Braids, hyperplane arrangements, and Milnor fibrations

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Workshop on Braids, Resolvent Degree and Hilbert's 13th Problem

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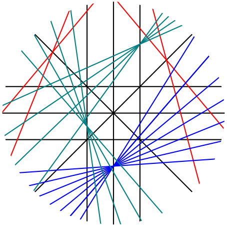


POLYNOMIAL COVERS

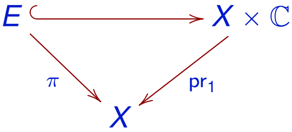
- Let X be a path-connected space. A *simple Weierstrass polynomial* of degree n on X is a map $f\colon X \times \mathbb{C} \to \mathbb{C}$ given by

$$f(x,z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i},$$

  with continuous coefficient maps $a_i\colon X \to \mathbb{C}$, and with no multiple roots for any $x \in X$.

- Let $E = E(f) = \{(x,z) \in X \times \mathbb{C} \mid f(x,z) = 0\}$.

- The restriction of $\mathrm{pr}_1\colon X \times \mathbb{C} \to X$ to E defines an n-fold cover $\pi = \pi_f\colon E \to X$, the *polynomial covering map* associated to f.



CONFIGURATION SPACES

- Let $\mathrm{Conf}_n(\mathbb{C}) = \{z \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$ and $\mathrm{UConf}_n(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})/S_n$.
- Since $f\colon X \times \mathbb{C} \to \mathbb{C}$ has no multiple roots, the *coefficient map* $a = (a_1, \dots, a_n)\colon X \to \mathbb{C}^n$ takes values in

$$\mathbb{C}^n \backslash \Delta_n = \mathrm{UConf}_n(\mathbb{C}).$$

- Over $\mathrm{UConf}_n(\mathbb{C})$, there is a canonical n-fold polynomial covering map, $\pi_n\colon E(f_n) \to \mathrm{UConf}_n(\mathbb{C})$, determined by the W-polynomial

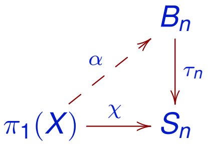
$$f_n(x, z) = z^n + \sum_{i=1}^{n} x_i z^{n-i}.$$

- We get a pullback diagram of covers,

$$\begin{array}{ccc} E(f) & \longrightarrow & E(f_n) \\ \big\downarrow{\scriptstyle \pi_f} & & \big\downarrow{\scriptstyle \pi_n} \\ X & \xrightarrow{\;a\;} & B^n \end{array}$$

BRAID GROUPS

- Let B_n be the Artin braid group on n strands. Then $B_n = \pi_1(\mathrm{UConf}_n(\mathbb{C}))$.
- We let $\psi_n \colon B_n \hookrightarrow \mathrm{Aut}(F_n)$ be the Artin representation.
- The *coefficient homomorphism*, $\alpha = a_* \colon \pi_1(X) \to B_n$, is well-defined up to conjugacy.
- Polynomial covers are those covers $\pi \colon E \to X$ for which the characteristic homomorphism $\chi \colon \pi_1(X) \to S_n$ factors through the canonical surjection $\tau_n \colon B_n \twoheadrightarrow S_n$,



THE ROOT MAP

- Now assume that the W-polynomial f completely factors as
$$f(x,z) = \prod_{i=1}^{n} (z - b_i(x)),$$
with continuous roots $b_i\colon X \to \mathbb{C}$.
- Since f is simple, the *root map* $b = (b_1, \dots, b_n)\colon X \to \mathbb{C}^n$ takes values in $\operatorname{Conf}_n(\mathbb{C})$.
- Over $\operatorname{Conf}_n(\mathbb{C})$, there is a canonical n-fold cover, $\pi_{Q_n}\colon E(Q_n) \to \operatorname{Conf}_n(\mathbb{C})$, where
$$Q_n(w,z) = (z - w_1)\cdots(z - w_n).$$
- We get a pullback diagram of covers,
$$\begin{array}{ccc} E(f) & \longrightarrow & E(Q_n) \\ {\scriptstyle \pi_f}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{Q_n}} \\ X & \xrightarrow{\ b\ } & \operatorname{Conf}_n(\mathbb{C}) \end{array}$$

BRAID BUNDLES

- Let $P_n = \ker(\tau_n \colon B_n \twoheadrightarrow S_n)$ be the pure braid group. Then $P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$.
- The map $\beta = b_* \colon \pi_1(X) \to P_n$ is well-defined up to conjugacy.
- The polynomial covers which are trivial covers are precisely those for which $\alpha = \iota_n \circ \beta$, where $\iota_n \colon P_n \hookrightarrow B_n$ is the inclusion map.

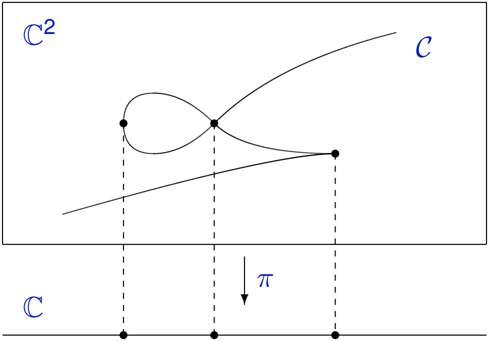
THEOREM (D. COHEN, A.S. 1997)

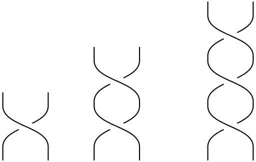
Let $f \colon X \times \mathbb{C} \to \mathbb{C}$ be a simple W-polynomial. Let $Y = X \times \mathbb{C} \setminus E(f)$ and let $p \colon Y \to X$ be the restriction of $\mathrm{pr}_1 \colon X \times \mathbb{C} \to X$ to Y.

- *The map $p \colon Y \to X$ is a locally trivial bundle, with structure group B_n and fiber $\mathbb{C}_n = \mathbb{C} \setminus \{n \text{ points}\}$. Upon identifying $\pi_1(\mathbb{C}_n)$ with F_n, the monodromy of this bundle is $\psi_n \circ \alpha \colon \pi_1(X) \to \mathrm{Aut}(F_n)$.*
- *If f completely factors into linear factors, the structure group reduces to P_n, and the monodromy factors as $\psi_n \circ \iota_n \circ \beta$.*

BRAID MONODROMY OF PLANE ALGEBRAIC CURVES

- Let $\mathcal{C}$ be a reduced algebraic curve in $\mathbb{C}^2$, defined by a polynomial $f = f(z_1, z_2)$ of degree n.
- Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ be a linear projection, and let $\mathcal{Y} = \{y_1, \dots, y_s\}$ be the set of points in $\mathbb{C}$ for which the fibers of π contain singular points of $\mathcal{C}$, or are tangent to $\mathcal{C}$.
- WLOG, we may assume that $\pi = \mathrm{pr}_1$ is generic with respect to $\mathcal{C}$. That is, for each k, the line $\mathcal{L}_k = \pi^{-1}(y_k)$ contains at most one singular point v_k of $\mathcal{C}$ and does not belong to the tangent cone of $\mathcal{C}$ at v_k, and, moreover, all tangencies are simple.
- Let $\mathcal{L} = \bigcup \mathcal{L}_k$.


$\mathbb{C}^2$
$\mathcal{C}$
π
$\mathbb{C}$



- In the chosen coordinates, the defining polynomial f of $\mathcal{C}$ may be written as $f(x,z) = z^n + \sum_{i=1}^{n} a_i(x) z^{n-i}$.
- Since $\mathcal{C}$ is reduced, for each $x \notin \mathcal{Y}$, the equation $f(x,z) = 0$ has n distinct roots. Thus, f is a simple W-polynomial over $\mathbb{C}\setminus\mathcal{Y}$, and

$$\pi = \pi_f \colon \mathcal{C}\setminus\mathcal{C} \cap \mathcal{L} \to \mathbb{C}\setminus\mathcal{Y}$$

  is the associated polynomial n-fold cover.
- Note that $Y(f) = ((\mathbb{C}\setminus\mathcal{Y}) \times \mathbb{C})\setminus(\mathcal{C}\setminus\mathcal{C} \cap \mathcal{L}) = \mathbb{C}^2\setminus\mathcal{C} \cup \mathcal{L}$.
- Thus, the restriction of pr_1 to $Y(f)$,

$$p \colon \mathbb{C}^2\setminus\mathcal{C} \cup \mathcal{L} \to \mathbb{C}\setminus\mathcal{Y},$$

  is a bundle map, with structure group B_n, fiber $\mathbb{C}_n$, and monodromy homomorphism $\alpha = a_* \colon \pi_1(\mathbb{C}\setminus\mathcal{Y}) \to B_n$.

BRAID MONODROMY PRESENTATION

- The homotopy exact sequence of fibration $p\colon \mathbb{C}^2\backslash\mathcal{C}\cup\mathcal{L}\to\mathbb{C}\backslash\mathcal{Y}$:

$$1 \longrightarrow \pi_1(\mathbb{C}_n) \longrightarrow \pi_1(\mathbb{C}^2\backslash\mathcal{C}\cup\mathcal{L}) \xrightarrow{p_*} \pi_1(\mathbb{C}\backslash\mathcal{Y}) \longrightarrow 1 .$$

- This sequence is split exact, with action given by the braid monodromy homomorphism $\alpha\colon \pi_1(\mathbb{C}\backslash\mathcal{Y})\to \mathrm{Aut}(\pi_1(\mathbb{C}_n))$.
- Order the points of $\mathcal{Y}$ by decreasing real part, and pick the basepoint y_0 in $\mathbb{C}\backslash\mathcal{Y}$ with $\mathrm{Re}(y_0) > \max\{\mathrm{Re}(y_k)\}$.
- Choose loops $\xi_k\colon [0,1]\to\mathbb{C}\backslash\mathcal{Y}$ based at y_0, and going around y_k.
- Setting $x_k=[\xi_k]$, identify $\pi_1(\mathbb{C}\backslash\mathcal{Y},y_0)$ with $F_s=\langle x_1,\dots,x_s\rangle$. Similarly, identify $\pi_1(\mathbb{C}_n,\hat{y}_0)$ with $F_n=\langle t_1,\dots,t_n\rangle$.
- Then $\pi_1(\mathbb{C}^2\backslash\mathcal{C}\cup\mathcal{L},\hat{y}_0)=F_n\rtimes_\alpha F_s$.

- The corresponding presentation is

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C} \cup \mathcal{L}) = \langle t_1, \dots t_n, x_1 \dots, x_s \mid x_k^{-1} t_i x_k = \alpha(x_k)(t_i) \rangle.$$

- The group $\pi_1(\mathbb{C}^2 \backslash \mathcal{C})$ is the quotient of $\pi_1(\mathbb{C}^2 \backslash \mathcal{C} \cup \mathcal{L})$ by the normal closure of $F_s = \langle x_1, \dots, x_s \rangle$. Thus,

$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}) = \langle t_1, \dots, t_n \mid t_i = \alpha(x_k)(t_i) \rangle.$$

- This presentation can be simplified by Tietze-II moves to eliminate redundant relations. This yields the *braid monodromy presentation*

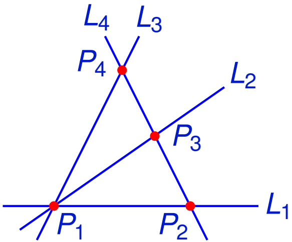
$$\pi_1(\mathbb{C}^2 \backslash \mathcal{C}) = \langle t_1, \dots, t_n \mid t_i = \alpha(x_k)(t_i), i = j_1, \dots, j_{m_k - 1}; k = 1, \dots, s \rangle.$$

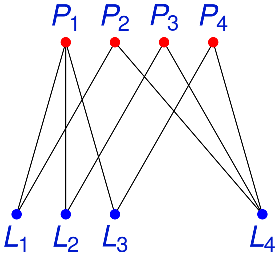
  where m_k is the multiplicity of the singular point y_k.

- (Libgober 1986) The 2-complex modeled on this presentation is homotopy equivalent to $\mathbb{C}^2 \backslash \mathcal{C}$.

Hyperplane arrangements

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$. It is a smooth, quasi-projective variety and also a Stein manifold. It has the homotopy type of a finite, connected, ℓ-dimensional CW-complex.

FUNDAMENTAL GROUP

EXAMPLE (THE BOOLEAN ARRANGEMENT)

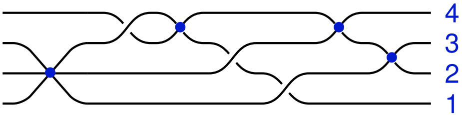
- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n \simeq K(\mathbb{Z}^n, 1)$.

EXAMPLE (THE BRAID ARRANGEMENT)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n) = \mathrm{Conf}_n(\mathbb{C}) \simeq K(P_n, 1)$.

- For an arbitrary (central) arrangement $\mathcal{A}$, let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar slice. Then the arrangement group, $\pi = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.

- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$.
- Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$; the braids $\alpha_r \in P_n$ can be read off the associated braided wiring diagram,



- The group $\pi = \pi_1(M(\mathcal{A}))$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j (x_i)^{-1}$.
- Let $\pi/\gamma_k(\pi)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^n$.
  - π/γ_3 is determined by $L(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$. (Rybnikov).

Cohomology ring

- The Betti numbers of the complement are given by

$$\sum_{q=0}^{\ell} b_q(M(\mathcal{A}))t^q = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

with $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ given by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y\supsetneq X} \mu(Y)$.

- Let $E = \bigwedge(\mathcal{A})$ be the $\mathbb{Z}$-exterior algebra on degree-1 classes e_H dual to the meridians around the hyperplanes $H \in \mathcal{A}$.

- Let $\partial\colon E^\bullet \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_\mathcal{B} = \prod_{H\in\mathcal{B}} e_H$ for each $\mathcal{B} \subset \mathcal{A}$.

- Building on work of Arnold & Brieskorn, Orlik and Solomon described the cohomology ring of $M(\mathcal{A})$ solely in terms of $L(\mathcal{A})$:

$$H^*(M(\mathcal{A}), \mathbb{Z}) \cong E/\big\langle \partial e_\mathcal{B} \mid \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}| \big\rangle.$$

- The space $M(\mathcal{A})$ is $\mathbb{Q}$-formal but not $\mathbb{F}_p$-formal in general.

RESONANCE VARIETIES

- Let X be a connected, finite cell complex,
- Let $A = H^*(X, \Bbbk)$, where $\operatorname{char} \Bbbk \neq 2$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon\ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

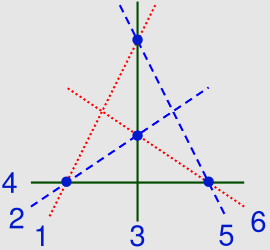
- E.g., $\mathcal{R}^1_1(X, \Bbbk) = \{a \in A^1 \mid \exists b \in A^1,\ b \neq \lambda a,\ ab = 0\}$.
- These loci are homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$. In general, they can be arbitrarily complicated.

Resonance varieties of arrangements

Work of Arapura, Falk, D.Cohen, A.S., Libgober, and Yuzvinsky, completely describes the varieties $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}^1_s(M(\mathcal{A}), \mathbb{C})$.

- $\mathcal{R}_1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A})$ is the union of those linear subspaces that have dimension at least $s+1$.
- Each k*-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}_1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}_1(\mathcal{A})$ arise in this way.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 local components (from the triple points), and one essential component, from the above $(3,2)$-net:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, let $\pi = \pi_1(X, x_0)$, and let $\operatorname{Hom}(\pi, \mathbb{C}^*)$ be the character variety of X (the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on π).
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X) = \{\rho \in \operatorname{Hom}(\pi, \mathbb{C}^*) \mid \dim H_q(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These loci are Zariski closed subsets of the character variety. In general, they can be arbitrarily complicated.
- The sets $\mathcal{V}^1_s(X)$ depend only on π/π''.

Characteristic varieties of arrangements

- Let $\mathcal{A}$ be an arrangement of n hyperplanes, and let $\operatorname{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*) = (\mathbb{C}^*)^n$ be the character torus.
- The characteristic variety $\mathcal{V}_1(\mathcal{A}) := \mathcal{V}^1_1(M(\mathcal{A}))$ lies in the subtorus $\{t \in (\mathbb{C}^*)^n \mid t_1 \cdots t_n = 1\}$; it is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_1(\mathcal{A})$.
- All components of $\mathcal{V}_1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}_1(\mathcal{A})$.

MILNOR FIBRATION

- Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^\ell$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H \in \mathcal{A}} \alpha_H$.
- $Q\colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a smooth fibration, $Q\colon M(\mathcal{A}) \to \mathbb{C}^*$. The *Milnor fiber* of the arrangement is $F(\mathcal{A}) := Q^{-1}(1)$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $\ell - 1$.
- In general, F is not $\mathbb{Q}$-formal, and $H_*(F, \mathbb{Z})$ may have torsion.
- F is the regular, $\mathbb{Z}_n$-cover of $U = \mathbb{P}(M(\mathcal{A}))$, classified by the morphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each loop x_H to 1 (where $n = |\mathcal{A}|$).

MODULAR INEQUALITIES

- The monodromy diffeo, $h\colon F \to F$, is given by $h(z) = e^{2\pi \mathrm{i}/n} z$.
- Let $\Delta(t)$ be the characteristic polynomial of $h_*\colon H_1(F,\mathbb{C}) \circlearrowleft$. Since $h^n = \mathrm{id}$, we have
$$\Delta(t) = \prod_{r|n} \Phi_r(t)^{e_r(\mathcal{A})},$$
where $\Phi_r(t)$ is the r-th cyclotomic polynomial, and $e_r(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- To compute h_*, we may assume $\ell = 3$, so that $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$ is an arrangement of lines in $\mathbb{CP}^2$.
- If there is no point of $\bar{\mathcal{A}}$ of multiplicity $q \geqslant 3$ such that $r \mid q$, then $e_r(\mathcal{A}) = 0$ (Libgober 2002).
- In particular, if $\bar{\mathcal{A}}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$. If multiplicity 4 appears, then we also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $A = H^{\bullet}(M(\mathcal{A}), \Bbbk)$, and let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^m}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, *for all* $m \geqslant 1$.

THEOREM (PAPADIMA–S. 2017)

- *Suppose* $\mathcal{A}$ *admits a* k*-net. Then* $\beta_p(\mathcal{A}) = 0$ *if* $p \nmid k$ *and* $\beta_p(\mathcal{A}) \geqslant k - 2$, *otherwise.*
- *If* $\mathcal{A}$ *admits a reduced* k*-multinet, then* $e_k(\mathcal{A}) \geqslant k - 2$.

Combinatorics and monodromy

Theorem (PS)

Suppose $\mathcal{A}$ has no points of multiplicity $3r$ with $r > 1$. Then $\mathcal{A}$ admits a reduced 3-multinet iff $\mathcal{A}$ admits a 3-net iff $\beta_3(\mathcal{A}) \neq 0$. Moreover,

- $\beta_3(\mathcal{A}) \leqslant 2$.
- $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$*, and thus* $e_3(\mathcal{A})$ *is combinatorially determined.*

Corollary

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

Theorem (PS)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

Conjecture (PS)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

- The conjecture has been verified for
  - All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
  - All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
  - Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).
- A counterexample has been announced by Yoshinaga (2019): there is an arrangement of 16 planes in $\mathbb{C}^3$ with $e_2 = 0$ but $\beta_2 = 1$.