Some inequalities related to functional calculus

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- General problem : How to Obtain a result on S_p from one on S_q via functional calculus ?
- To understand the Schatten p-classes.
  (more generally n.c. L_p with von Neumann algebras).
- To define new bounded maps $S_p \to S_q$.

- S_p^n, S_p : Schatten p-classes $0 < p \leqslant \infty$ with norm $\|.\|_p$.
  For $p = \infty$, $(M_n, \|.\|_\infty)$.
  $S_p^{n,sa}$, S_p^{n+}
- A Markov map $T : M_n \to M_m$ is a unital completely positive maps that preserves the normalized trace (to deal with n.c. probability spaces)
  Thus $T : S_p^n \to S_p^m$ is contractive.
- $\|.\|$ will denote a unitarily norm on M_n
  Recall that $\|A\| \leqslant \|B\|$ for all $\|.\|$ iff there is a Markov map T such that $|A| \leqslant T(|B|)$.
- Most of the results will hold for semi-finite (or type III) von Neumann algebras

The basic inequality

Ando (88) ; Birman, Koplienko, Solomjak (75)

If $f : [0, \infty[\to [0, \infty[$ is an operator monotone function, then for $a, b \in M_n^+$

$$\|f(a) - f(b)\| \leqslant \|f(|a - b|)\|$$

The inequality reverses if f^{-1} is operator monotone.

In particular for $f(x) = x^{p/q}$, $q \geqslant p$, an extension of the Power-Stormer inequality

$$\|a^{p/q} - b^{p/q}\|_q \leqslant \|a - b\|_p^{p/q}.$$

Based on integral decompositions : $f(x) = \alpha + \beta x + \int \frac{x}{x+s} d\mu(s)$
It gives the modulus of continuity of $x \mapsto x^{p/q}$ from $S_p^+ \to S_q^+$.

There are many possible variations

R. (19)

If $f : [0, \infty[\to [0, \infty[$ is operator monotone, $a, b \in M_n^+$

$$\|(a-b)exp(f(a)-f(b))\| \leqslant \|(a-b)exp(f(|a-b|))\|.$$

If $g, h : [0, \infty[\to [0, \infty[$, g operator convex and h non decreasing

$$\|hf(|a-b|)\| \leqslant \|h(|b-a|)(f(b)-f(a))\|$$

In the case of power functions

R. (16)

If $p \geqslant 2$, $a, b \in M_n^+$, then

$$\mathrm{tr}(|a-b|^p) \leqslant \mathrm{tr}\big((a-b)(a^{p-1}-b^{p-1})\big)$$

Original motivation

Corollary

If $E : M_n \to M_n$ is a conditional expectation then for all $x \in M_n^+$ and $p \geqslant 2$

$$\|x - E(x)\|_p \leqslant \|x\|_p.$$

False if $p < 2$ even in the commutative case.

Ball-Carlen-Lieb (94)

If $1 < p < 2$, $x, y \in M_n$

$$\|x+y\|_p^2 + \|x-y\|_p^2 \geqslant 2\|x\|_p^2 + 2(p-1)\|y\|_p^2$$

R.-Xu (16), Cond. version

If $1 < p < 2$, $E : M_n \to M_n$ a cond. expectation, $x \in M_n$

$$\|x\|_p^2 \geqslant \|\mathbb{E}(x)\|_p^2 + (p-1)\|x - \mathbb{E}(x)\|_p^2$$

It reverses if $p > 2$.

Applications to hypercontractivity $L_2 \to L_p$ of Markov semi-groups by iterations.

Ando's inequality only deals wih positive operators and operator monotone functions

Aleksandrov-Peller (2010)

If $1 < p < q < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is p/q-Hölder, then for $a, b \in M_n^{sa}$

$$\|f(a) - f(b)\|_q \leqslant C_{p,q} \|a - b\|_p^{p/q}$$

Mazur maps $M_{p,q}$ on M_n

Let $0 < p, q < \infty$, if a has polar decomposition $a = u|a|$

$$M_{p,q}(a) = u|a|^{p/q}.$$

It is an homeomorphism from $S_p^n \to S_q^n$.

Raynaud (02) showed that there are uniformly continuous uniformly in n but without precise estimates (ultrapower techniques). The same holds for $f_{p/q} : x \mapsto |x|^{p/q}$

Aleksandrov-Peller with a 2×2 trick says that

If $1 < p < q < \infty$, $M_{p,q}, f_{p/q}$ are p/q-Hölder as in the commutative case.

The proof works use a Cesaro operator and $C_{p,q} \to_{p \to 1} \infty$.
False with $p = 1$, weak-type.
On the opposite direction, we have using basic algebra

$M_{2,1}$ and f_2 are Lipschitz on balls (as in the commutative case).

Question : What happen for other values of p, q ?

R. (2015, 2018, 2021)

If $0 < p < q < \infty$, $M_{p,q}, f_{p/q}$ are p/q-Hölder, more precisely with $f = M_{p,q}$ or $f = f_{p/q}$

$$\|f(a) - f(b)\|_q \leqslant C_{p,q}\|a - b\|_p^{p/q}.$$

If $0 < q < p < \infty$, $M_{p,q}, f_{p/q}$ are Lipschitz on balls, more preciselly

$$\|f(a) - f(b)\|_q \leqslant C'_{p,q}\|a - b\|_p(\|a\|_p + \|b\|_p)^{p/q-1}.$$

Strange behaviour the constant $C_{1,\theta} \to_{\theta\to 1} \infty$.
Much more involved for exponents <1

R. (2021)

Fix $\alpha > 0$, $0 < s < \infty$ and $0 < r \leqslant \infty$. Let p be so that $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$ and q so that $\frac{1}{q} = \frac{1+\alpha}{s} + \frac{1}{r}$. If $d \in M_n^+$, $x \in M_n$:

$$\|xd^{1+\alpha}\|_q \leqslant C_{\alpha,q}\|d\|_s^{\alpha}.\|dx + xd\|_p.$$

Application to Markov Maps

Quantitative estimates on almost multiplicative domains

Caspers, Parcet, Perrin, R. (2015)

Let $T : M_n \to M_n$ be a Markov map for any $1 \leqslant p \leqslant \infty$ and $x \in M_n^+$

$$\big\| T(x) - T(\sqrt{x})^2 \big\|_{2p} \leqslant \frac{1}{2} \big\| T(x^2) - T(x)^2 \big\|_p^{\frac{1}{2}}.$$

Application to local approximation

Corollary

There is $C > 0$, such that if T is Markov, $y \in M_n$ has polar decomposition $y = u|y|$ and any $0 < \theta \leqslant 1$

$$\big\| T(u|y|^\theta) - u|y|^\theta \big\|_{\frac{2}{\theta}} \leqslant C \big\| T(y) - y \big\|_2^{\frac{\theta}{4}} \|y\|_2^{\frac{3\theta}{4}}.$$

Another statement if $\theta > 1$.

Possible application :

$x \in L_\infty(\mathbb{R}) \cap L_1(\mathbb{R})$, ε small

$$supp(\hat{x}) \subset [-\varepsilon, \varepsilon] \quad \Rightarrow \quad supp(\widehat{x^2}) \subset [-2\varepsilon, 2\varepsilon]$$

What about the opposite ?
Can we say something on $supp(\hat{x})$ assuming that $supp(\widehat{x^2}) \subset [-\varepsilon, \varepsilon]$?

Application to Markov Maps

If $T : M_n \to M_n$ is Markov,

$$N = \{x \mid T(x) = x\} \text{ is a subalgebra with cond. exp } \mathbb{E}$$

We set $M_n^0 = \mathrm{Ker}\,\mathbb{E} = N^\perp$,

Spectral gaps

We say that T has a p-spectral gap if there is $0 < \delta_p < 1$ such that

$$\forall x \in M_n^0, \quad \|T(x)\| \leqslant (1-\delta_p)\|x\|_p$$

Factorizable maps

T is factorizable if there is a Markov representation π and a Markov conditional expectation E such that $T = E\pi$

Conde-Alonso, Parcet, R. (18)

1. If T has a 2-spectral gap then it has a p-spectral gap for $1 < p < \infty$.
2. If T is factorizable with a p-spectral gap some $1 < p < \infty$ then it has a 2-spectral gap

1. has also been obtained by Heilman, Mossel (17), Oleszkiewcz using only interpolation. Their quantitive estimate is better for $p > 2$.