Asymptotic of invariants of quasi-projective fundamental groups.

Anatoly Libgober

Univ. of Illinois at Chicago

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Problem Understand distribution of invariants of the fundamental groups of smooth quasi-projective varieties.

A. Ranks of free quotients

B. Ranks of abelianized commutator: π_1'/π_1''.

References (on math.uic.edu/ libgober or/and arxiv)

1. (with J.I. Cogolludo) Free quotients of fundamental groups of smooth quasi-projective varieties (preprint)

2. On Mordell-Weil groups of isotrivial abelian varieties over function fields (Math. Ann. 2013)

3.(with.J.I.Cogolludo, Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves. (Crelle,2014).

Precursor: **Theorem**(L-Yuzvinsky,Falk,Pereira)
Let $\mathcal{A}$ an arrangement of lines in $\mathbb{P}^2$. If $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ has **essential** surjection onto $F_r, r > 3$ then $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) = F_r$ and arrangement is the union of concurrent lines.

Essential: each meridian taken to a congugate of standard generator of F_r. Given an essential surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to F_r$ one can trivially construct infinitely many non-essential ones: $\forall \mathcal{A}'$:
$\pi_1(\mathbb{P}^2 \setminus \mathcal{A} \bigcup \mathcal{A}') \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to F_r$.

Free quotients of quasi-projective groups

One has exact sequence:

$$1 \to Ker \to \pi_1(V \setminus \mathcal{D}) \to \pi_1(V) \to 1$$

Here compactification is assumed simply connected.

Theorem Let V be a smooth simply connected projective surface.

Let $\Delta \subset NS(V)$ be subset of effective cone with the property d_1, d_2 are effective and $d_1 + d_2 \in \Delta$ then $d_1, d_2 \in \Delta$ (*saturated subset*)

Let $\mathcal{D}$ be a curve which components have classes in Δ.
Technical assumption: assume that all singular points belonging to more than one component are ordinary multiple points i.e. locally are transversal intersection of smooth germs.

Then there is a constant $M(V,\Delta)$ such that if $\pi_1(V \setminus \mathcal{D})$ admits surjection onto $F_r, r > M(V,\Delta)$ taking each meridian of a component of D to one r generators of F_r, then $\mathcal{D}$ is composed of a pencil of curves in a class from Δ.

Moreover, if $r > 10$ then there are only finite number $N(V,\Delta)$ curves $\mathcal{D}$ with components from Δ and not composed of a pencil but admitting surjection $\pi_1(V \setminus \mathcal{D}) \to F_r$.

Given set of classes of components in $NS(V)$ there is trichotomy for existence of surjections of $\pi_1(V \setminus \mathcal{D})$ onto a free group F_r

A. There are infinitely many (isotopy classes of) curves admitting essential surjections $\pi_1(V \setminus \mathcal{D}) \to F_r$. This is happening only for $r \le 10$.

B.There are finitely many isotopy classes of curves $\mathcal{D}$ admitting surjections $\pi_1(V \setminus \mathcal{D}) \to F_r, 10 < r < M(V, \Delta)$.

C. If there is surjection $\pi_1(V \setminus \mathcal{D}) \to F_r, r > M(\Delta)$ then $\mathcal{D}$ is composed of curves of a pencil from one of the classes in Δ and $\pi_1(V \setminus \mathcal{D}) = H *_{\pi_1(\Sigma)} G$ where Σ is Riemann surface (possibly punctured) and G is an extension:

$$0 \to \pi_1(\Sigma) \to G \to F_r \to 0$$

Example

Free quotients of small rank $V = \mathbb{P}^2, \Delta = [1]$ $M(\Delta) = 3$ For any d there is pencil of curves of degree d with 3 completely reducible fibers:

$$a(x^d - y^d) + b(y^d - z^d) = 0$$

(at $(a, b) = (1, 0), (0, 1), (1, 1)$) inducing map

$$\mathbb{P}^2 \setminus \mathcal{A}) \to \mathbb{P}^1 \setminus 3pts$$

there are infinitely many curves having surjectino onto F_2 with all components being lines.

Example

F_3-quotient for Δ=[1]. C is a smooth cubic, There are 12 lines, each containing triple of inflection points of C. There are 4 groups of triple each containing all 9 inflection points and there is pencil of cubics for which each group is a completely reducible fiber.

Examples Case $\Delta = [1], [2]$ For $d = 2$, $M(\Delta) = 5$

$$\lambda_0 x_0(x_1^2 - x_2^2) + \lambda_1 x_1(x_2^2 - x_0^2) + \lambda_2 x_2(x_0^2 - x_1^2) = 0$$

is the pencil of cubics and gives reducible curve $\mathcal{D}$ which is the union of 6 lines and 6 quadrics.

One has surjection:

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \to F_5$$

Example

Let $\Delta_k = [1], ..., [k]$. $lim_{\to\infty} M(\Delta_k) = \infty$

There is a pencil of curves of degree $k+1$ with all components having degree not exceeding k, not composed of a pencil of curves of degree k and having $3k$ reducible fibers i.e. $M(k) > 3k$. It is a generic pencil in the following net:

$$\lambda_0 x_0(x_1^k - x_2^k) + \lambda_1 x_1(x_2^k - x_0^k) + \lambda_2 x_2(x_0^k - x_1^k) = 0$$

Finiteness of the isotopy classes of curves having surjection onto $F_r, r > 10$ for $\mathbb{P}^2$ takes place already for $r > 6$ In other words: For $\Delta = \{[1], ..., [k]\} \in Pic(\mathbb{P}^2)$ existence of surjection $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}) \to F_r, r > 6$ where degree of each component of $\mathcal{D}$ does not exceed k implies that degrees of curves of pencil less than $A(k)$. Hence there is a finite number $N(k, r)$ pencils of curves having more than 6 fibers with irreducible components having degree k or less and hence a finite number the curves $\mathcal{D}$ with large free quotients which do not form a union of elements of a pencil of curves of degree k.

For $\mathbb{P}^2$ data describing asymptotic can be packaged as follows: let $P(d,k,r)$ be the number of curves of degree d with irreducible components having degree at most k, fundamental group having essential surjections onto F_r. $N(k,r) = \sum_d P(d,k,r)$ is finite for $r > 6$.

Remark

Above example of pencil of curves of degree d with 3 maximally reducible fibers shows: $lim_{n\to} P(d,1,1) = \infty$

Additional information about $P(d, k, r)$ in the case of $\mathbb{P}^2$:

If $d > 2k$ then pencil of curves of degree d has no more than 12 fibers with all irreducible components having degree at most k. In particular

the number of fundamental groups of the reducible curves and having essentail surjection onto $F_r, r > 11$ (and not composed of a pencil) which are unions of curves of degree at most k does not exceed the number of isotopy classes of pencils of degree $d, d \leq 2k$ which is finite.

However, only small fraction of them has fibers with components having degree at most d i.e. can give a curve with components of degree at most d and having surjection on $F_r, r \geq 11$.

In particular $P(d, k, r) = 0$ for $r \geq 12, d > 2kr$ for curves not composed of a pencil (since the total degree of union of r fibers of pencil of degree m is $n = mr$ and hence for $n > 2dr$ one has $m > 2d$)

Other surfaces:

$\mathbb{P}^1 \times \mathbb{P}^1$, $\Delta = (1,0),(0,1),(1,1)$. Then for the threshold for ranks of free essential quotients for curves not composed of a pencil one has: $M(\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \geq 4$.

$V_3 \subset \mathbb{P}^3$, Δ_1 (resp. Δ_2) is the set of 27 exceptional curves (resp. the set of exceptional curves and residual quadrics). Then $M(V_3, \Delta_1) = 5$ (resp. $M(V_3, \Delta_2) = 6$).

Arrangements on lines on surfaces having arbitrary large free quotients For any positive integer d there is a surface $S_{f,g} \subset \mathbb{P}^3$ of degree d and a pencil on it containing at least d completely reducible curves consisting of lines (i.e. $M(S_{f,g}, \Delta) \geq d$ where Δ is the set of lines on $S_{f,d}, degf = degg = d$).
Consider $f(x, y)$ and $g(z, t)$ two homogeneous polynomials of degree d with no multiple roots, then the surface

$$S_{f,g} = \{[x : y : z : t] \in \mathbb{P}^3 \mid f(x, y) = g(z, t)\}$$

contains at least d^2 lines, namely all the lines $L_{i,j}$, $i, j = 1, ..., d$ joining a point $P_i = [x_i : y_i : 0 : 0]$ and a point $Q_j = [0 : 0 : z_j : t_j]$ where $f(x_i : y_i) = g(z_j, t_j) = 0$.
The pencil of hyperplanes containing the line $L = \{x = y = 0\}$ induces a pencil of curves on $S_{f,g}$. Given any point P_i, the hyperplane $H_i = \{y_i x = x_i y\}$ containing P_i has to contain the lines $L_{i,1}, ..., L_{i,d}$.
Therefore this pencil contains at least d completely reducible fibers.

Steps of the proof:
1. Consider set of rank one local systems on $V \setminus \mathcal{D}$ i.e.
$\chi : Hom(\pi_1(V \setminus \mathcal{D}), \mathbb{C}^*)$. Existence of surjection $\pi_1(V \setminus \mathcal{D}) \to F_r, r \geq 2$ implies that

$$\{\chi | dimH^1(V \setminus \mathcal{D}, \chi) \neq 0\}$$

has positive dimension.
2. A theorem due to D.Arapura implies that there is holomorphic map $\Phi : V \setminus \mathcal{D} \to \mathbb{P}^1 \setminus \mathcal{S} \quad Card\mathcal{S} \geq r - 1$.

3. Φ can be extended to $V \setminus \mathcal{B}, dim\mathcal{B} = 0$. Generic fiber of Φ is a curve smooth outside of the set $\mathcal{B}$ of base points of the pencil. Fibers over $\mathcal{S}$ are unions of irreducible components of $\mathcal{D}$ and assumption that surjections of fundamental group is essential implies that $\mathcal{D}$ is not bigger than the union of reducible fibers. Meridians go to conjugates of generators impies that the fibers of Φ are reduced.

4. Blowing up base points (eliminating indeterminacy points of Φ) gives holomorphic map $\tilde{\Phi} : \tilde{V} \to \mathbb{P}^1$ and the main step is comparison of the euler characteristic of $\tilde{V}$ with the euler characteristic calculated from fibration. Each completely reducible fiber produces "many" singular points (in the case of arrangements of lines if d is the degree of generic fiber and completely reducible fiber has only double points then $\frac{d(d-1)}{2}$ singular points).

Two inequalities If r is the number of fibers of the pencil in a linear system D, components in the classes $d_i \in NS(V)$, then for any m_i such that $D = \sum m_i d_i$ is a singular fiber of the pencil then

$$r \leq 2\frac{e(V) + 3D^2 + 2KD}{D^2 + KD + \sum m_i e(d_i)}$$

where $e(d_i) = K_V d_i + d_i^2$

Let Δ be a saturated subset of $Eff(V)$ and $r = Card\Delta$. For $\alpha > \frac{5}{3}$ and all but finitely many $(m_1, ..., m_r) \in \mathbb{Z}^r_{\geq 0}$ such that $D = \sum_i m_i d'_i$ where d'_i are curves having classes $d_i \in \Delta \subset NS(V)$ moves in a pencil with all its reducible fibers being reduced, one has

$$\frac{\frac{e(V)}{3} + D^2 + \frac{2}{3}KD}{D^2 + \sum_i m_i e(d'_i) + KD} < \alpha$$

(uses Riemann Roch and Kollar-Fujita results on growth of H^1, H^2 of linear systems).

Problems

1. Find asymptotic or bounds of $M(V,\Delta)$ (threshold and for the order of free quotients) $N(V,\Delta)$ (the number of curves having essential free quotients of rank greater than 10).

2.How sharp are estimates in examples above.

3.New pencils with many reducible fibers.

4. What is complexity of quasi-projective groups without surjections onto $F_r, r \geq 2$

Mordell-Weil ranks of iso-trivial abelian varieties over $\mathbb{C}(x, y)$

and fundamental groups of the complements to discriminants

C_d irreducible plane curve of degree d. There is relation between the the rank $\pi_1(\mathbb{P}^2 \setminus C_d)'/\pi_1(\mathbb{P}^2 \setminus C_d)''$ and the Mordell-Weil rank of associated iso-trivial abelian variety (depending on the local type of singularities of C_d. Asymptotic of MW ranks implies asymptotic for the ranks of these quotients.

In the case when singularities are only ordinary nodes, ordinary cusps or ordinary triple points one get abelian variety of dimension 1. Let $C_d \subset \mathbb{C}^2$ be given by equation $f(x,y)=0$. Consider

$$W_f: \; u^2+v^3=f(x,y) \subset \mathbb{C}^4$$

This elliptic threefold is isotrivial elliptic curve over $\mathbb{C}(x,y)$ with zero set of discriminant given by $f(x,y)=0$ becoming trivial over $\mathbb{C}(x,y)(f^{\frac{1}{6}})$. Pull back is trivial over $z^6=f(x,y)$.

Example

Let E_0 elliptic curve with j-invariant zero, μ_3 the cyclic group of order 3. Elliptic fibration $E_0^3/\mu_3 \to E_0^2/\mu_3$ is birational to the iso-trivial with discriminant being the arrangement of 9 lines in $\mathbb{P}^2$ dual to 9 inflection points of smooth cubic. (S.Yang thesis, UIC).

Theorem (Cogolludo-Libgober)

$$rkMW(W_f) = rk\pi_1(\mathbb{P}^2 \setminus C_d)'/\pi_1(\mathbb{P}^2 \setminus C_d)'' =$$
$$rkH^1(z^6 = f(x,y))$$

All do not exceed $\frac{5}{3}d - 2$.

Problem:

What is the optimal bound? Can $rkMW_f$ be arbitrary large?

Largest known $rkMW_f$ is 8 (for curve of degree 12 with 39 cusps).

Many examples with $rkMW_f = 2$ e.g. $f_{2k}^3 + f_{3k}^2 = 0$

Restriction of W_f on generic line in $\mathbb{C}^2$ gives an isotrivial elliptic sufrace $S_{f=0\cap L}$. One has

$$rkMW(W_f) \leq rkMW(S_{f=0\cap L})$$

Theorem

(L-, 2012) Let $\mathcal{E} \to \mathbb{P}^1$ be an isotrivial elliptic surface over $\mathbb{C}$. Denote by E_0 a generic fiber of this fibration and let $\Gamma = \mu_6$. Denote by C_Γ the cyclic cover of $\mathbb{P}^1$ branched over the zero set of the discriminant of $\mathcal{E}$ such that the pullback of $\mathcal{E}$ to C_Γ of $\mathcal{E}$ is biholomorphic to a direct product. Let $Jac(C_\Gamma)$ be the Jacobian of C_Γ and let $r = \{max\ k | Jac(C_\Gamma) \sim_\Gamma E_0^k \times A\}$ be the number of equivariant isogeny E_0-factors of the Jacobian (i.e. $\sim_\Gamma$ denotes equivariant isogeny of abelian varieties with Γ-action). Then:

$$rkMW(\mathcal{E}) = 2r \qquad (1)$$

Example

Conisder the surfaces $y^2 = x^3 + t^{360k} - 1$. Case k was considered by Shioda who showed that $rkMW(\mathcal{E}) = 68$ (the largest known).

We need to know the number of factors of the Jacobian, up to isogeny, for the curve $C_{6,360} : s^6 = t^{360k} - 1$ which are isomorphic to the curve E_0 with j-invariant zero. Those are among the factors of the Jacobian of Fermat curve $F_{360k} : s^{360k} = t^{360k} - 1$ with $N = 360k$. Calculations using Koblitz results on Jacobians of Fermat curves give $JacC_{6,360} = E_0^{34} \times A$ which shows the claim on $rkMW(\mathcal{E})$.

Problem:

How many component of type E_0 the Jacobian of a cyclic cover of $\mathbb{P}^1$ can have (in particular 6-fold cyclic cover)?

It is interesting to compare such question on decomposability of Jacobians with the problem posed by T.Ekedahl and J.P.Serre (1993) Do exist curves of arbitrary large genus for which the Jacobian is a product of elliptic curves?

Given a pure Hodge structure $(H_{\mathbb{Z}}, F)$ of weight -1, one associates to it a complex torus as (a more general case of mixed Hodge structures of type $(0,0), (0,-1), (-1,0), (-1,-1)$ in Deligne's Hodge Theore III):

$$A_H = H_{\mathbb{Z}} \backslash H_{\mathbb{C}} / F^0 H_{\mathbb{C}} \tag{2}$$

In the case when the Hodge structure is polarized, A_H is an abelian variety.

Definition

Local Albanese variety Alb_f of a plane curve singularity $f(x,y) = 0$ is the abelian variety (2) corresponding to the Hodge structure on homology $H_1(M_f, \mathbb{Z})$ of the Milnor fiber which is dual to the limit cohomological mixed Hodge structure.

Definition

A plane curve singularity is called a singularity of CM type if its local Albanese variety is isogenous to a product of simple abelian varieties of CM type.

Example

1. The local Albanese if $x^2 + y^3$ is the elliptic curve E_0.

2. Consider the singularity $x^2 + y^5$. One can identify the local Albanese with the Jacobian of the genus two curve:

$$y^{10} = x^4(x - z)z^5$$

Theorem

(L-2013) Theorem: $Alb(V_f)$ *is quotient of product of local Albanese varieties In particular for the curves with cusps and nodes* $Alb(V_f) = E_0^r$.

Back to the relation between Mordell Weil ranks and ranks of abelianized commutators of the fundamental groups.

Key step is the isogeny:
$Alb(z^6 = f(x,y)) = E_0^s, s = \frac{1}{2} rkH^1(z^6 = f(x,y)) = \frac{1}{2} rk\pi_1'/\pi_1''$.

Proof of relation between Mordell Weil ranks of 3-folds and ranks of abelianized commutators

1.$\frac{1}{2} rk \pi_1' / \pi_1''$ is equal to the dimension of Albanese of the surface V_f and hence the number of factors E_0.

2. The pullback of W_f onto V_f via $V_f \to \mathbb{P}^2$ trivializes W_f i.e. is birational to $V_f \times E_0$.

3.Elements of $MW(W_f)$ correspond to the maps $V_f \to E_0$

4.$Maps(V_f, E_0) = Hom(Alb(V_f), E_0) = Hom(E_0^{\frac{1}{2} rk \pi_1' / \pi_1''}, E_0) = \mathbb{Z}[\omega_6]^{\frac{1}{2} rk \pi_1' / \pi_1''}$