Abnormal Hamilton-Jacobi Equations arising in Infinite Horizon Problems

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Infinite Horizon Optimal Control Problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

over **(viable)** trajectory-control pairs (x, u),
subject to the state equation and state constraint

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \geq t_0 \\ x(t_0) = x_0, & x(t) \in K \quad \text{for all } t \geq t_0 \end{cases}$$

$U : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^m$ is measurable with closed $\neq \emptyset$ values,
$L : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are Carathéodory,
$K \subset \mathbb{R}^n$ is closed, $x_0 \in K$,
Assume L is bounded from below by an **integrable** on $\mathbb{R}_+$ function.

Controls $u(t) \in U(t)$ are **Lebesgue measurable** selections.
Set $V(t_0, x_0) = +\infty$ if there is no viable (feasible) trajectory.

Classical Problem

A discounted infinite horizon optimal control problem

$$W(x_0) = \text{minimize} \int_0^\infty e^{-\lambda t} \ell(x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u) subject to

$$\begin{cases} x'(t) = f(x(t), u(t)), & u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0, & x(t) \in K \quad \text{for all } t \geq 0 \end{cases}$$

Controls $u(\cdot)$ are Lebesgue measurable, $\lambda > 0$, $K = \overline{\Omega}$, Ω-open.

The economic literature addressing this problem deals with traditional questions of **existence** of optimal solutions, **regularity** of W, **necessary and sufficient** optimality conditions.
A. Seierstad and K. Sydsaeter,
Optimal control theory with economic applications, 1986.

Stationary Hamilton-Jacobi Equation

Under some technical assumptions W is the unique **lower semicontinuous solution** of the Hamilton-Jacobi equation

$$\lambda W(x) + H(x, -\nabla W(x)) = 0,$$

where $H(x,p) = \sup_{u \in U} \left(\langle p, f(x,u) \rangle - \ell(x,u) \right)$ in the sense:

$$\lambda W(x) + H(x, -p) = 0 \quad \forall\, p \in \partial^- W(x), \;\; x \in \Omega$$

$$\lambda W(x) + H(x, -p) \geq 0 \quad \forall\, p \in \partial^- W(x), \;\; x \in \partial\Omega$$

$$\lambda W(x) + \sup_{-f(x,u) \in \mathrm{Int}\, C_K(x),\, u \in U} \left(\langle p, f(x,u) \rangle - \ell(x,u) \right) \leq 0, \; x \in \partial\Omega$$

$\forall\, p \in \partial^- W(x)$, where $\partial^- W(x)$ is the **Fréchet subdifferential** of W at x and $C_K(x)$ - Clarke tangent cone. HF and Plaskacz 1999.
Earlier results by Soner 1986,
with smooth compact **state constraint** and BUC solutions.

Fréchet Subdifferential

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $x \in \mathbb{R}^n$, $\varphi(x) \neq +\infty$.
$\partial^-\varphi(x)$ - **Fréchet subdifferential** of φ at $x \in \mathrm{dom}(\varphi)$.

$$p \in \partial^-\varphi(x) \iff \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle p, y - x \rangle}{|y - x|} \geq 0$$

For $Q \subset \mathbb{R}^n$ and $x \in Q$ the **contingent** (Peano) cone to Q at x

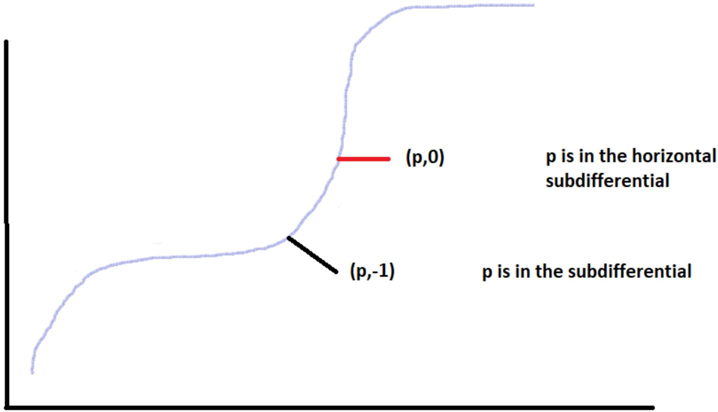
$$T_Q(x) := \left\{ u \in X \mid \liminf_{\varepsilon \to 0+} \frac{\mathrm{dist}(x + \varepsilon u, Q)}{\varepsilon} = 0 \right\}$$

$$p \in \partial^-\varphi(x) \iff (p, -1) \in [T_{\mathrm{epi}\,\varphi}(x, \varphi(x))]^-$$

epi φ – epigraph of φ

If $(p, q) \in [T_{\mathrm{epi}\,\varphi}(x, \varphi(x))]^-$ and $q \neq 0$, then $\frac{p}{|q|} \in \partial^-\varphi(x)$.

Horizontal Subdifferentials



Uniqueness for HJB Equation, HF 2020, AME

Under some technical assumptions V is the unique **locally Lipschitz solution** of the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t}(t,x) + H(t,x,-\frac{\partial V}{\partial x}(t,x)) = 0,$$

where $H(t,x,p) = \sup_{u \in U(t)} \left(\langle p, f(t,x,u)\rangle - L(t,x,u)\right)$, satisfying the final condition

$$\lim_{t\to\infty} \sup_{y \in K} |V(t,y)| = 0$$

in the following sense: for a.e. $t > 0$ and for all $x \in \text{Int } K$

$$-p_t + H(t,x,-p_x) = 0 \quad \forall\, (p_t,p_x) \in \partial^- V(t,x)$$

and for all $x \in \partial K$ (boundary of K)

$$-p_t + H(t,x,-p_x) \geq 0 \quad \forall\, (p_t,p_x) \in \partial^- V(t,x)$$

Outline

Assumptions (H1)

i) $\exists\, c > 0$ such that for a.e. $t \geq 0$

$$|f(t,x,u)| \leq c(|x|+1), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

ii) $\exists\, c_1 > 0$ such that for a.e. $t \geq 0$, $\forall\, x, y \in \mathbb{R}^n$, $\forall\, u \in U(t)$

$$|f(t,x,u) - f(t,y,u)| + |L(t,x,u) - L(t,y,u)| \leq c_1|x-y|;$$

iii) $\sup_{u \in U(t)} L(t,0,u)$ is locally integrable;

iv) For a.e. $t \geq 0$, $\forall\, x \in \mathbb{R}^n$ the set $F(t,x)$ is closed and convex

$$F(t,x) := \{(f(t,x,u), L(t,x,u)+r) \,:\, u \in U(t) \text{ and } r \geq 0\}$$

Existence and Lower Semicontinuity of Value

Fix t_0, x_0 with $V(t_0, x_0) < +\infty$. A viable trajectory-control pair $(\bar{x}, \bar{u})$ is called **optimal** for the infinite horizon problem at (t_0, x_0) if for every viable trajectory-control pair (x, u) with $x(t_0) = x_0$

$$\int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt \leq \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

Proposition

Assume (H1). Then V is lower semicontinuous and for every $(t_0, x_0) \in dom\ V$, there exists a viable in K trajectory-control pair $(\bar{x}, \bar{u})$ satisfying $V(t_0, x_0) = \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt$.

Finite Horizon Bolza Problem

Question: Can the infinite horizon problem be seen as limit of finite horizon **Bolza** type optimal control problems when $T \to \infty$

$$\inf \int_0^T L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \in [0, T] \\ x(0) = x_0 & x(t) \in K \quad \forall\, t \in [0, T] \end{cases}$$

If $(\bar{x}, \bar{u})$ is optimal for the infinite horizon problem with $x(0) = x_0$, then, in general, its restriction to the time interval $[0, T]$ is not optimal for the above Bolza problem.

Maximum Principle for the Bolza Problem

$$H(t,x,p) := \sup_{u \in U(t)} \left(\langle p, f(t,x,u) \rangle - L(t,x,u) \right)$$

Case without state constraint.

If $(\bar{x}, \bar{u})$ is optimal for the Bolza problem, then, under mild assumptions, the solution $p : [0,T] \to \mathbb{R}^n$ of the **adjoint system**

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)), \quad p(T) = 0$$

satisfies the **maximality condition** for a.e. $t \in [0,T]$

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t))$$

If restrictions of an optimal pair $(\bar{x}, \bar{u})$ for the infinite horizon problem to $[0,T]$, $T > 0$ were optimal also for the Bolza problems, then we could pass to the limit when $T \to \infty$ and get the maximum principle also for the **infinite horizon problem**.

Maximum Principle for the Infinite Horizon Problem

If $(\bar{x}, \bar{u})$ is optimal, then $\exists\ p_0 \in \{0, 1\}$ and a locally absolutely continuous $p : [0, \infty[\to \mathbb{R}^n$ with $(p_0, p) \neq 0$, solving the **adjoint system**

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - p_0 L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{for a.e.} \quad t \geq 0$$

and satisfying the **maximality condition**

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - p_0 L(t, \bar{x}(t), \bar{u}(t)) =$$

$$\max_{u \in U(t)} (\langle p(t), f(t, \bar{x}(t), u) \rangle - p_0 L(t, \bar{x}(t), u)) \quad \text{for a.e.} \quad t \geq 0$$

If $p_0 = 0$ this maximum principle (MP) is called abnormal.

Transversality condition like $\lim_{t \to \infty} p(t) = 0$ is, in general, absent, cf. Halkin 1974 for a counterexample.

Main Differences with the Finite Horizon Case

Even in the absence of state constraint

- The maximum principle may be **abnormal**
- Transversality conditions are absent :
  some authors, under appropriate assumptions, obtain a **transversality condition** at infinity in the form

$$\lim_{t\to\infty} p(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \langle p(t), \bar{x}(t)\rangle = 0$$

  However they are a consequence of the growth assumptions on f, L.

Main difficulty behind : Restriction of an optimal solution to a finite time interval is no longer optimal.

Reduction to the Bolza Problem with Finite Horizon

Introducing $g_T(y) := V(T, y)$ we get, using the **dynamic programming principle**, the **Bolza** type problem

$$V^B(t_0, x_0) := \inf \left(g_T(x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt \right)$$

over all trajectory-control pairs (x, u) subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0, & x(t) \in K \quad \text{for all } t \in [t_0, T] \end{cases}$$

Under assumptions (H1) $i) - iii)$, $V^B(s_0, y_0) = V(s_0, y_0)$ for all $s_0 \in [0, T]$, $y_0 \in K$. Furthermore, if $(\bar{x}, \bar{u})$ is optimal for the infinite horizon problem at (t_0, x_0) then the restriction of $(\bar{x}, \bar{u})$ to $[t_0, T]$ is optimal for the above Bolza problem.

Absolute Continuity of Maps

A set-valued map $P : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^k$ is **locally absolutely continuous** if it takes nonempty closed images and for any $[S, T] \subset \mathbb{R}_+$, $\varepsilon > 0$, and any compact $Q \subset \mathbb{R}^k$, $\exists\, \delta > 0$ such that for any finite partition $S \leq t_1 < \tau_1 \leq t_2 < \tau_2 \leq ... \leq t_m < \tau_m \leq T$ of $[S, T]$,

$$\sum_{i=1}^{m}(\tau_i - t_i) < \delta \quad \Longrightarrow \quad \sum_{i=1}^{m} \max\{d_{P(t_i)}(P(\tau_i) \cap Q), d_{P(\tau_i)}(P(t_i) \cap Q)\} < \varepsilon$$

where $d_E(E') := \inf\{r > 0 \;:\; E' \subset E + rB\}$ for any $E, E' \subset \mathbb{R}^k$ (the infimum over an empty set is $= +\infty$).

Absolute Continuity of the Epigraph of Value

Lemma

If (H1) holds and for a.e. $t \geq 0$

$$-f(t,x,U(t)) \cap \overline{co}\, T_K(x) \neq \emptyset \quad \forall\, x \in \partial K,$$

then $t \rightsquigarrow \mathrm{epi}\, V(t,\cdot)$ is locally absolutely continuous.

Define the abnormal Hamiltonian

$$\mathcal{H}(t,x,p,q) := \sup_{u \in U(t)} \left(\langle p, f(t,x,u) \rangle - qL(t,x,u) \right)$$

Weak Solutions to HJ equation

A function $W : \mathbb{R}_+ \times K \to \mathbb{R} \cup \{+\infty\}$ is called a weak solution of HJ equation on $(0, \infty) \times K$ if $t \rightsquigarrow \mathrm{epi}\, W(t, \cdot)$ is locally absolutely continuous and there exists a set $A \subset (0, \infty)$, with $\mu(A) = 0$ such that for all $(t, x) \in \mathrm{dom}(W) \cap ((0, \infty) \times \mathrm{Int}\ K)$, $t \notin A$

$$-p_t + \mathcal{H}(t, x, -p_x, -q) = 0 \quad \forall\, (p_t, p_x, q) \in [T_{\mathrm{epi}\, W}(t, x, W(t, x))]^-$$

and $\forall\, (t, x) \in \mathrm{dom}(W) \cap ((0, \infty) \times \partial K)$, $t \notin A$

$$-p_t + \mathcal{H}(t, x, -p_x, -q) \geq 0 \quad \forall\, (p_t, p_x, q) \in [T_{\mathrm{epi}\, W}(t, x, W(t, x))]^-$$

Outward Pointing Condition

If f, U are continuous and bounded, K is bounded and $\partial K \in C^1$, then (OPC) : $\exists\ r > 0\ \forall\ t \in \mathbb{R}_+, \forall\ x \in \partial K, \exists\ u \in U(t)$

$$\langle n_x, f(t,x,u)\rangle \geq r$$

where n_x is the unit outward normal to K at x.

In the general case (OPC) becomes: $\exists\ \eta > 0$, $r > 0$, $M \geq 0$ such that for a.e. $t > 0$ and any $y \in \partial K + \eta B$, and any $v \in f(t,y,U(t))$, with $\min_{n \in N^1_{y,\eta}} \langle n, v\rangle \leq 0$, we can find $w \in\ f(t,y,U(t)) \cap B(v,M)$ satisfying

$$\min_{n \in N^1_{y,\eta}} \{\langle n, w\rangle, \langle n, w-v\rangle\} \geq r$$

where $N^1_{y,\eta} := \{n \in N^1_K(x)\ :\ x \in \partial K \cap B(y,\eta)\}$,
$N^1_K(x) := N_K(x) \cap S^{n-1}$
and $N_K(x)$ denotes the Clarke normal cone to K at x.

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$$\min_{n \in N^1_{y,\eta}} \{\langle n, w\rangle, \langle n, w - v\rangle\} \geq r$$

where $N^1_{y,\eta} := \{n \in N^1_K(x) \ : \ x \in \partial K \cap B(y,\eta)\}$,
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and $N_K(x)$ denotes the Clarke normal cone to K at x.

Uniqueness of Weak Solutions to HJ Equation

Theorem (V. Basco, HF. 2019) Assume (OPC) and (H1) and that for an uniformly integrable $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and a.e. $t > 0$

$$\sup_{u \in U(t)} \left(|f(t,x,u)| + |L(t,x,u)| \right) \leq \gamma(t) \qquad \forall x \in \partial K.$$

Let $W : \mathbb{R}_+ \times K \to \mathbb{R} \cup \{+\infty\}$ be lsc such that for all large $t > 0$, $\operatorname{dom} V(t,\cdot) \subset \operatorname{dom} W(t,\cdot) \neq \emptyset$ and

$$\lim_{t \to \infty} \sup_{y \in \operatorname{dom} W(t,\cdot)} |W(t,y)| = 0. \qquad (*)$$

Then the following statements are equivalent:
(i) W is a weak solution of (HJ) equation on $(0,\infty) \times K$;
(ii) W=V.

Uniqueness of Weak Solutions

Corollary

Under the same assumptions, suppose that $\mathrm{dom}\, V \neq \emptyset$ *and* $\exists\, T > 0$ *and* $\psi \in L^1([T, \infty); \mathbb{R}_+)$ *such that* $|L(t, x, u)| \leq \psi(t)$ *for a.e.* $t \geq T$ *and all* $x \in K,\ u \in U(t)$.

Then V *is a weak solution of (HJ) satisfying*

$$\lim_{t \to \infty} \sup_{y \in \mathrm{dom}\, V(t, \cdot)} |V(t, y)| = 0.$$

(we say $V(t, \cdot)$ *vanishes at infinity)*
and for any lsc weak solution $W : \mathbb{R}_+ \times K \to \mathbb{R}$ *of (HJ) satisfying* $\lim_{t \to \infty} \sup_{y \in K} |W(t, y)| = 0$ *we have* $W = V$.

Assumptions for Lipschitz Continuity of $V(t,\cdot)$

We denote by (H2) the following assumptions

- for some $\lambda > 0$, $L(t,x,u) = e^{-\lambda t}\ell(t,x,u)$
- $\{(f(t,x,u),\ell(t,x,u)) \,:\, u \in U(t)\}$ is closed $\forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$
- $\sup\{|f(t,x,u)| + |\ell(t,x,u)| : u \in U(t), (t,x) \in \mathbb{R}_+ \times \partial K\} < \infty$
- for some uniformly integrable $k : \mathbb{R}_+ \to \mathbb{R}$
  $(f(t,\cdot,u),\ell(t,\cdot,u))$ is $k(t)$-Lipschitz for a.e. $t \in \mathbb{R}_+$, $\forall\, u \in U(t)$
- for some locally integrable $c : \mathbb{R}_+ \to \mathbb{R}_+$ and all $x \in \mathbb{R}^n$

$$\sup\{|f(t,x,u)| + |\ell(t,x,u)| \,:\, u \in U(t)\} \leq c(t)(1+|x|)$$

- $\limsup_{t\to\infty} \frac{1}{t}\int_0^t (c(s)+k(s))\,ds < \infty$

Lipschitz Continuity of $V(t,\cdot)$

(IPC) $\exists\, \eta > 0,\ r > 0$ such that for a.e. $t \in \mathbb{R}_+$, $\forall\, y \in \partial K + \eta B$, $\forall\, v \in f(t,y,U(t))$ with $\max_{n \in N^1_{y,\eta}} \langle n, v\rangle \geq 0$, there exists $w \in f(t,y,U(t))$ such that

$$\max_{n \in N^1_{y,\eta}} \{\langle n, w\rangle, \langle n, w - v\rangle\} \leq -r,$$

where $N^1_{y,\eta} := \{n \in N^1_K(x) \ :\ x \in \partial K \cap B(y,\eta)\}$.

Theorem

If (H2) and (IPC) hold, then there exist $b > 1$, $C > 0$ such that for all $\lambda > C$ and every $t \geq 0$ the function $V(t,\cdot)$ is $\gamma(t)$-Lipschitz continuous on K with $\gamma(t) = be^{-(\lambda - C)t}$

Open Loop Optimal Controls

Consider the **augmented control system** with state constraint

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & x(0) = x_0 \\ z'(t) = -L(t, x(t), u(t)), & z(0) = V(0, x_0) \\ u(t) \in U(t) \ \text{ for a.e. } \ t \geq 0 & \\ (t, x(t), z(t)) \in \operatorname{epi} V \ \text{ for all } t \geq 0 & \end{cases}$$

Theorem

Assume $V(0, x_0) \neq \infty$ and $V(t, \cdot)$ vanishes at infinity.
Let $(\bar{x}, \bar{u})$ solve the above augmented constrained system.
Then $\bar{u}$ is optimal for the infinite horizon problem at $(0, x_0)$.

Applying Differential Inclusions

Consider the **differential inclusion** with state constraint

$$\begin{cases} (x'(t), z'(t)) \in G(t, x(t), z(t)) \\ x(0) = x_0, \quad z(0) = V(0, x_0) \\ (t, x(t), z(t)) \in \operatorname{epi} V \ \text{ for all } t \geq 0, \end{cases}$$

with $G(t, x, z) = \{(f(t, x, u), -L(t, x, u) - r) \ : \ u \in U(t), \ r \geq 0\}$.

Theorem

Assume $V(0, x_0) \neq \infty$ and $V(t, \cdot)$ vanishes at infinity. Let $(\bar{x}, z)$ solve the above constrained differential inclusion. Then $r(\cdot) = 0$ and for a measurable $\bar{u}(t) \in U(t)$ we have

$$(\bar{x}'(t), z'(t)) = (f(t, \bar{x}(t), \bar{u}(t)), -L(t, \bar{x}(t), \bar{u}(t))$$

Moreover $\bar{u}$ is optimal for the infinite horizon problem at $(0, x_0)$.

Set-Valued Optimal Feedback

For every $t \geq 0$ and $x \in \mathbb{R}^n$ define the **set-valued feedback map** :

$$C(t,x) := \{u \in U(t) : (1, f(t,x,u), -L(t,x,u)) \in T_{\mathrm{epi}\, V}(t,x,V(t,x))\}$$

Theorem

Let $(\bar{x}, \bar{u})$ be optimal for the infinite horizon problem at $(0, x_0)$. Then $\bar{u}(t) \in C(t, \bar{x}(t))$ for a.e. $t > 0$. Furthermore, if V ***is locally Lipschitz*** *on $\mathbb{R}_+ \times K$ and and $V(t,\cdot)$ vanishes at infinity, then every trajectory-control pair $(\bar{x}, \bar{u})$ of*

$$\begin{cases} x'(t) = f(t,x(t),u(t)), & x(0) = x_0 \\ u(t) \in C(t,x(t)) & \text{for a.e. } t \geq 0 \end{cases}$$

is optimal for the infinite horizon problem at $(0, x_0)$.

Conclusions

- When data are discontinuous in time, there is a need to consider the abnormal Hamiltonian

$$\mathcal{H}(t,x,p,q) := \sup_{u \in U(t)} \left(\langle p, f(t,x,u) \rangle - qL(t,x,u) \right)$$

  to investigate uniqueness of solutions to HJB equation.
- The inward and outward pointing conditions can not be removed - it is possible to write down counter-examples to theorems.
- Open loop controls and augmented constrained control system is a good alternative to the optimal feedback map, since no additional regularity of V is requested.

Wishing a good health to all you guys.

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