Characterizing quantum correlations of fixed dimension

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Motivation

- Entanglement leads to quantum correlations stronger than classical correlations [Bell 64]:

  Bell inequalities

Questions

- How to quantify classical vs quantum correlations?
- How does this depend on the underlying dimension?
- What is the computational complexity of correlations?

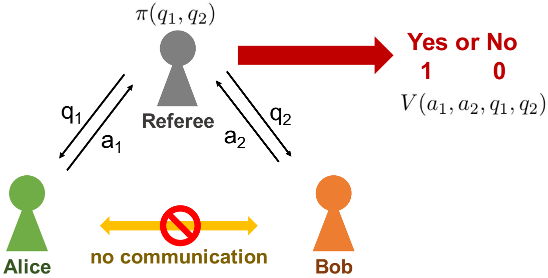
- Quantifying correlations in multipartite systems from computer science perspective:

  Non-local games

Non-local games

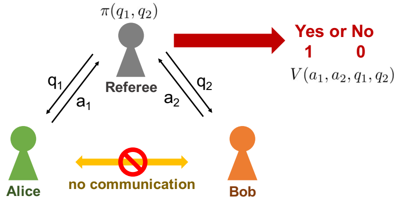
Setting

- Two-prover one-round games:



- Referee gives Alice and Bob **questions** $q_1 \in Q_1$ and $q_2 \in Q_2$ according to the probability distribution $\pi(q_1, q_2)$
- Alice and Bob give **answers** $a_1 \in A_1$ and $a_2 \in A_2$ back to the referee
- Alice and Bob win or lose according to the **rule function** $V : A_1 \times A_2 \times Q_1 \times Q_2 \to \{0, 1\}$

Maximum winning probabilities



- **CHSH game** $|A_1| = |A_2| = |Q_1| = |Q_2| = 2, \pi(q_1, q_2) = \frac{1}{4}$ [Clauser *et al.* 69]:

$V_{\text{CHSH}}(a_1, a_2, q_1, q_2) = 1$ if $q_1 \cdot q_2 = a_1 \oplus a_2$, and $V_{\text{CHSH}}(a_1, a_2, q_1, q_2) = 0$ otherwise

- Correlations quantified by respective **maximum winning probabilities**:

$$\omega_C(V,\pi) := \sup_{(e,d)} \sum_{q_1,q_2} \pi(q_1,q_2) \sum_{a_1,a_2} V(a_1,a_2,q_1,q_2) e(a_1|q_1) d(a_2|q_2)$$

$$\omega_{Q(T)}(V,\pi) := \sup_{(E,D,\rho)} \sum_{q_1,q_2} \pi(q_1,q_2) \sum_{a_1,a_2} V(a_1,a_2,q_1,q_2) \mathrm{Tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right]$$

$$\omega_Q(V,\pi) := \sup_{(\mathcal{H},E,D,\rho)} \sum_{q_1,q_2} \pi(q_1,q_2) \sum_{a_1,a_2} V(a_1,a_2,q_1,q_2) \mathrm{Tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right]$$

Previous results

- **CHSH game** $\omega_C(V_{\text{CHSH}}) \equiv \omega_C(V_{\text{CHSH}}, 1/4)$ [Clauser *et al.* 69, Tsirelson 80]:

$$\omega_C(V_{\text{CHSH}}) = 0.75 < 0.85 \approx \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \omega_{Q(2)}(V_{\text{CHSH}}) = \omega_Q(V_{\text{CHSH}})$$

Hardness:

- Approx. $\omega_C(V, \pi)$ to constant multiplicative factor NP-hard [Arora *et al.* 98]
- Approx. $\omega_Q(V, \pi)$ not possible for an algorithm running in finite time [Ji *et al.* 20]

Algorithms:

- Quasi-polynomial time algorithm to approximate $\omega_C(V, \pi)$ to constant additive error for free games [Aaronson *et al.* 14, Brandão & Harrow 17]
- Polynomial time algorithms to compute $\omega_Q(V, \pi)$ for XOR [Cleve *et al.* 04] and unique games [Kempe *et al.* 10]
- NPA hierarchy: strong heuristics to approximate $\omega_Q(V, \pi)$ [Navascués *et al.* 08]

Question

What about $\omega_{Q(T)}(V, \pi)$ for fixed dimension T?

Quantum correlations of fixed dimension

Quantum correlations of fixed dimension

Motivation:

- Quantum information processing, e.g., quantum error correction [B. *et al.* 18]
- Device-independence: **dimension witness** for $\omega_{Q(T)}(V, \pi) < \omega_Q(V, \pi)$

Algorithms:

- Various heuristics to approximate $\omega_{Q(T)}(V, \pi)$ [Navascués *et al.* 14/15], but known worst case guarantee is **exponential**

Main result

For free games, i.e., $\pi(q_1, q_2) = \pi(q_1) \times \pi(q_2)$, with $|Q_1| = |Q_2| =: Q$ and $|A_1| = |A_2| =: A$, we give an approximation algorithm with complexity

$$\exp\left(\mathcal{O}\left(\frac{T^{12}}{\varepsilon^2}\log(\mathbf{A}T)\left(\log(\mathbf{Q}) + \log(\mathbf{A}T)\right)\right)\right)$$

to compute additive ε-approximations of $\omega_{Q(T)}(V, \pi)$. That is, for fixed dimension T, the complexity scales **polynomially** in Q and **quasi-polynomially** in A.

Connection to quantum separability problem

$$\omega_{Q(T)}(V,\pi) := \sup_{(E,D,\rho)} \sum_{q_1,q_2} \pi(q_1,q_2) \sum_{a_1,a_2} V(a_1,a_2,q_1,q_2) \mathrm{Tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right]$$

$$\omega_{Q(T)}(V,\pi) = |T|^2 \cdot \sup_{(E,D,\rho)} \mathrm{Tr}\left[\left(V_{A_1A_2Q_1Q_2} \otimes \Phi_{T\hat{T}|S\hat{S}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}} \otimes \rho_{S\hat{S}}\right)\right]$$

s.t. $\rho_{S\hat{S}} \geq 0\,, \mathrm{Tr}[\rho_{S\hat{S}}] = 1$

$$E_{A_1Q_1T} = \sum_{a_1,q_1} \pi_1(q_1)|a_1q_1\rangle\langle a_1q_1|_{A_1Q_1} \otimes \frac{E_T(a_1|q_1)}{|T|} \geq 0\,, E_{Q_1T} = \sum_{q_1} \pi_1(q_1)|q_1\rangle\langle q_1|_{Q_1} \otimes \frac{\mathrm{id}_T}{|T|}$$

$$D_{A_2Q_2\hat{T}} = \sum_{a_2,q_2} \pi_2(q_2)|a_2q_2\rangle\langle a_2q_2|_{A_2Q_2} \otimes \frac{D_{\hat{T}}(a_2|q_2)}{|T|} \geq 0\,, D_{Q_2\hat{T}} = \sum_{q_2} \pi_2(q_2)|q_2\rangle\langle q_2|_{Q_2} \otimes \frac{\mathrm{id}_{\hat{T}}}{|T|}$$

- By linearity, equivalently optimize over

$$\sum_i p_i \cdot E^i_{A_1Q_1T} \otimes D^i_{A_2Q_2\hat{T}} \otimes \rho^i_{S\hat{S}}$$

  $\Rightarrow$ tripartite **quantum separability problem** with linear constraints!

- Hard problem [Gharibian 10], but DPS hierarchy [Doherty *et al.* 02] gives approximation algorithms via **quantum de Finetti theorems**

Quantum de Finetti theorems

Monogamous entanglement

- Quantum states ρ_{AB} are called ***n*-shareable** on B with respect to A if

$$\rho_{AB_1^n} \equiv \rho_{AB_1 \ldots B_n} \text{ with } \rho_{AB_j} = \rho_{AB} \; \forall j \in [n]$$

$\Rightarrow$ characterizes separable states [Stoermer 69]

Quantum de Finetti

For n-shareable quantum states ρ_{AB}, there exist probabilities $\{p_i\}_{i \in I}$ and quantum states $\sigma_A^i, \omega_B^i \; \forall i \in I$ such that [Christandl *et al.* 07]

$$\left\| \rho_{AB} - \sum_{i \in I} p_i \cdot \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \frac{|B|^2}{n}.$$

- ***n*-shareable is efficient criteria** to check (positive semi-definite) — though $n \gg |B|^2$ needed for good approximation on the set of separable states
- Tripartite quantum de Finetti with linear constraints?

Adapted quantum de Finetti

Tripartite with linear constraints

For ρ_{ABC} **n-shareable** on B wrt AC^n and n-shareable on C wrt AB^n, there exist probabilities $\{p_i\}_{i\in I}$ and quantum states σ_A^i, ω_B^i and γ_C^i $\forall i \in I$ such that

$$\left\| \rho_{ABC} - \sum_{i\in I} p_i \cdot \sigma_A^i \otimes \omega_B^i \otimes \gamma_C^i \right\|_1 \leq O\left(|B||C| \cdot \sqrt{\frac{\log|A| + \log|B|}{n} + \frac{\log|A|}{n}} \right)$$

where for linear maps $\mathcal{E}_{A\to\tilde{A}}$, $\Lambda_{B\to\tilde{B}}$, and $\Gamma_{C\to\tilde{C}}$ and operators $\mathcal{X}_{\tilde{A}}$, $\mathcal{Y}_{\tilde{B}}$, and $\mathcal{Z}_{\tilde{C}}$

$$\mathcal{E}_{A\to\tilde{A}}\left(\sigma_A^i\right) = \mathcal{X}_{\tilde{A}}, \quad \Lambda_{B\to\tilde{B}}\left(\omega_B^i\right) = \mathcal{Y}_{\tilde{B}}, \quad \Gamma_{C\to\tilde{C}}\left(\gamma_C^i\right) = \mathcal{Z}_{\tilde{C}} \qquad \forall i \in I$$

whenever for the n-shareable extension $\rho_{AB^nC^n}$ we ask that

$$\begin{aligned} \left(\mathcal{E}_{A\to\tilde{A}} \otimes \mathcal{I}_{B^nC^n}\right)\left(\rho_{AB^nC^n}\right) &= \mathcal{X}_{\tilde{A}} \otimes \rho_{B^nC^n} \\ \left(\Lambda_{B\to\tilde{B}} \otimes \mathcal{I}_{B^{n-1}C^n}\right)\left(\rho_{B^nC^n}\right) &= \mathcal{Y}_{\tilde{B}} \otimes \rho_{B^{n-1}C^n} \\ \left(\mathcal{I}_{B^nC^{n-1}} \otimes \Gamma_{C\to\tilde{C}}\right)\left(\rho_{B^nC^n}\right) &= \mathcal{Z}_{\tilde{C}} \otimes \rho_{B^nC^{n-1}}. \end{aligned}$$

- NB: Standard quantum de Finetti not sufficient for linear constraints [B. *et al.* 18]

Proof: quantum entropy inequalities

- Ingredient I: by quantum Pinsker's inequality

$$2\|\rho - \sigma\|_1^2 \leq D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

work with **quantum relative entropy** distance [Brandão & Harrow 16/17]

- Ingredient II: multipartite quantum mutual information [Yang *et al.* 09]

$$I(A_1 : A_2 : \ldots : A_n|R)_\rho := \sum_{i=1}^{n} H(A_i R) - H(A_1 A_2 \ldots A_n R) - H(R)$$

$$H(R)_\rho := -\mathrm{Tr}\left[\rho_R \log \rho_R\right] \quad \text{von Neumann entropy}$$

and corresponding **quantum entropy inequalities**

$$I(A_1 : \ldots : A_n|R)_\rho = I(A_1 : A_2|R)_\rho + I(A_1 A_2 : A_3|R)_\rho + \ldots + I(A_1 \ldots A_{n-1} : A_k|R)_\rho$$

$$I(A_1 A_2 : A_3|R)_\rho = I(A_2 : A_3|R)_\rho + I(A_1 : A_3|A_2 R)_\rho$$

- Ingredient III: measurement $\mathcal{M}_B$ with at most $|B|^6$ outcomes such that

$$\|\left(\mathcal{I}_A \otimes \mathcal{M}_B\right)\left(\rho_{AB} - \sigma_{AB}\right)\|_1 \leq \|\rho_{AB} - \sigma_{AB}\|_1 \leq 2|B| \cdot \|\left(\mathcal{I}_A \otimes \mathcal{M}_B\right)\left(\rho_{AB} - \sigma_{AB}\right)\|_1$$

$\Rightarrow$ optimal distortion relative to quantum side information [Lami *et al.* 18]

Main result: approximation algorithm

Approximation algorithm for free games

We get semi-definite program approximations $\text{sdp}_n(V, \pi, T)$ on the **maximum winning probability** of free games with the guarantee

$$0 \le \text{sdp}_n(V, \pi, T) - \omega_{Q(T)}(V, \pi) \le O\left(T^6 \cdot \sqrt{\frac{\log(TA)}{n}}\right)$$

in the form

$$\text{sdp}_n(V, \pi, T) := |T|^2 \max_{\rho} \mathrm{Tr}\left[\left(V_{A_1A_2Q_1Q_2} \otimes \Phi_{T\hat{T}|S\hat{S}}\right) \rho_{(A_1Q_1T)(A_2Q_2\hat{T})(S\hat{S})}\right]$$

$$\text{s.t. } \rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \ge 0\,, \quad \mathrm{Tr}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right] = 1$$

$$\rho_{(A_1Q_1T)(A_2Q_2\hat{T})(S\hat{S})} \quad n\text{-shareable on } (A_2Q_2\hat{T}) \text{ wrt } (A_1Q_1T)(S\hat{S})^n$$

$$\rho_{(A_1Q_1T)(A_2Q_2\hat{T})(S\hat{S})} \quad n\text{-shareable on } (S\hat{S}) \text{ wrt } (A_1Q_1T)(A_2Q_2\hat{T})^n$$

$$\mathrm{Tr}_{A_1}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right] = \left(\sum_{q_1} \pi_1(q_1)|q_1\rangle\langle q_1|_{Q_1} \otimes \frac{\mathrm{id}_T}{|T|}\right) \otimes \mathrm{Tr}_{A_1Q_1T}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right]$$

$$\mathrm{Tr}_{A_2}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right] = \left(\sum_{q_2} \pi_2(q_2)|q_2\rangle\langle q_2|_{Q_2} \otimes \frac{\mathrm{id}_{\hat{T}}}{|T|}\right) \otimes \mathrm{Tr}_{A_2Q_2\hat{T}}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right]$$

- NB: positive partial transpose and dimension agnostic NPA criteria to add

Conclusion

- Additive ε-approximations on the maximum winning probability $\omega_{Q(T)}(V, \pi)$ of free games with quantum assistance of fixed dimension T with complexity

$$\exp\left(\mathcal{O}\left(\frac{T^{12}}{\varepsilon^2}\log(AT)\left(\log(Q)+\log(AT)\right)\right)\right)$$

  where Q denotes the number of question and A the number of answers

- In terms of worst case guarantees, this is an **exponential improvement** over previous work [Navascués *et al.* 14/15]

- Based on novel **multipartite quantum de Finetti** with linear constraints + optimal distortion measurement relative to quantum side information

- Dependence on T?
  - Matching hardness for $T = 1$ classical case [Aaronson *et al.* 14]
  - Diverges for $T \to \infty$ consistent with [Ji *et al.* 20]