# Matching

July 14, 2015

# TU Co-op Game

$N$ = set of players

Value function $V : 2^N \rightarrow \Re$ is monetary value of subset $S$ forming a coalition.

$V(N) \geq \max_{S \subset N} V(S)$.

# TU Co-op Game

An allocation $z$ specifies a division of the total surplus:

$$\sum_{i \in N} z_i = V(N).$$

An allocation $z$ is **blocked** by a coalition $S \subset N$ if

$$\sum_{i \in S} z_i < V(S).$$

The **Core** of $(v, N)$ is the set of unblocked allocations:

$$C(V,N) = \{z \in \Re^n : \sum_{j \in N} z_j = V(N), \sum_{j \in S} z_j \geq V(S)\ \forall S \subset N\}.$$

Let $B(N)$ be feasible solutions to the following:

$$\sum_{S:i\in S}\delta_S = 1 \quad \forall i \in N$$

$$\delta_S \geq 0 \quad \forall S \subset N$$

Each $\delta \in B(N)$ are called balancing weights. $C(V,N) \neq \emptyset$ iff

$$V(N) \geq \sum_{S\subset N} V(S)\delta_S \quad \forall \delta \in B(N).$$

# Allocation of a Single Good

A seller, $\sigma$ with a single good and opportunity cost zero.

$N = \{1, 2, \ldots, n\}$ a set of potential buyers.

$v_i =$ is $i$'s monetary value for the good.

# Allocation of a Single Good

Let $x_j = 1$ if good is allocated to buyer $j \in N$ and zero otherwise.

$$v(S) = \max \sum_{j \in S} v_j x_j$$

$$\text{s.t.} \ \sum_{j \in S} x_j \leq 1$$

$$0 \leq x_j \leq 1 \ \forall j \in S$$

$$v(S) = \max_{i \in S} v_i$$

# Allocation of a Single Good

Co-op game defined on $N \cup \sigma$.

$$V(S \cup \sigma) = v(S)\ \forall S \subseteq N$$

$$V(S) = 0\ \forall S \subseteq N$$

$$C(V, N \cup \sigma) \neq \emptyset$$

# Allocation of a Single Good. $|N| = 2$

Suppose $v_1 > v_2$.

$$z_1 + z_2 + z_\sigma = \max\{v_1, v_2\} = v_1$$

$$z_1 + z_\sigma \geq v_1$$

$$z_2 + z_\sigma \geq v_2$$

$$z_1, z_2, z_\sigma \geq 0$$

$z_1 = v_1 - v_2, z_2 = 0, z_\sigma = v_2$

# Allocation of a Single Good: Prices

$$v(N) = \max \sum_{j \in N} v_j x_j$$

$$\text{s.t.} \sum_{j \in N} x_j \leq 1$$

$$0 \leq x_j \leq 1 \ \forall j \in N$$

$$\min p + \sum_{j \in N} s_j$$

$$\text{s.t. } p + s_j \geq v_j \ \forall j \in N$$

$$p, s_j \geq 0 \ \forall j \in N$$

# Allocation of a Single Good: Prices

$$v(N) = \min p + \sum_{j \in N} s_j$$

$$\text{s.t. } \; p + s_j \geq v_j \;\forall j \in N$$

$$p, s_j \geq 0 \;\forall j \in N$$

$$v_1 > v_2 > v_3 \ldots > v_{|N|}$$

$$p \in [v_2, v_1], \; s_1 = v_1 - p, \; s_j = 0 \;\forall j > 1$$

# Allocation of a Single Good: Prices

$$v(N) = \min p + \sum_{j \in N} s_j$$

$$\text{s.t. } p + s_j \geq v_j \ \forall j \in N$$

$$p, s_j \geq 0 \ \forall j \in N$$

$$z_\sigma + \sum_{j \in N} z_j = V(N \cup \sigma) = v(N)$$

$$z_\sigma + \sum_{j \in S} z_j \geq V(S \cup \sigma) = v(S)$$

# Shapley-Shubik

- $B$ = set of buyers.
- $G$ = set of sellers each selling one unit of a good (could be divisible).
- $v_{ij} \geq 0$ is the monetary value that buyer $i \in B$ assigns to seller $j \in G$'s object.
- Unit demand.
- $|B| = |G|$.

# Shapley-Shubik

- Bilateral Trade
  $v_{ij} = u_i - c_j$.
  $u_i$ is the value to buyer $i$ of acquiring a good irrespective who sells it.
  $c_j$ is the opportunity cost to seller $j$.
- Productivity
  $v_{ij} = a_i b_j$ is output generated when worker of type $a_i$ is matched with firm of type $b_j$.
- Keyword auction
  $b_j$ is the the click through rate on slot $i$ and $a_i$ is the value per click to advertiser $i$.

$x_{ij} = 1$ if buyer $i$ allocated to seller $j$. For any $N \subseteq B$ and $M \subseteq G$, let

$$v(N, M) = \max \sum_{i \in N} \sum_{j \in M} v_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j \in M} x_{ij} \leq 1 \quad \forall i \in N$$

$$\sum_{i \in N} x_{ij} \leq 1 \quad \forall j \in M$$

$$x_{ij} \geq 0 \quad , \forall i \in N, \, j \in M$$

$v(B, G) =$ value of an efficient allocation.

# Shapley-Shubik: Bipartite Matching

Complete bipartite graph on vertex set $B \cup G$.

For each edge $(i, j)$ where $i \in B$ and $j \in G$, there is a weight $v_{ij}$.

A set of edges is a matching if no two edges share a vertex.

Efficiency $=$ maximum weight matching

# Shapley-Shubik: Dual

$s_i$ = the dual variable associated with

$$\sum_{j \in M} x_{ij} \leq 1 \quad \forall i \in N$$

$p_j$ = dual variable associated with

$$\sum_{i \in N} x_{ij} \leq 1 \quad \forall j \in M$$

# Shapley-Shubik: Dual

$$\min \sum_{i \in N} s_i + \sum_{j \in M} p_j$$

subject to

$$s_i + p_j \geq v_{ij} \; \forall i \in N, \forall j \in M$$

$$s_i, p_j \geq 0 \; \forall j \in M, \forall i \in N$$

$$s_i = \{\max_{j \in M}(v_{ij} - p_j), 0\}^+.$$

# Shapley-Shubik: Competitive Equilibrium

- $x^*$ is an optimal primal solution.
- $(s^*, p^*)$ an optimal dual solution.
- Prices $p^*$ 'support' efficient allocation $x^*$.

Post a price $p_j^*$ for each $j \in M$.

Each buyer points to all goods that maximize surplus.

Resulting bipartite graph has a perfect matching; supply $=$ demand.

# Shapley-Shubik: Competitive Equilibrium

Complementary Slackness:

$$(s^*i + p_j^* - v_{ij})x_{ij}^* = 0.$$

$$x_{ij}^* = 1 \;\Rightarrow\; s_i^* = v_{ij} - p_j^* = \{\max_{r \in M}(v_{ir} - p_r^*), 0\}^+.$$

# Total Unimodularity

$A$ is called **totally unimodular** (TUM) iff the determinant of each square submatrix has value 1, -1 or 0.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

# Total Unimodularity

If $A$ is TUM then so is $A^T$.

If $A$ and $E$ are TUM, then so is $AE$.

If $A$ is a $m \times n$ TUM matrix and $b$ a $m \times 1$ integral vector. Then, every extreme point of $\{Ax = b, x \geq 0\}$ is integral.

# Total Unimodularity

$A$ is a **network** matrix if $a_{ij} = 0, 1, -1$ for all $i, j$ and each column contains at most two non-zero entires of opposite sign.

If $A$ is a network matrix, then $A$ is TUM.

# Shapley-Shubik: Core

Assume all goods owned by a single seller, $\sigma$.

For any $N \subseteq B$, let $v(N) = v(N, G)$.

$v(N)$ is non-decreasing and sub modular:

$$\forall N' \subset N \subseteq B, j \in B \setminus N \;\; \Rightarrow$$

$$v(N \cup j) - v(N) \leq v(N' \cup j) - v(N')$$

# Shapley-Shubik: Core

The characteristic function $V$ is defined as follows:

1. $V(S) = 0 \ \forall S \subseteq B$
2. $V(S \cup \sigma) = v(S) \ \forall S \subseteq B$

Core, $C(V, B \cup \{\sigma\})$, of this game is:

$$\sum_{i \in B} z_i + z_\sigma = V(B \cup \sigma) = v(B)$$

$$\sum_{i \in \{S \cup \sigma\}} z_i + z_\sigma \geq V(S \cup \sigma) = v(S)$$

$$\sum_{i \in S} z_i \geq V(S) = 0 \quad \forall S \subset B$$

Outcome in core is a division of value that no coalition of agents can 'block'.

# Shapley-Shubik: Core

$C(V, B \cup \{\sigma\})$ is non-empty.

If $(s^*, p^*)$ is an optimal dual solution, an efficient assignment produces an outcome in $C(V, B \cup \{\sigma\})$

$z_i = s_i^*$ and $z_\sigma = \sum_{j \in G} p_j^*$

Every point in the core corresponds to an optimal dual solution.

Marginal product of $k \in B$ is $V(B \cup \sigma) - V(B \cup \sigma \setminus k)$.

$$\sum_{i \in B} z_j + z_\sigma = V(B \cup \sigma)$$

$$\sum_{i \in B \setminus k} z_i + z_\sigma \geq V(B \cup \sigma \setminus k)$$

Submodularity of $V$ implies a point $z \in C(V, B \cup \{\sigma\})$ such that $z_i = V(B \cup \sigma) - V(B \cup \sigma \setminus i)$ for all $i \in B$.

# Scarf-Shapley Housing Model

$N$ = set of agents

Each agent $i \in N$ owns a house.

Each agent $i$ has a strict preference ordering $\succ_i$ over all houses.

An assignment $x$ of homes to agents can be blocked by a subset $S \subseteq N$ if the agents in $S$ can by trading among themselves each do strictly better than the house each gets under $x$.

Top trading cycle.

# Quinzii Housing Model

$N = \{1, 2, \ldots, n\}$ is a set of agents.

Each $i \in \{1, \ldots, q\}$ owns house $h_i$.

Agents $i \in \{q+1, \ldots, n\}$ do not own a house.

Each $i \in N$ has money in the amount $w_i$.

Utility of agent $i$ for a monetary amount $y$ and house $h_j$ is denoted $u_i(y, h_j)$.

Each $u_i$ is continuous and non-decreasing in money for each house.

# NTU Co-op Game

A game with nontransferable utility is a pair $(N, V)$ where $N$ is a finite set of players, and, for every coalition $S \subseteq N$, $V(S)$ is a subset of $\Re^n$ satisfying:

1. If $S \neq \emptyset$, then $V(S)$ is non-empty and closed; and $V(\emptyset) = \emptyset$.
2. For every $i \in N$ there is a $V_i$ such that for all $x \in \Re^n$, $x \in V(i)$ if and only if $x_i \leq V_i$.
3. If $x \in V(S)$ and $y \in \Re^n$ with $y_i \leq x_i$ for all $i \in S$ then $y \in V(S)$ (lower comprehensive).
4. The set $\{x \in V(N) : x_i \leq V_i\}$ is compact.

# NTU Co-op Game

Core of an NTU-game $(N, V)$ is all payoff vectors that are feasible for the grand coalition $N$ and that cannot be improved upon by any coalition, including $N$ itself.

If $x \in V(N)$, then coalition $S$ can improve upon $x$ if there is a $y \in V(S)$ with $y_i > x_i$ for all $i \in S$.

Core of the game $(N, V)$ is

$$V(N) \setminus \cup_{S \subseteq N} int V(S).$$

NTU game $(N, V)$ is balanced if for any balanced collection $\mathcal{C}$ of subsets of $N$,

$$\cap_{S \in \mathcal{C}} V(S) \subseteq V(N).$$

Scarf's lemma: a balanced NTU game has a non-empty core.

# Quinzii Housing Model

For any $S \subseteq N$ let $F_S$ be the set of feasible solutions to

$$\sum_{j \in S} x_{ij} = 1 \ \forall i \in S$$

$$\sum_{i \in S} x_{ij} = 1 \ \forall j \in S$$

$$x_{ij} \in \{0, 1\} \ \forall i, j \in N$$

$x_{ir} = 1$ means agent $i$ receives agent $r$'s house.

# Quinzii Housing Model

Allocation $(y, x)$ is **feasible** for $S \subseteq N$ if $x \in F_S$ and $\sum_{i \in S} y_i \leq \sum_{i \in S} w_i$.

$A(S)$ = set of feasible allocations for $S \subseteq N$.

Core is set of allocations in $A(N)$ such that no $S \subseteq N$ can find an allocation in $A(S)$ that gives *each* member of $S$ *strictly* more utility.

# Quinzii Housing Model

$$V(S) = \{\{v_i\}_{i \in S} : \exists (y, x) \in A(S), v_i \leq u_i(y_i, \sum_{j \in S} x_{ij}), \ \forall i \in S\}$$

1. $V(S)$ is closed.
2. $V(S)$ is lower comprehensive.
3. The projection of $V(S) \setminus \cup_{i \in S} int V(i)$ onto $\Re^S$ is non-empty and bounded (this is individual rationality).

# Quinzii Housing Model

$$m_{ij}(v) = \inf\{m \in \Re_+ : u_i(m, h_j) \geq v_i\}.$$

$m_{ij}(v)$ is least amount of money given to agent $i$ to guarantee a utility of at least $v_i$ when agent $i$ holds house $h_j$.

$v \in V(S)$ iff $\exists x^S \in F_S$ such that

$$\sum_{i \in S} \sum_{j \in S} m_{ij}(v) x_{ij}^S \leq \sum_{i \in S} w_i.$$

Complete Bipartite Graph

$D \cup H$ = set of vertices (single doctors and hospitals with capacity 1)

$E$ = complete set of edges

$\delta(v) \subseteq E$ set of edges incident to $v \in D \cup H$

Each $v \in D \cup H$ has a strict ordering $\succ_v$ over edges in $\delta(v)$

Let $x_e = 1$ if we select edge $e = (d, h)$ and 0 otherwise.

Selecting edge $e = (d, h)$ corresponds to matching doctor $d$ to hospital $h$.

The convex hull of all feasible matchings is given by

$$\sum_{e \in \delta(v)} x_e \leq 1 \; \forall v \in D \cup H$$

A matching $x$ is blocked by a pair $e = (d, h)$ if

1. $x_e = 0$
2. either $h \succ_d h'$ where $x_{dh'} = 1$, *and*,
3. $d \succ_h d'$ where $x_{d'h} = 1$.

A matching $x$ is called stable if it cannot be blocked by any pair.

# Scarf's Lemma (1967)

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$\mathcal{P} = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$.

Each row $i \in [m]$ of $A$ has a strict order $\succ_i$ over the set of columns $j$ for which $a_{ij} > 0$.

A vector $x \in \mathcal{P}$ **dominates** column $r$ if there exists a row $i$ such that $\sum_j a_{ij}x_j = b_i$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

$\mathcal{P}$ has an extreme point that dominates every column of $A$.

# Stable Matching w/o Couples

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$$\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in D \cup H$$

Each row $i \in [m]$ of $A$ has a strict order $\succ_i$ over the set of columns $j$ for which $a_{ij} > 0$.

Each $v \in D \cup H$ has a strict ordering $\succ_v$ over edges in $\delta(v)$.

$x \in \mathcal{P}$ **dominates** column $r$ if $\exists i$ such that $\sum_j a_{ij} x_j = b_i$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

For all $e \in E$ there is a $v \in D \cup H$ such that $e \in \delta(v)$ and

$$\sum_{f \succ_v e} x_f + x_e = 1$$

# Stable Matching with Couples

$D$ = set of single doctors

$C$ = set of couples, each couple $c \in C$ is denoted $c = (f_c, m_c)$

$D^* = D \cup \{m_c | c \in C\} \cup \{f_c | c \in C\}$.

$H$ = set of hospitals

Each $s \in D$ has a strict preference relation $\succ_s$ over $H \cup \{\emptyset\}$

Each $c \in C$ has a strict preference relation $\succ_c$ over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$

# Stable Matching with Couples

$k_h$ = capacity of hospital $h \in H$

Preference of hospital $h$ over subsets of $D^*$ is modeled by choice function $ch_h(.) : 2^{D^*} \to 2^{D^*}$.

$ch_h(.)$ is responsive

$h$ has a strict priority ordering $\succ_h$ over elements of $D \cup \{\emptyset\}$.

$ch_h(R)$ consists of the (upto) $\min\{|R|, k_h\}$ highest priority doctors among the feasible doctors in $R$.

# Blocking

$\mu =$ matching

$\mu_h =$ the subset of doctors matched to $h$.

$\mu_s =$ position that single doctor $s$ receives.

$\mu_{f_c}, \mu_{m_c} =$ positions that female member and male member of the couple $c$ obtain, respectively.

# Individual Rationality

$\mu$ is individually rational if

- $ch_h(\mu_h) = \mu_h$ for any hospital $h$
- $\mu_s \succeq_s \emptyset$ for any single doctor $s$
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \mu_{m_c})$
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\mu_{f_c}, \emptyset)$
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \emptyset)$

# Blocking #1

A matching $\mu$ can be blocked in one of three ways.

By a single doctor, $d \in D$ and a lone hospital, $h \in H$

- $h \succ_d \mu(d)$
- $d \in ch_h(\mu(h) \cup d)$

# Blocking #2

By a couple $c \in C$ and a pair of distinct hospitals $h, h' \in H$

- $(h, h') \succ_c \mu(c)$
- $f_c \in ch_h(\mu(h) \cup f_c)$ when $h \neq \emptyset$
- $m_c \in ch_{h'}(\mu(h') \cup m_c)$ when $h' \neq \emptyset$

# Blocking #3

By a couple $c \in C$ and a single hospital $h \in H$

- $(h, h) \succ_c \mu(c)$
- $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$

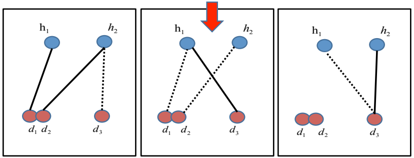
# Non-Existence: Roth (84), Klaus-Klijn (05)

Hospital 1: $d_1 \succ_{h_1} \succ d_3 \succ_{h_1} \emptyset \succ_{h_1} d_2$, $k_{h_1} = 1$

Hospital 2: $d_3 \succ_{h_2} \succ d_2 \succ_{h_2} \emptyset \succ_{h_2} d_1$, $k_{h_2} = 1$

Couple $\{1, 2\}$: $(h_1, h_2) \succ_{(d_1, d_2)} \emptyset$

Single doctor $d_3$: $h_1 \succ h_2$



# Matching with Couples

Given an instance of a matching problem with couples, determining if it has a stable matching is NP-hard.

Roth & Peranson algorithm (1999): Heuristic modification of Gale-Shapley

On all recorded instances in NRMP, it returns a matching that is stable wrt reported preferences.

Kojima-Pathak-Roth (QJE 2013); Ashlagi-Braverman-Hassidim (OR 2014)

- Randomized preferences,
- Market size increases to infinity
- Fraction of couples goes to 0

Probability that Roth & Peranson gives a stable matching approaches 1.

# Matching with Couples

US: resident matching: 40,000 doctors participate every year, fraction of couples couples can be upto 10%.

Ashlagi et al: If the fraction of couple does *not* goes to 0, the probability of no stable matching is positive.

Simulations by Biro et. al.: when the number of couples is large, Roth & Perason algorithm fails to find a stable match.

# Matching with Couples

(Nguyen & Vohra)

Given any instance of a matching problem with couples, there is a 'nearby' instance that is guaranteed to have stable matching.

For any capacity vector $k$, there exists a $k'$ and a stable matching with respect to $k'$, such that

1. $|k_h - k'_h| \leq 4 \ \forall h \in H$
2. $\sum_{h \in H} k_h \leq \sum_{h \in H} k'_h \leq \sum_{h \in H} k_h + 9.$

# Matching with Couples

$x(d, h) = 1$ if single doctor $d$ is assigned to hospital $h \in H$ and zero otherwise.

$x(c, h, h') = 1$ if $f_c$ is assigned to $h$ and $m_c$ is assigned to $h'$ and zero otherwise.

$x(c, h, h) = 1$ if $f_c$ and $m_c$ are assigned to hospital $h \in H$ and zero otherwise.

# Matching with Couples

Every 0-1 solution to the following system is a feasible matching and vice-versa.

$$\sum_{h \in H} x(d,h) \leq 1 \ \forall d \in D \tag{1}$$

$$\sum_{h,h' \in H} x(c,h,h') \leq 1 \ \forall c \in D \tag{2}$$

$$\sum_{d \in D} x(d,h) + \sum_{c \in C} \sum_{h' \neq h} x(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} x(c,h',h) + \sum_{c \in C} 2x(c,h,h) \leq k_h \ \forall h \in H \tag{3}$$

# Matching with Couples

Constraint matrix and RHS satisfy conditions of Scarf's lemma.

Each row associated with single doctor or couple (1-2) has an ordering over the variables that 'include' them from their preference ordering.

Row associated with each hospital (3) does not have a natural ordering over the variables that 'include' them.

(1-3) *not* guaranteed to have integral extreme points.

# Matching with Couples

Use choice function to induce an ordering over variables for each hospital.

Construct ordering so that a dominating solution wrt this ordering will correspond to a stable matching.

Must show that a dominating solution under this made-up ordering corresponds to a stable matching.

Apply Scarf's Lemma to get a 'fractionally' stable solution.

Round the fractionally stable solution into an integer solution that preserves stability (sharpening of Shapley-Folkman-Starr Lemma).

# Shapley-Folkman-Starr (Starr (1969))

$\{S^j\}_{j=1}^n$ be a collection of sets in $\Re^m$ with $n > m$.

$S = \sum_{i=1}^m conv(S^i)$

Every $b \in S$ can be expressed as $\sum_{j=1}^n x^j$ where $x^j \in conv(S^j)$ for all $j = 1, \ldots, n$ and $|\{j : x^j \in S^j\}| \geq n - m$.

Let $A$ be an $m \times n$ 0-1 matrix and $b \in \Re^m$ with $n > m$.

Denote each column $j$ of the $A$ matrix by $a^j$.

$S^j = \{a^j, 0\}$.

Suppose $b = Ax^* = \sum_{j=1}^{n} a^j x_j^*$ where $0 \leq x_j^* \leq 1 \ \forall j$.

SFS $\Rightarrow b = \sum_{j=1}^{n} a^j y_j$ where each $y_j \in [0, 1]$ with at at least $n - m$ of them being 0-1.

$y$ has at most $m$ fractional components.

Let $y^*$ be obtained by rounding up each fractional component.

$||Ay^* - b||_\infty \leq m$

# Shapley-Folkman-Starr

Suppose $b = \sum_{j=1}^{n} a^j y_j$ with at least $m+1$ components of $y$ being fractional.

Let $\bar{C}$ be submatrix of $A$ that corresponds to integer components of $y$.

Let $C$ be submatrix of $A$ that corresponds to fractional components of $y$.

$$b = \bar{C} y_{\bar{C}} + C y_C$$

$$b - \bar{C}y_{\bar{C}} = Cy_C$$

Suppose (for a contradiction) that $C$ has more columns ($\geq m+1$) than rows ($m$).

$\Rightarrow \; ker(C) \neq 0 \;\Rightarrow\; \exists z \in ker(C)$

Consider $y_C + \lambda z$.

$$b - \bar{C}y_{\bar{C}} = C[y_C + \lambda z]$$

Choose $\lambda$ to make at least one component of $y_C$ take value 0 or 1.

# Shapley-Folkman-Starr

Exchange economy with non-convex preferences i.e., upper contour sets of utility functions are non-convex.

$n$ agents and $m$ goods with $n \geq m$.

Starr (1969) identifies a price vector $p^*$ and a feasible allocation with the property that at most $m$ agents do not receiving a utility maximizing bundle at the price vector $p^*$.

# Shapley-Folkman-Starr

$u_i$ is agent $i$'s utility function.

$e^i$ is agent $i$'s endowment

Replace the upper contour set associated with $u_i$ for each $i$ by its convex hull.

Let $u_i^*$ be the quasi-concave utility function associated with the convex hull.

$p^*$ is the Walrasian equilibrium prices wrt $\{u_i^*\}_{i=1}^n$.

$x_i^*$ be the allocation to agent $i$ in the associated Walrasian equilibrium.

# Shapley-Folkman-Starr

For each agent $i$ let

$$S^i = \arg\max\{u_i(x) : p^* \cdot x \leq p^* \cdot e^i\}$$

$w =$ vector of total endowments and $S^{n+1} = \{-w\}$.

Let $z^* = \sum_{i=1}^n x_i^* - w = 0$ be the excess demand with respect to $p^*$ and $\{u_i^*\}_{i=1}^n$.

$z^*$ is in convex hull of the Minkowski sum of $\{S^1, \ldots, S^n, S^{n+1}\}$.

By the SFS lemma $\exists x_i \in conv(S^i)$ for $i = 1, \ldots, n$, such that $|\{i : x_i \in S^i\}| \geq n - m$ and $0 = z^* = \sum_{i=1}^n x_i - w$.

$$\max \sum_{j=1}^{n} f_j(y_j)$$

$$\text{s.t. } Ay = b$$

$$y \geq 0$$

$A$ is an $m \times n$ matrix with $n > m$.

$f_j^*(\cdot)$ is the smallest concave function such that $f_j^*(t) \geq f_j(t)$ for all $t \geq 0$

# Shapley-Folkman-Starr

Solve the following to get $y^*$:

$$\max \sum_{j=1}^{n} f_j^*(y_j)$$

$$\text{s.t. } Ay = b$$

$$y \geq 0$$

$e_j = \sup_t [f_j^*(t) - f_j(t)]$

Sort $e_j$'s in decreasing order.

$$\sum_{j=1}^{n} f_j(y_j^*) \geq \sum_{j=1}^{n} f_j^*(y_j^*) - \sum_{j=1}^{m} e_j$$

# Shapley-Folkman-Starr: Weakness

Does not allow you to control which constraints to violate.

Want to satisfy (1-2) but are willing to violate (3).

Degree of violation is large because it makes no use of information about $A$ matrix. In our case each variable intersects exactly two constraints.

We use this sparsity to show that no constraint can contain many occurences of a fractional variable.

# Iterative Rounding

Kiralyi, Lau & Singh (2008)

Gandhi, Khuller, Parthasarathy & Srinivasan (2006)

# Iterative Rounding

Step 0: Choose extreme point $x^* \in \arg\max\{w \cdot x : Ax \leq b, x \geq 0\}$.

Step 1: If $x^*$ is integral, output $x^*$, otherwise continue to either Step 2a or 2b.

Step 2a: If any coordinate of $x^*$ is integral, fix the value of those coordinates, and update the linear program and move to step 3.

# Iterative Rounding: Updating LP

$C =$ columns of $A$ that correspond to the non-integer valued coordinates of $x^*$.

$\overline{C} =$ columns of $A$ that correspond to the integer valued coordinates of $x^*$.

$A_C$ and $A_{\overline{C}}$ be the sub-matrices of $A$ that consists of columns in $C$ and the complement $\overline{C}$, respectively.

Let $x_C$ and $x_{\overline{C}}$ be the sub-vector of $x$ that consists of all coordinates in $C$ and $\overline{C}$. The updated LP is:

$$\max\{w_C \cdot x_C : \text{ s.t. } D_C \cdot x_C \leq d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}\}.$$

Step 2b: If all coordinates of $x^*$ fractional, delete *certain* rows of $A$ (to be specified later) from the linear program. Update the linear program, move to Step 3.

Step 3: Solve the updated linear program $\max\{w \cdot x \text{ s.t. } Ax \leq b\}$ to get an extreme point solution. Let this be the new $x^*$ and return to Step 1.

# Rounding & Matching

Start with an extreme point solution $x^*$ to (1-3)

$$\sum_{h \in H} x(d,h) \leq 1 \ \forall d \in D$$

$$\sum_{h,h' \in H} x(c,h,h') \leq 1 \ \forall c \in D$$

$$\sum_{d \in D} x(d,h) + \sum_{c \in C} \sum_{h' \neq h} x(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} x(c,h',h) + \sum_{c \in C} 2x(c,h,h) \leq k_h \ \forall h \in H$$

Round $x^*$ into a 0-1 solution $\bar{x}$ such that

$$\sum_{h \in H} \bar{x}(d,h) \leq 1 \ \forall d \in D$$

$$\sum_{h,h' \in H} \bar{x}(c,h,h') \leq 1 \ \forall c \in D$$

$$\sum_{d \in D} \bar{x}(d,h) + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h',h) + \sum_{c \in C} 2\bar{x}(c,h,h) \leq k_h + 3 \ \forall h \in H$$

If every component of $x^*$ is $< 1$ (all fractional), there must be a hospital $h$ where

$$\sum_{d \in D} \lceil x^*(d,h) \rceil + \sum_{c \in C} \sum_{h' \neq h} \lceil x^*(c,h,h') \rceil + \sum_{c \in C} \sum_{h' \neq h} \lceil x^*(c,h',h) \rceil + \sum_{c \in C} 2 \lceil x^*(c,h,h) \rceil \leq k_h + 3$$

In setp 2(b) of the iterative rounding method, delete this row/constraint.

If false, then for *every* hospital $h$

$$\sum_{d\in D}\lceil x^*(d,h)\rceil+\sum_{c\in C}\sum_{h'\neq h}\lceil x^*(c,h,h')\rceil+\sum_{c\in C}\sum_{h'\neq h}\lceil x^*(c,h',h)\rceil+\sum_{c\in C}2\lceil x^*(c,h,h)\rceil \geq k_h+4$$

1. Every column of $A$ contains exactly two non-zero entries.
2. The columns of $A$ that correspond to non-zero entries of $x^*$ are linearly independent and form a basis.
3. The number of non-zero entries that intersect row $h$ is at least $k_h + 4$.

# Proof of Scarf's Lemma

$A$ is an $m \times n$ *positive* matrix.

$\mathcal{P} = \{x \geq 0 : Ax \leq e\}$.

$U = \{u_{ij}\}$ is an $m \times n$ positive matrix.

$x \in F$ is **dominating** if for each column index $k$ there is a row index $i$ such that

1. $\sum_{j=1}^{n} a_{ij} x_j = 1$ and
2. $u_{ik} \leq \min_{j : x_j > 0} u_{ij}$.

# Proof of Scarf's Lemma

Associate a 2 person game with the pair $(U, A)$.

Fix a large integer $t$, let $w_{ij} = -\frac{1}{u_{ij}^t}$.

Payoff matrix for row player will be $A$.

Payoff matrix for the column player will be $W$.

# Proof of Scarf's Lemma

$(x^*, y^*)$ is an equilibrium pair of mixed strategies for the game.

$x^*$ is mixed strategy for column (payoff matrix $W$).

$y^*$ is mixed strategy for row player (payoff matrix $A$).

For all pure strategies $q$ for row:

$$\sum_{i=1}^{m} y_i^* [\sum_{j=1}^{n} a_{ij} x_j^*] \geq \sum_{j=1}^{n} a_{qj} x_j^*, \ \forall q. \tag{4}$$

(4) will bind when $y_q^* > 0$.

# Proof of Scarf's Lemma

For all pure strategies $r$ for column

$$\sum_{i=1}^{m} y_i^*[\sum_{j=1}^{n}(u_{ij})^{-t}x_j^*]\sum_{j=1}^{n} x_j^*[\sum_{i=1}^{m}(u_{ij})^{-t}y_i^*] \leq \sum_{i=1}^{m}(u_{ir})^{-t}y_i^*, \ \forall r \tag{5}$$

Therefore, for each column $r$ there is a row index $i_r$ with $y_{i_r}^* > 0$ such that

$$\sum_{j=1}^{n}(u_{i_rj})^{-t}x_j^* \leq (u_{i_rr})^{-t} \Rightarrow u_{i_rr} \leq (\frac{1}{\sum_{j=1}^{n}(u_{i_rj})^{-t}x_j^*})^{1/t}.$$

# Proof of Scarf's Lemma

$u_{i_r r} \leq \frac{u_{i_r k}}{(x_k^*)^{1/t}}$ for all $x_k^* > 0$.

$t \to \infty$, $x^*$ and $y^*$ will converge to some $\bar{x}$ and $\bar{y}$ respectively.

For sufficiently large $t$, $\bar{x}_j > 0 \Rightarrow x_j^* > 0$ and $\bar{y}_j > 0 \Rightarrow y_j^* > 0$.

$$u_{i_r r} \leq \frac{u_{i_r k}}{(x_k^*)^{1/t}} \to u_{i_r k}.$$

Recall that for index $i_r$ we have $y_{i_r}^* > 0$. Therefore,
$\sum_{j=1}^m a_{i_r j} x_j^* = \sum_{i=1}^n y_i^* [\sum_{j=1}^m a_{ij} x_j^*] = v$.

$x = \frac{\bar{x}}{v} \in F$ and $\sum_{j=1}^n a_{i_r j} x_j = 1$.