phi_{t+h}^k} \phi(g_t^m\left(x_t^{k-1}, u_t^{k-1}, W_t^h\right))$, and deduce the quadratic form

$$w(x', u') := \mathbb{E}\left[h\ell^m(x', u') + \tilde{Z}(g_t^m\left(x', u', W_t^h\right))\right] \quad ,$$

   and $\phi(x') = \sup_{u' \in \mathcal{U}} w(x', u')$. Set

$$\Phi_t^k = \Phi_t^{k-1} \cup \{\phi\} \quad .$$

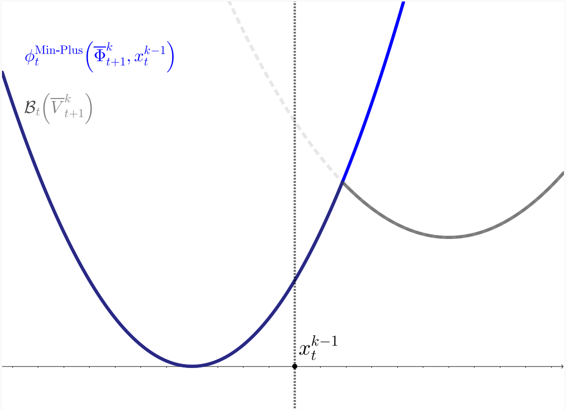
- When there is no control u, one only need to draw the states $\left(x_t^{k-1}\right)_t$ on $\mathbb{R}^d$.
- The same holds, when considering the discrete time model:

$$
\begin{aligned}
\tilde{\mathcal{B}}_t(\phi)(x, w) &= \min_{m,u}\left(c_t^m(x, u, w) + \phi\left(g_t^m(x, u, w)\right)\right) \\
\mathcal{B}_t(\phi)(x) &= \mathbb{E}\left[\tilde{\mathcal{B}}_t(\phi)(x, W_{t+1})\right] \quad ,
\end{aligned}
$$

- Indeed, $\tilde{\mathcal{B}}$ is min-plus linear, so it transforms a minimum of a finite number of quadratic forms into a minimum of a finite number of quadratic forms. Then, one applies the previous method with x fixed.
- Moreover, we get (at least when the sampling of the noise W_{t+1} is exact, so its support is finite) that $\mathcal{B}_t(V_{t+h}^k)(x_t^{k-1}) = V_t^k(x_t^{k-1})$.
- This is the same property as in SDDP algorithm.

4. Comparison with the SDDP method (A., Chancelier, Tran, 2018)

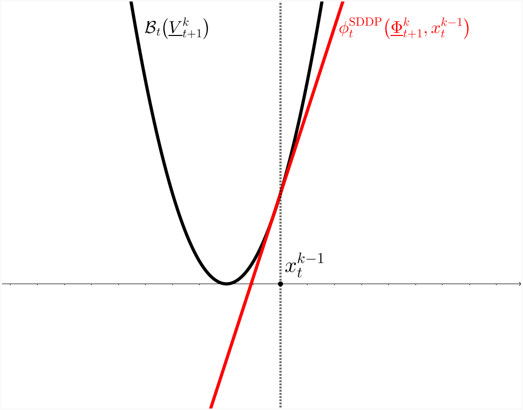
Trial points and selection functions: Min-Plus exemple



Min-Plus Exemple

- $\overline{\Phi}_{t+1}^k$ are sets of Quadratic functions
- Upper approximations opt = inf
- $\overline{V}_{t+1}^k := \inf_{\phi \in \overline{\Phi}_{t+1}^k} \phi$

Trial points and selection functions: SDDP exemple



SDDP Exemple

- $\underline{\Phi}_{t+1}^k$ are sets of Affine functions
- Lower approximations
  opt $=$ sup
- $\underline{V}_{t+1}^k := \sup_{\phi \in \underline{\Phi}_{t+1}^k} \phi$

Tight and Valid selection functions

Tightness Assumption

$$\underbrace{\overbrace{\phi_t^{\text{SDDP/Min-Plus}}}^{\text{Selection function}}\left(\overbrace{\underline{\Phi}_{t+1}^k}^{\text{Set of basic functions}}, x_t^{k-1}\right)}_{\text{Basic function}}\Big(\overbrace{x_t^{k-1}}^{\text{Trial point}}\Big) = \mathcal{B}_t\left(\underline{V}_{t+1}^k\right)\left(x_t^{k-1}\right)$$

It is a local property.

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It is a local property.

Validity Assumption

$$\phi_t^{\text{SDDP}}\left(\underline{\Phi}_{t+1}^{k}, x_t^{k-1}\right) \leq \mathcal{B}_t\left(\underline{V}_{t+1}^{k}\right) \quad \text{(SDDP)} \quad \mathrm{opt} = \sup$$

$$\phi_t^{\text{Min-Plus}}\left(\overline{\Phi}_{t+1}^{k}, x_t^{k-1}\right) \geq \mathcal{B}_t\left(\overline{V}_{t+1}^{k}\right) \quad \text{(Min-Plus)} \quad \mathrm{opt} = \inf$$

It is a global property.

Scheme of Tropical Dynamic Programming (TDP) algorithm

Compute at each step $k \geq 0$, the sets Φ_t^k, $t = 0, h, \ldots, T$, representing the approximate value functions: $V_t^k(x) := \operatorname{opt}_{\phi \in \Phi_t^k} \phi(x)$, as follows:

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3. **Backward phase:** backward in time, evaluate the selection function at Φ_{t+1}^k and the trial point x_t^{k-1}, which gives a new basic function ϕ that is added to the current set of approximations

$$\Phi_t^k = \Phi_t^{k-1} \cup \{\phi\}\,.$$

Almost sure uniform convergence to a limit V_t^*

If the Bellman operators $\mathcal{B}_t$ are order-preserving "+" mild technical assumptions on $\mathcal{B}_t$ and the basic functions, we have

Existence of an approximating limit

Let $t \in [\![0, T]\!]$ be fixed. The sequence of functions $\left(V_t^k\right)_{k \in \mathbb{N}}$ generated by TDP μ-a.s. converges uniformly on every compact set included in the domain of V_t to a function V_t^*.

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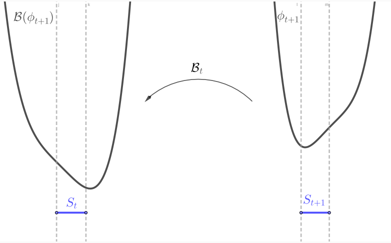
Is V_t^* equal to V_t ?

Optimal sets: the trial points need to be rich enough

Optimal sets

Let $(\phi_t)_{t\in[\![0,T]\!]}$ be $T+1$ functions. A sequence of sets $(S_t)_{t\in[\![0,T]\!]}$ is said to be (ϕ_t)-optimal if for every $t \in [\![0, T-1]\!]$

$$\mathcal{B}_t\left(\phi_{t+1} + \delta_{S_{t+1}}\right) + \delta_{S_t} = \mathcal{B}_t\left(\phi_{t+1}\right) + \delta_{S_t}.$$



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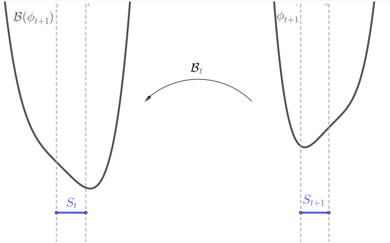

In order to compute $\mathcal{B}_\mathbf{t}\left(\phi_{\mathbf{t+1}}\right)$ restricted to $\mathbf{S_t}$, one only needs to know $\phi_{\mathbf{t+1}}$ restricted to $\mathbf{S_{t+1}}$.

Almost sure convergence towards V_t

Almost surely, the approximations $\left(V_t^k\right)_k$ converges uniformly to V_t^*, which is equal to V_t on a set of interest.

Theorem (Convergence of TDP A., Chancelier, Tran, 2018)

Define $K_t^* := \limsup_k \text{supp}\left(\mu_t^k\right)$*, for every time* $t \in [\![0, T]\!]$*. Assume that,* μ*-a.s the sets* $(K_t^*)_{t\in[\![0,T]\!]}$ *are*

- (V_t)*-optimal if* $\mathrm{opt} = \inf$*,*
- (V_t^*)*-optimal if* $\mathrm{opt} = \sup$*.*

Then, μ*-a.s. for every* $t \in [\![0, T]\!]$ *the function* V_t^* *is equal to the value function* V_t *on* K_t^**.*

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This is the usual convergence result for SDDP, new for a Min-Plus method

Deterministic linear-quadratic optimal control with one constrained control

Let β, γ be such that $\beta < \gamma$, we study the following Multistage convex optimization problem involving a constraint on one of the controls denoted by v:

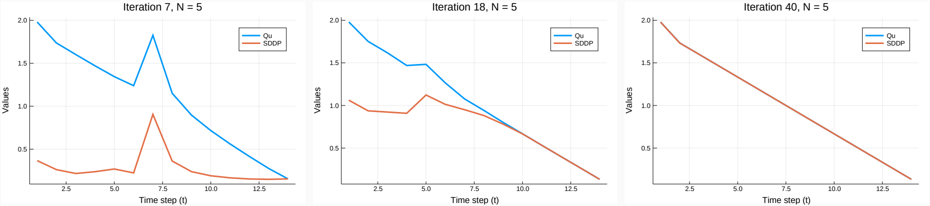
$$\min_{\substack{x=(x_0,\dots,x_T)\\ u=(u_0,\dots,u_{T-1})\\ v=(v_0,\dots,v_{T-1})}} \sum_{t=0}^{T-1} c_t(x_t, u_t, v_t) + \psi(x_T)$$

$$\text{s.t.} \begin{cases} x_0 \in \mathcal{X} \text{ is given}, \\ \forall t \in [\![0, T-1]\!],\ x_{t+1} = f_t(x_t, u_t, v_t) \\ \forall t \in [\![0, T-1]\!], (u_t, v_t) \in \mathcal{U} \times [\beta, \gamma], \end{cases}$$

where f_t is linear, c_t and ψ are convex quadratic.

Numerical illustration on a toy example: converging gap

The gap between upper and lower approximations converges to 0 along the current optimal trajectories of SDDP



- $d = 25$, $p = 3$, $[\beta, \gamma] = [-3, 5]$,
- Plots of $\overline{V}_t^k(x_t^k)$ and $\underline{V}_t^k(x_t^k)$ with t in abscisses
- After 7 iterations (left), 18 iterations (middle) and 40 iterations (right)
- Discretization of the control v to apply the Min-Plus algorithm (A., Chancelier, Tran, CDC 2019).

Converging upper and lower approximations along current optimal trajectories

- In SDDP algorithm for deterministic MSP, one can drawn the optimal trajectories x_t^k associated to the previous value functions V_t^k, to obtain the convergence.
- This is not enough to obtain the convergence of the Min-Plus method.

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- This is not enough to obtain the convergence of the Min-Plus method.
- One can use the optimal trajectories of SDDP to draw points both for the upper approximations and lower approximations.
- This should be extended to the stochastic case (with finite noise), by using a method of (Baucke, Downward and Zackeri 2018). (Work in progress).

5. A new probabilistic scheme for HJB equations (A., Fodjo, 2018)

The algorithm of Fahim, Touzi and Warin

Decompose the Hamiltonian $\mathcal{H}$ of HJB as $\mathcal{H} = \mathcal{L} + \mathcal{G}$ with

$$\mathcal{L}(x, r, p, \Gamma) := \frac{1}{2}\,\mathrm{tr}\,(a(x)\Gamma) + \underline{f}(x) \cdot p \ , \quad a(x) = \underline{\sigma}(x)\underline{\sigma}(x)^{\mathsf{T}} > 0 \ ,$$

and $\partial_\Gamma \mathcal{G} \geq 0$, for all $x \in \mathbb{R}^d$, $r \in \mathbb{R}, p \in \mathbb{R}^d, \Gamma \in \mathbb{S}_d$.

Theorem (Cheridito, Soner, Touzi and Victoir, 2007**)**

If v is the viscosity solution of HJB, X_t is the diffusion with generator $\mathcal{L}$:

$$dX_t = \underline{f}(X_t)dt + \underline{\sigma}(X_t)dW_t \ , \quad X_0 = x$$

then $Y_t = v(t, X_t)$, $Z_t = Dv(t, X_t)$ and $\Gamma_t = D^2 v(t, X_t)$ satisfy the second-order backward stochastic diff. eq.:

$$\begin{aligned} dY_t &= -\mathcal{G}(X_t, Y_t, Z_t, \Gamma_t)dt + Z_t^{\mathsf{T}}\,\underline{\sigma}(X_t)dW_t \\ dZ_t &= A_t dt + \Gamma_t dX_t \\ Y_T &= \psi(X_T) \ . \end{aligned}$$

Idea of the algorithm of Fahim, Touzi and Warin: after time discretization, simulate X_t, then apply a regression estimator to compute Y_t

Denote by $\hat{X}$ the Euler discretization of X_t:

$$\hat{X}(t+h) = \hat{X}(t) + \underline{f}(\hat{X}(t))h + \underline{\sigma}(\hat{X}(t))(W_t^h) \ .$$

The following is a time discretization of HJB:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h := \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$T_{t,h}(\phi)(x) = \mathcal{D}^0_{t,h}(\phi)(x) + h\mathcal{G}(x, \mathcal{D}^0_{t,h}(\phi)(x), \mathcal{D}^1_{t,h}(\phi)(x), \mathcal{D}^2_{t,h}(\phi)(x)) \ ,$$

and $\mathcal{D}^i_{t,h}(\phi)$ the approximation of the ith differential of $e^{h\mathcal{L}}\phi$ given by:

$$\begin{aligned}
\mathcal{D}^i_{t,h}(\phi)(x) :=& \mathbb{E}(D^i\phi(\hat{X}(t+h)) \mid \hat{X}(t) = x) \\
=& \mathbb{E}(\phi(\hat{X}(t+h))\mathcal{P}^i_{t,x,h}(W_t^h) \mid \hat{X}(t) = x), \quad i = 0,1,2 \ , \\
\mathcal{P}^0_{t,x,h}(w) =& 1 \ , \\
\mathcal{P}^1_{t,x,h}(w) =& (\underline{\sigma}(x)^{\mathsf{T}})^{-1} h^{-1} w \ , \\
\mathcal{P}^2_{t,x,h}(w) =& (\underline{\sigma}(x)^{\mathsf{T}})^{-1} h^{-2} (ww^{\mathsf{T}} - hI)(\underline{\sigma}(x))^{-1} \ .
\end{aligned}$$

Lemma (Fahim, Touzi and Warin, 2011)

When $\mathrm{tr}(a(x)^{-1}\partial_\Gamma \mathcal{G}) \leq 1$, $\partial_\Gamma \mathcal{G}$ is lower bounded by some > 0 matrix and $\mathcal{G}$ is Lipschitz continuous, $T_{t,h}$ is L-almost monotone on the set $\mathcal{F}$ of Lipschitz continuous functions $\mathbb{R}^d \to \mathbb{R}$, for some constant $L = O(h)$:

$$\phi, \psi \in \mathcal{F},\ \phi \leq \psi \implies T_{t,h}(\phi) \leq T_{t,h}(\psi) + L \sup(\psi - \phi)\ .$$

- Then Barles and Souganidis (90) $\Rightarrow$ convergence and error estimation of the time discretization scheme.
- Under these conditions, and given the convergence of the regression estimator approximating the $\mathcal{D}^i_{t,h}(\phi)$, the full Fahim, Touzi and Warin algorithm converges.
- Note that theoretically, the sample size necessary to obtain the convergence of the estimator is at least in the order of $1/h^{d/2}$. Also the dimension of the linear regression space should be in this order.

- The critical constraint $\mathrm{tr}(a(x)^{-1}\partial_\Gamma \mathcal{G}) \leq 1$ does not allow to handle variations in $\sigma^m(x,u)$.
- When discretizing W_t^h, the previous scheme becomes a finite difference space discretization with a small stencil.
- We change the polynomial function $\mathcal{P}^2$ so that the stencil becomes larger.
- We also change the polynomial $\mathcal{P}^1$ so that the discretization becomes similar to upwind discretization.

A monotone probabilistic scheme for $D^2 v$

- Let $\Sigma^m(x,u) \in \mathbb{R}^{d\times\ell}$ be such that

$$\sigma^m(x,u)\sigma^m(x,u)^\mathsf{T} - a(x) = \underline{\sigma}(x)\Sigma^m(x,u)\Sigma^m(x,u)^\mathsf{T}\,\underline{\sigma}(x)^\mathsf{T} \quad .$$

- For any $\Sigma \in \mathbb{R}^{d\times\ell}$, denote

$$\mathcal{P}^2_{\Sigma,k}(w) = \sum_{j=1}^{\ell} \|\Sigma_{.j}\|_2^2 \left(c_k \left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2} \right)^{4k+2} - d_k \right) \quad ,$$

with

$$c_k := \frac{1}{(4k+2)\mathbb{E}\left[N^{4k+2}\right]} \quad , \quad d_k := \frac{1}{4k+2} \quad , \quad N = N(0,1).$$

- Then change $\frac{1}{2}\operatorname{tr}((\sigma^m(x,u)\sigma^m(x,u)^\mathsf{T} - a(x))\mathcal{P}^2_{t,x,h}(w))$ into $h^{-1}\mathcal{P}^2_{\Sigma^m(x,u),k}(h^{-1/2}w)$
- and so $\frac{1}{2}\operatorname{tr}((\sigma^m(x,u)\sigma^m(x,u)^\mathsf{T} - a(x))\mathcal{D}^2_{t,h}(\phi)(x))$ into

$$\mathbb{E}(\phi(\hat{X}(t+h))h^{-1}\mathcal{P}^2_{\Sigma^m(x,u),k}(h^{-1/2}W^h_t) \mid \hat{X}(t) = x) \quad .$$

A monotone probabilistic scheme for $D^1 v$

- Let $g^m(x,u)$ be the d-dimensional vector such that

$$f^m(x,u) - \underline{f}(x) = \underline{\sigma}^m(x) g^m(x,u) \ .$$

- Denote

$$\mathcal{P}^1_g(w) := 2 \sum_{i=1}^{d} ((g_i)_+ (w_i)_+ + (g_i)_- (w_i)_-) \ ,$$

and

$$\mathcal{D}^1_{t,h,g}(\phi)(t,x) := \mathbb{E}\left[(\phi(t+h, \hat{X}(t+h)) - \phi(t,x)) \mathcal{P}^1_g(h^{-1}(W^h_t)) \mid \hat{X}(t) = x \right] \ .$$

- Then change $(f^m(x,u) - \underline{f}(x)) \cdot \mathcal{D}^1_{t,h}(\phi)(x)$ into $\mathcal{D}^1_{t,h,g^m(x,u)}(\phi)(t,x)$.

Lemma

Denote

$$T^N_{t,h,m,u}(\phi)(x) = h\ell^m(x,u) + \mathbb{E}\left[\phi(\hat{X}(t+h))\mathcal{P}^N_{h,t,m,u,x}(W^h_t) \mid \hat{X}(t) = x\right] \ .$$

with

$$\mathcal{P}^N_{h,t,m,u,x} = 1 + h\mathcal{P}^1_{g^m(x,u)}(h^{-1}w) + h^{-1}\mathcal{P}^2_{\Sigma^m(x,u),k}(h^{-1/2}w)$$

and

$$T^D_{t,h,m,u}(x) = 1 + h\delta^m(x,u) + h\mathbb{E}\left[\mathcal{P}^1_{g^m(x,u)}(h^{-1}(W^h_t)\right] \ .$$

If $\delta^m \geq 0$, or if δ^m is lower bounded and h is small enough, then $T^D_{t,h,m,u}(x) \geq 1/2$ for all $x \in \mathbb{R}^d$ and we can define $T_{t,h}$ as:

$$T_{t,h}(\phi)(x) = \sup_{m\in\ M,\ u\in\mathcal{U}} \frac{T^N_{t,h,m,u}(\phi)(x)}{T^D_{t,h,m,u}(x)} \ .$$

Moreover, the induced time discretization is equivalent to the recurrence equation:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h := \{0, h, 2h, \dots, T-h\} \ .$$

Theorem

Under suitable assumptions, the discretization is consistent, stable, monotone, and satisfies estimations.

So if v is the unique viscosity solution of the HJB equation, and v^h is the solution of the discretized equation with the initial condition $v^h(T, x) = \psi(x)$ for all $x \in \mathbb{R}^d$, we have, for all $(t, x) \in \{0, h, \ldots, T\} \times \mathbb{R}^d$,

$$-C_1 h^{1/10} \leq (v^h - v)(t, x) \leq C_2 h^{1/4} \ .$$

6. A probabilistic idempotent method for stochastic control problems (A., Fodjo, 2018)

The probabilistic max-plus method

The monotone probabilistic scheme can be written as

$$T_{t,h}(\phi)(x) = \sup_{m\in\ M,\ u\in\mathcal{U}} T_{t,h,m,u}(\phi)(x) \ ,$$

with

$$T_{t,h,m,u}(\phi)(x) = \mathbb{E}\left[c_h^m(x,u) + \phi(\hat{X}(t+h))\alpha_{h,t,m,u,x}(W_t^h) \mid \hat{X}(t) = x\right] \ ,$$

and $\alpha_{h,t,m,u,x}(W) \geq 0$.

Let $\mathcal{W} = \mathbb{R}^d$. Then

$$T_{t,h}(\phi)(x) = G_{t,h,x}(\tilde{\phi}_{t,h,x}) \qquad x \in \mathbb{R}^d \ ,$$

where

$$\tilde{\phi}_{t,h,x} = \phi(S_{t,h}(x,\cdot)) \ ,$$

$$S_{t,h} : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}^d,\ (x,W) \mapsto S_{t,h}(x,W) = x + \underline{f}(x)h + \underline{\sigma}(x)W \ ,$$

$$G_{t,h,x}(\tilde{\phi}) = \max_{m\in\mathcal{M},\ u\in\mathcal{U}} \left\{\mathbb{E}\left[c_h^m(x,u) + \tilde{\phi}\alpha_{h,t,m,u,x}(W_t^h)\right]\right\} \ .$$

Let $\mathcal{D}$ be the set of measurable functions from $\mathcal{W}$ to $\mathbb{R}$ with at most some given growth or growth rate. One can observe that

- $G_{t,h,x}$ is an operator from $\mathcal{D}$ to $\mathbb{R}$ and $\tilde{\phi}_{t,h,x} \in \mathcal{D}$ if $\phi \in \mathcal{D}$;
- The operator $G_{t,h,x}$ is monotone additively $(1 + Ch)$-subhomogeneous from $\mathcal{D}$ to $\mathbb{R}$, for $h \leq h_0$.
- Assume that $\mathcal{L}$ corresponds to a linear dynamics, then $x \mapsto \tilde{\phi}_{t,h,x}$ is a *random quadratic form* if ϕ is a quadratic form;
- Assume that $\mathcal{H}^m$ corresponds to a LQ problem, then

$$G_{t,h,x}(\tilde{\phi}) = \max_{m \in \mathcal{M}} G^m_{t,h,x}(\tilde{\phi})$$

  with

$$x \mapsto \tilde{\phi}_x \text{ random quadratic} \Longrightarrow G^m_{t,h,x}(\tilde{\phi}_x) \text{ quadratic.}$$

Theorem (A., Fodjo, 2016)

Let G be a monotone additively α-subhomogeneous operator from $\mathcal{D} \to \mathbb{R}$, for some constant $\alpha > 0$. Let $(Z, \mathfrak{A})$ be a measurable space, and let $\mathcal{W}$ be endowed with its Borel σ-algebra. Let $\phi : \mathcal{W} \times Z \to \mathbb{R}$ be a measurable map such that for all $z \in Z$, $\phi(\cdot, z)$ is continuous and belongs to $\mathcal{D}$. Let $v \in \mathcal{D}$ be such that $v(W) = \sup_{z \in Z} \phi(W, z)$. Assume that v is continuous and bounded. Then,

$$G(v) = \sup_{\bar{z} \in \overline{Z}} G(\bar{\phi}^{\bar{z}})$$

where $\bar{\phi}^{\bar{z}} : \mathcal{W} \to \mathbb{R}, \ W \mapsto \phi(W, \bar{z}(W))$, and

$$\overline{Z} = \{\bar{z} : \mathcal{W} \to Z, \ \text{measurable and such that } \bar{\phi}^{\bar{z}} \in \mathcal{D}\}.$$

This says that any monotone continuous map distributes over $\max$ and generalizes the max-plus distributivity.

Formally, we have $G(v) = G(\bar{\phi}^{\bar{z}})$, when $v(W) = \phi(W, \bar{z}(W))$.

Theorem (A., Fodjo, 2016, compare with McEneaney, Kaise and Han, 2011)

Assume that, for each $m \in \mathcal{M}$, δ^m and σ^m are constant, f^m is affine with respect to (x, u), ℓ^m is concave quadratic with respect to (x, u), and that ψ is the supremum of a finite number of concave quadratic forms.
Consider the monotone probabilistic scheme with $T_{t,h}$ as above.
Assume that the operators $G^m_{t,h,x}$ are monotone additively α_h-subhomogeneous from $\mathcal{D}$ to $\mathbb{R}$, for some constant $\alpha_h = 1 + Ch$ with $C \geq 0$.
Assume also that the value function v^h belongs to $\mathcal{D}$ and is locally Lipschitz continuous with respect to x.
Then, for all $t \in \mathcal{T}_h$, there exists a set Z_t and a map $q_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ such that for all $z \in Z_t$, $q_t(\cdot, z)$ is a concave quadratic form and

$$v^h(t, x) = \sup_{z \in Z_t} q_t(x, z) \ .$$

Moreover, the sets Z_t satisfy

$$Z_t = \mathcal{M} \times \{\bar{z}_{t+h} : \mathcal{W} \to Z_{t+h} \mid \text{Borel measurable}\} \ .$$

The probabilistic max-plus method: the sampling algorithm

- Apply the same idea as in the random idempotent method for stochastic control problems of Section 4, without sampling u and with a sampling of the states x_t^k obtained from the process $\hat{X}(t)$.

The probabilistic max-plus method: the sampling algorithm

Denote $q(x,z) := \frac{1}{2}x^{\mathsf{T}} Qx + b\cdot x + c$ for $z=(Q,b,c)\in\mathcal{Q}_d=\mathbb{S}_d^-\times\mathbb{R}^d\times\mathbb{R}$.
Input: $M=\#\mathcal{M}$, $\epsilon>0$, $Z_T\subset\mathcal{Q}_d$ such that $|\psi(x)-\max_{z\in Z_T} q(x,z)|\leq\epsilon$ and $\#Z_T\leq N_{\mathrm{in}}$,
$N=(N_{\mathrm{in}},N_x,N_w)$ (the numbers of samples with $N_x\leq N_{\mathrm{in}}$).
Output: $Z_t\subset\mathcal{Q}_d$, $t\in\mathcal{T}_h\cup\{T\}$, and $v^{h,N}$.
Initialization: Define $v^{h,N}(T,x)=\max_{z\in Z_T} q(x,z)$. Construct a sample of $(\hat{X}(0),(W_t^h)_{t\in\mathcal{T}_h})$ of size N_{in} indexed by $\omega\in\Omega_{N_{\mathrm{in}}}$, and deduce $\hat{X}(t,\omega)$.

For $t=T-h,T-2h,\dots,0$ *do*

1. Construct independent subsamples of sizes N_x and N_w of $\Omega_{N_{\mathrm{in}}}$, then take the product of samplings,
   leading to $(\omega_\ell,\omega'_\ell)$ for $\ell\in\Omega_{N_{\mathrm{rg}}}:=[N_x]\times[N_w]$.
   Induce the sample $\hat{X}(t,\omega_\ell)$ (resp. $(W_t^h)(\omega'_\ell)$) for $\ell\in\Omega_{N_{\mathrm{rg}}}$ of $\hat{X}(t)$ (resp. W_t^h).
   Denote by $\mathcal{W}_t^N\subset\mathcal{W}$ the set of $(W_t^h)(\omega'_\ell)$ for $\ell\in\Omega_{N_{\mathrm{rg}}}$.

The probabilistic max-plus method: the sampling algorithm cont.

2. For each $\omega \in \Omega_{N_{\rm in}}$ denote $x_t = \hat{X}(t,\omega)$.
   (a) Choose $\bar{z}_{t+h} : \mathcal{W}_t^N \to Z_{t+h}$ such that, for all $\ell \in \Omega_{N_{\rm rg}}$, we have
   $$\bar{z}_{t+h}((W_t^h)(\omega'_\ell)) \in \operatorname*{Argmax}_{z\in Z_{t+h}} q\big(S_{t,h}(x_t,(W_t^h)(\omega'_\ell)), z\big) \quad .$$
   Let $\tilde{q}_{t,h,x}$ be the element of $\mathcal{D}$ given by $W \in \mathcal{W} \mapsto q(S_{t,h}(x,W), \bar{z}_{t+h}(W))$.
   (b) For each m, approximate $x \mapsto G^m_{t,h,x}(\tilde{q}_{t,h,x})$ by a linear regression estimation on the set of quadratic forms using the sample $(\hat{X}(t,\omega_\ell),(W_t^h)(\omega'_\ell))$, with $\ell \in \Omega_{N_{\rm rg}}$, and denote by $z_t^m \in \mathcal{Q}_d$ the parameter of the resulting quadratic form.
   (c) Choose $z_t \in \mathcal{Q}_d$ optimal among the $z_t^m \in \mathcal{Q}_d$ at the point x_t, that is such that $q(x_t,z_t) = \max_{m\in\mathcal{M}} q(x_t,z_t^m)$.
3. Denote by Z_t the set of the parameters $z_t \in \mathcal{Q}_d$ obtained in this way, and define
   $$v^{h,N}(t,x) = \max_{z\in Z_t} q(x,z) \quad \forall x \in \mathbb{R}^d \quad .$$

The probabilistic max-plus method: the sampling algorithm cont.

Computational time:

$$O(d^2 N_{\mathrm{in}}^2 \times N_w + d^3 M \times N_{\mathrm{in}} \times N_x \times N_w)$$

where the first term corresponds to step (a) and the second one to steps (b) and (c).

Note also that N_x can be choosen to be in the order of a polynomial in d since the regression is done on the set of quadratic forms,
so in general the second term is negligeable.

A Finance example

Problem: *pricing and hedging an option with uncertain volatility and several underlying stock processes.*

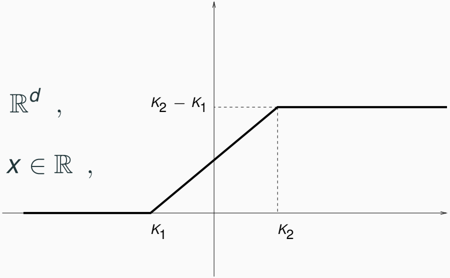
- *The dynamics:* $d\xi_i = \sigma_i \xi_i dB_i$, where the Brownians B_i have uncertain correlations: $\langle dB_i, dB_j \rangle = \mu_{i,j} ds$.
- We know: $\mu \in \mathrm{cvx}(\mathcal{M})$ with $\mathcal{M}$ a finite set.
- *Maximize*

$$J(t,x,\mu) := \quad \mathbb{E}\left[\psi(\xi(T)) \mid \xi(t) = x\right] \quad , \quad \text{with}$$

$$\psi(x) = \phi(\max_{i \text{ odd}} x_i - \min_{j \text{ even}} x_j), \quad x \in \mathbb{R}^d \quad ,$$

$$\phi(x) = (x - K_1)^+ - (x - K_2)^+, \quad x \in \mathbb{R} \quad ,$$

$$x^+ = \max(x, 0), \quad K_1 < K_2 \quad .$$



A Finance example

- Since the dynamics is linear, we can reduce to $\mu_s \in \mathcal{M}$.
- The parameters with respect to the previous model: $\mathcal{M}$ is a finite subset of the set of positive definite symmetric matrices with 1 on the diagonal and

$$\begin{aligned} f^m &= 0 \\ \delta^m &= 0 \\ \ell^m &= 0 \\ [\sigma^m(\xi)\sigma^m(\xi)^\mathsf{T}]_{i,j} &= \sigma_i \xi_i \sigma_j \xi_j \mu_{i,j} \quad . \end{aligned}$$

- Proposed with 2 stocks in Kharroubi, Langrené, Pham (2013) and solved using randomized control+regression.
- Solved in dimension 2 in A., Fodjo (CDC 2016) with a probabilistic max-plus method.
- In both cases: $\sigma_1 = 0.4,\ \sigma_2 = 0.3,\ K_1 = -5,\ K_2 = 5,\ T = 0.25$, and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} \\ m_{12} & 1 \end{bmatrix} \mid m_{12} = \pm\rho\} \qquad \rho = 0.8 \quad .$$

The pricing and hedging an option example

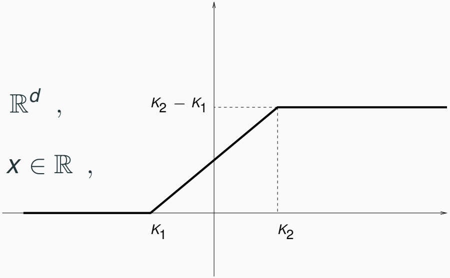
- *The dynamics:* $d\xi_i = \sigma_i \xi_i dB_i$, where the Brownians B_i have uncertain correlations: $\langle dB_i, dB_j \rangle = \mu_{i,j} ds$.
- We know: $\mu \in \mathrm{cvx}(\mathcal{M})$ with $\mathcal{M}$ a finite set.
- *Maximize*

$$J(t, x, \mu) := \quad \mathbb{E}\left[\psi(\xi(T)) \mid \xi(t) = x\right] \ , \quad \text{with}$$

$$\psi(x) = \phi(\max_{i \text{ odd}} x_i - \min_{j \text{ even}} x_j), \quad x \in \mathbb{R}^d \ ,$$

$$\phi(x) = (x - K_1)^+ - (x - K_2)^+, \quad x \in \mathbb{R} \ ,$$

$$x^+ = \max(x, 0), \quad K_1 < K_2 \ .$$



- $\mathcal{M}$ is a finite subset of the set of positive definite symmetric matrices with 1 on the diagonal and

$$[\sigma^m(\xi)\sigma^m(\xi)^\mathsf{T}]_{i,j} = \sigma_i\xi_i\sigma_j\xi_j\mu_{i,j} \ .$$

- We take $K_1 = -5$, $K_2 = 5$, $T = 0.25$, and $h = 0.01$.
- In dimension 2, we take $\sigma = (0.4, 0.3)$, and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} \\ m_{12} & 1 \end{bmatrix} \mid m_{12} = \pm\rho\} \ .$$

- In dimension 5, we take $\sigma = (0.4, 0.3, 0.2, 0.3, 0.4)$ and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} & 0 & 0 & 0 \\ m_{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & m_{45} \\ 0 & 0 & 0 & m_{45} & 1 \end{bmatrix} \mid m_{12} = \pm\rho,\ m_{45} = \pm\rho\} \ .$$

- We tested the cases $\rho = 0$, $\rho = 0.4$ and 0.8.

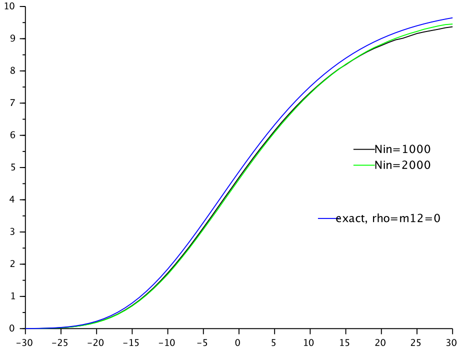


Figure 2: Value function obtained at $t = 0$, and $x_2 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0$, $N_{\text{in}} = 1000$, or 2000, $N_x = 10$, $N_w = 1000$.

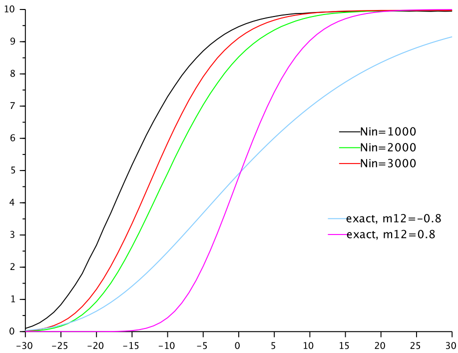


Figure 3: Value function obtained at $t = 0$, and $x_2 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0.8$, $N_{\text{in}} = 1000$, or 2000 or 3000, $N_x = 10$, $N_w = 1000$.

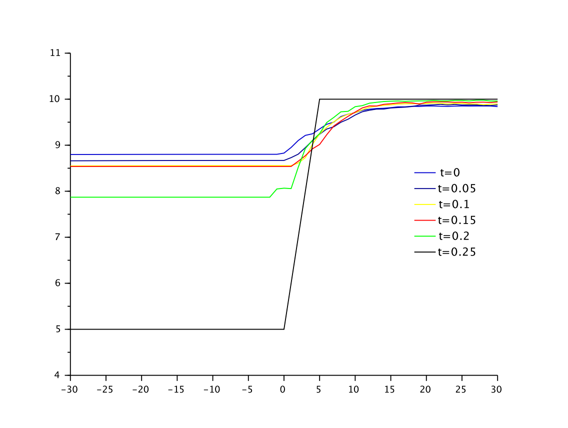


Figure 4: Value function obtained in dimension 5 at $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0.8$, $N_{\text{in}} = 3000$, $N_x = 50$, $N_w = 1000$. The time by time iteration is $\simeq 2500s$ and the total time is $\simeq 19h$ on a 12 core.

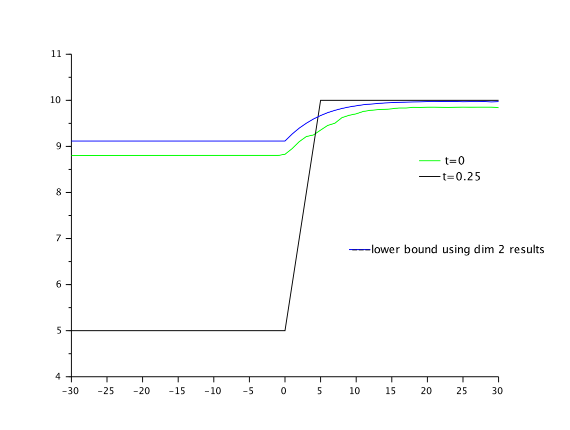


Figure 5: Comparizon between the value function obtained in dimension 5 at $t = 0$, and $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$, and a lower bound from the dimension 2. Here $\rho = 0.8$, $N_{\text{in}} = 3000$, $N_x = 50$, $N_w = 1000$.

Conclusion

- We proposed several “random” algorithms to solve HJB equations and Multistage stochastic programming problems, combining ideas included in the idempotent algorithm of McEneaney(2007), Zheng Qu (2014), McEneaney, Kaise and Han (2011), the probabilistic numerical scheme of Fahim, Touzi and Warin (2011) and the SDDP algorithm.
- The advantages with respect to the pure probabilistic scheme are that no regression is done or the regression estimation is over a linear space of small dimension.
- The advantages with respect to the pure idempotent scheme is that one avoid the pruning step: the number of quadratic forms generated by the algorithm is linear with respect to the sampling size times the number of discrete controls.
- The advantage with respect to SDDP algorithm is that we do not need the convexity of the value function.
- We improved the probabilistic numerical scheme of Fahim, Touzi and Warin (2011) to obtain a monotone scheme and so apply the probabilistic max-plus method in general situations.
- The theoretical results suggest that it can also be applied to Isaacs equations of zero-sum games.

Open

- Improve the optimization step to decrease the complexity.
- Find a “SDDP” algorithm in the non convex case, to obtain a“lower” approximation and also a way to compute the value function only along an optimal trajectory.

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Good health to all