Topological approach to 13th Hilbert problem

Workshop "Braids, Resolvent Degree and Hilberts 13th Problem"

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1.13-th HILBERT'S PROBLEM

The general degree n algebraic function $x(a_0, \ldots, a_{n-1})$ is the solution of the equation $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$.

Problem 1 (D. Hilbert)

Find the smallest $H(n)$ such that x can be represented by composition of algebraic functions of $H(n)$ variables. Which functions of n variables can be represented by composition of algebraic functions of $m < n$ variables?

Actually, the problem on compositions was formulated by Hilbert for continuous functions, not for algebraic functions.

Theorem 2 (A.N.Kolmogorov and V.I. Arnold, 1957)

Any continuous function of n variables can be represented as the composition of functions of a single variable with the help of addition.

2. NOMOGRAPHY

13-th Hilbert's problem was first presented in the context of **nomographic construction** – a process by which a function of several variables is constructed using functions of two variables.

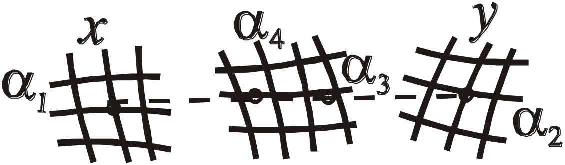

Figure: A nomographic construction of a relation between 6 variables

3. KNOWN RESULTS

Theorem 3 (D.Hilbert)

Generic analytic function of n variables can not be represented as a composition of analytic functions of fewer than n variables.

Theorem 4 (A.G. Vitushkin)

Generic function of n variables variables of smoothness p can not be represented as a composition of functions of m variables of smoothness r for which $m/r < n/p$.

Algebraic results.

1) if $n = 2, 3, 4$ then $H(n) = 1$ (equations of degree < 5 are solvable by radicals);
2) $H(5) = 1$; if $n > 5$ then $H(n) \leq (n - 4)$;
3) if $n > 8$ then $H(n) \leq (n - 5)$;
5) for any m there is N such that $H(n) \leq (n - m)$ for $n > N$. □

4. ALGEBRAIC FUNCTIONS OF ONE VARIABLE

In Kolmogorov-Arnold Theorem one can not replace continuous functions by entire algebraic functions.

Theorem 5 (Kh.,1969)

If an algebraic function can be represented as a composition of polynomials and entire algebraic functions of one variable, then its local monodromy group at each point is solvable.

Sketch of the proof.

At each point the local monodromy group of an algebraic function of one variable is a cyclic group. The operation of division that destroys locality is not an allowed operation in Theorem 5. Now Theorem 5 follows from the Galois theory type arguments. □

5. COROLLARY AND OPEN PROBLEM

Corollary 6

The function $y(a, b)$, defined by equation $y^5 + ay + b = 0$, cannot be expressed in terms of entire algebraic functions of a single variable by means of composition, addition and multiplication, since its local monodromy group at the origin is the unsolvable group $S(5)$ of all permutations of five elements.

It is easy to see that $y(a, b) = g(b/\sqrt[4]{a^5})\sqrt[4]{a}$, where $g(u)$ is defined by equation $g^5 + g + u = 0$.

Problem 7 (still open !)

Show that there is an algebraic function of two variables which cannot be expressed in terms of algebraic functions of a single variable by means of composition and arithmetic operations.

6. KLEIN'S RESOLVEN PROBLEM AND TOPOLOGY

KLEIN'S RESOLVENTS PROBLEM .

Find the smallest $K(n)$ such that the general degree n algebraic function can be represented as a composition of rational functions and one algebraic function of $K(n)$ variables. □

Theorem 8 (Buhler, Reichstein, based on an essential dimension of an algebraic action of a finite group on an algebrauc variety)

$K(n) \geq \lfloor \frac{n}{2} \rfloor$.

Serre reproved Theorem 8 using an algebraic version of characteristic classes with values in Galois cohomology. Burda reproved Theorem 8 in a pure topological way. He showed that a topological characteristic class of the covering defined by the general algebraic function of degree n does not vanish in dimension $\lfloor n/2 \rfloor$.

7. UNSOLVABILITY IN FINITE TERMS

A lot of beautiful results on unsolvabilily of equations in finite terms were obtained by **Abel, Galois, Liouville, Picard, Vessiot, Kolchin** and by other mathematicians.

What does it mean that an equation can not be solved explicitly?

One can fix a class of functions and say that an equation is solved explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

A class of functions can be introduced by specifying a list of basic functions and a list of admissible operations. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations.

8. CLASSICAL CLASSES OF FUNCTIONS

To define a classical class of functions we have to fix its list of basic functions and its list of admissible operations. All of them use the **list of basic elementary functions** and the **list of classical operations**.

LIST OF BASIC ELEMENTARY FUNCTIONS.

all constants, x (or $x_1, \ldots, x_n$);
exp, ln, $x \to x^\alpha$;
sin, cos, tan;
arcsin, arccos, arctan. □

LIST OF CLASSICAL OPERATIONS.

1) composition: $f, g \in L \Rightarrow f \circ g \in L$;
2) arithmetic operations: $f, g \in L \Rightarrow f \pm g, f \times g, f/g \in L$;
3) differentiation: $f \in L \Rightarrow f' \in L$;
4) integration: $f \in L$ and $y' = f$, i.e. $y = C + \int^x f(t)dt \Rightarrow y \in L$;
5) extension by exponent of integral: $f \in L$ and $y' = fy$, i.e. $y = C \exp \int^x f(t)dt \Rightarrow y \in L$;
6) algebraic extension: $f_1, \dots, f_n \in L$ and $y^n + f_1 y^{n-1} + \dots + f_n = 0 \Rightarrow y \in L$;
7) exponent: $f \in L$ and $y' = f'y$, i.e. $y = C \exp f \Rightarrow y \in L$;
8) logarithm: $f \in L$ and $dy = df/f$, i.e. $y = C + \ln f \Rightarrow y \in L$;
9) meromorphic operation: if $F : \mathbb{C}^n \to \mathbb{C}$ is a meromorphic function, $f_1 \dots, f_n \in L$, and $y = F(f_1, \dots, f_n) \Rightarrow y \in L$. □

The operations 2) and 7) are meromorphic operations.

10. RADICALS, QUADRATURES, etc.

I. Radicals. Basic functions: rational functions. Operations: arithmetic operations and extensions by radicals.

II. Elementary functions. Basic functions: basic elementary functions. Operations: composition, arithmetic operations, differentiation.

III. Generalized elementary functions. The same as elementary functions + algebraic extensions.

IV. Quadrature. Basic functions: basic elementary functions. Operations: composition, arithmetic operations, differentiation and integration.

IV'. "Liouville's quadratures". Basic functions: all complex constant. Operations: the arithmetic operations, integration, extension by the exponent of integral.

V. Generalized quadratures. The same as quadratures + algebraic extensions.

11. LIOUVILLE'S THEORY

SLOGAN OF LIOUVILLE'S THEORY.

"Sufficiently simple" equations have either "sufficiently simple" solutions or no explicit solutions at all. □

Theorem 9 (Liouville)

Class of "Liouville's quadratures" = class of quadratures.

Theorem 8 reduces the problem of solvability by quadratures to differential algebra.

Theorem 10 (Liouville 1833, M. Rosenlicht 1968 , Kh 2018)

An integral $y(x)$ of an algebraic function is a generalized elementary function if and only if $y(x) = A_0(x) + \sum_{i=1}^{n} \lambda_i \ln A_i(x)$ where $\lambda_i \in BbbC$ and A_i are algebraic functions.

Rosenlicht proof is pure algebraic, my proof is basically geometric.

12. SECOND LIOUVILLE'S THEOREM

Theorem 11 (Liouville, 1841)

. An equation $y'' + py' + qy = 0$, where p, q are rational functions, is solvable by generalized quadratures if and only if it has a solution $y_1(x) = \exp \int^x a(t)dt$, where $a(t)$ is an algebraic function.

Theorem 12 (Picard-Vessio 1910, M. Rosenlicht 1973, Kh. 2018)

A linear order n differential equation is solvable by generalized quadratures if and only if: 1) it has a solution $y_1 = \exp \int^x a(t)dt$ where $a(t)$ is an algebraic function, and 2) if the equation of order $(n-1)$ obtained from the original equation by the reduction of order is solvable by generalized quadratures.

To prove Theorem 12 Picard and Vessio developed the differential Galois theory. Rosenlicht used the valuation theory. My proof is based on the original ideas due to Liouville.

13. PICARD–VESSIOT THEORY

Picard discovered a similarity between linear differential equations and algebraic equations. He initiated the development of a differential of analogue of Galois theory.

Theorem 13 (Picard–Vessiot, 1910)

. *A linear differential equation is solvable by quadratures if and only if its differential Galois group is solvable. It is solvable by generalized quadratures if and only if the connected component of the identity in its differential Galois group is solvable.*

Picard–Vessiot theory has many applications. For example, for an equation whose coefficients are rational functions with integral coefficients it allows to determine explicitly is the equation solvable by generalized quadratures or not.

14. TOPOLOGICAL GALOIS THEORY

Theorem 14 (C.Jordan)

The Galois group of an algebraic equation over the field of rational functions is isomorphic to the monodromy group of the (multivalued) algebraic function defined by the same equation.

Corollary 15

If the monodromy group of an algebraic function is unsolvable then the function is not representable by radicals.

PROGRAM.

I. Find a wide class of functions which is closed under classical operations, such that for all functions from the class the monodromy group is well defined.

II. Use the monodromy group within this class instead of the Galois group. □

15. CLASS OF $\mathcal{S}$-FUNCTIONS

A multivalued analitic function of one complex variable is called $\mathcal{S}$-**function** if the set of its singular points is at most countable.

Theorem 16

The class of $\mathcal{S}$-functions is closed under composition, arithmetic operations, differentiation, integration, meromorthic operations, solving algebraic equations, solving linear differential equations.

Corollary 17

A function having an uncountable number of singular points can not be expressed by generalized quadratures.

Example. Consider $f = \ln(\sum_{i=1}^{n} \lambda_i \ln(x - a_i))$. If $n \geq 3$, λ_i – generic and $a_i \neq a_j$ if $i \neq j$ then: 1) the monodromy group of f contains continuum elements, 2) the set of singular points of f is everywhere dense on the complex line.

16. SOLVABLE MONODROMY GROUP

Theorem 18

The class of $\mathcal{S}$-functions whose monodromy group is solvable is closed under integration, differentiation, composition and meromorphic operations (in particular arithmetic operations).

Corollary 19

If the monodromy group of a function f is unsolvable, then f can not by represented via meromorphic functions using integration, differentiation, composition and meromorphic operations.

Theorem 20

If the monodromy group of an algebraic function is unsolvable one can not represent it by a formula which involves meromorphic functions and elementary functions and uses integration, composition and meromorphic operations.

17. FUCHS-TYPE LINEAR DIFFENTIAL EQUATIONS

Theorem 21

If the monodromy group of a Fuchs-type linear differential equation is solvable then this equation is solvable by quadratures. But if it is unsolvable one can not represent its solutions by a formula which involves integration, composition and meromorphic operations and uses meromorphic and elementary functions.

Corollary 22

Consider a system $y' = \sum \frac{A_i}{x - a_i} y$*, where* y *is n-vector and* A_i *are* $n \times n$ *matrices with constans entries. Assume that the matrices* A_i *have sufficiently small entries. Then the system is solvable if and only if all the matrices* A_i *are triangular in some basis.*

18. MAPPING FROM A BALL TO A CURVED POLYGON

Corollary 23

Let G be a polygon bounded by arcs of circles on the complex line. Let $f_G : B_1 \to G$ be a Riemann map from a unit ball onto G. One can classify all polygons G such that the function f_G is representable by quadratures.

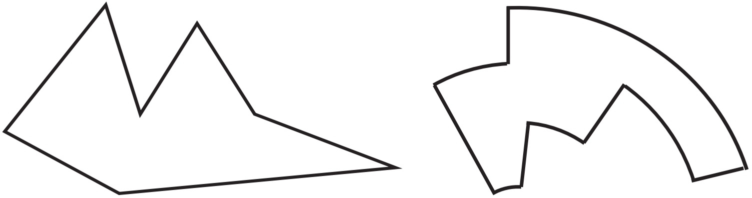

Figure: The first and the second cases of integrability

19. THIRD CASE OF INTEGRABILITY

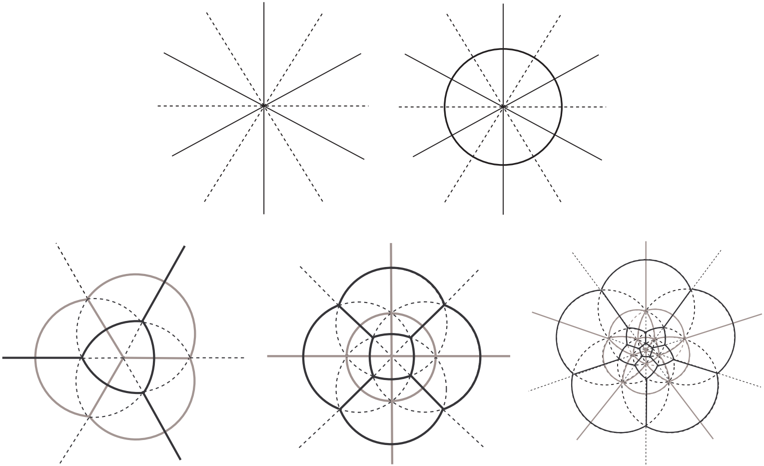

Figure: Finite nets of great circles

20. POLYNOMIALS INVERTIBLE IN RADICALS

Theorem 24 (Ritt 1922)

A polynomial invertible in radicals if and only if it is a composition of the power polynomials $z \to z^n$, Chebyshev polynomials and polynomials of degree ≤ 4.

Theorem 25 (Yu.Burda, Kh. 2012)

A polynomial invertible in radicals and solutions of equations of degree at most k is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most k and, if $k \leq 14$, certain exceptional polynomials polynomials (a description of these polynomials is given).

The proof is based on classification of finite simple groups and results on primitive polynomials obtained by Muller and many other authors.

21. MULTIDIMENSIONAL CASE

One can construct a class of functions of many complex variables containing all meromorphic functions and closed under composition, integration, meromorphic operations, such that for all functions from the class the monodromy group is well defined.

Theorem 26 (Kh. 2001)

If the monodromy group of an an algebraic equation whose coefficients are rational functions in many variables is solvable, then the equation is solvable by radicals.
If the monodromy group of a regular holonomic system of equation is solvable then this system is solvable by quadratures.

Otherwise one cannot represent solutions of the equation or of the system by a formula which involves all meromorphic functions, all elementary functions and which is written using integrations, differentiations, compositions and meromorphic operations.

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THANK YOU