On L^2 Fourier restriction

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Generalizations of Stein-Tomas [1967/75/77]

Setup in Mockenhaupt [2000] and Mitsis [2002]:

Given a compactly supported Borel probability measure μ satisfying

- $|\widehat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2}$,
- $|\mu(B(x,r))| \lesssim r^{\alpha}$.

Note $\dim(\mathrm{supp}\ \mu) \geq \beta$ and $\geq \alpha$.

Mockenhaupt's Fourier restriction estimate

$$\int |\widehat{f}(\xi)|^2 d\mu \lesssim \|f\|_p^2, \quad p < p_{\mathrm{cr}} = \frac{4(d-\alpha)+2\beta}{4(d-\alpha)+\beta}.$$

Unclear whether Stein's analytic families argument can give $L^{p_{\mathrm{cr}}} \to L^2(d\mu)$ in this generality.

J, Bak, S. [11]: The endpoint bound holds, with an improvement:

$$\mathcal{F} : L^{p_{\mathrm{cr}},2} \to L^2(d\mu)$$

Standard "T^*T" argument:

$$\mathcal{F} : L^{p,q} \to L^2(d\mu) \iff \widehat{\mu}* : L^{p,q} \to L^{p',q'} .$$

Following Tomas: $\widehat{\mu} = \sum_{j=0}^{\infty} \widehat{\mu}_j$,
supp $\widehat{\mu}_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$, $j \ge 1$.

- Fourier decay implies $\|\widehat{\mu}_j\|_\infty \lesssim 2^{-j\beta/2}$,

$$\|\widehat{\mu}_j * f\|_\infty \lesssim 2^{-j\beta/2} \|f\|_1 .$$

- α-upper regularity implies $\|\mu_j\|_\infty \lesssim 2^{j(d-\alpha)}$,

$$\|\widehat{\mu}_j * f\|_2 \lesssim 2^{j(d-\alpha)} \|f\|_2 .$$

By interpolation

$$\|\widehat{\mu}_j * f\|_{p'} \lesssim 2^{-j\varepsilon} \|f\|_p \text{ if } p < p_{\text{cr}} .$$

Sum in j. At the endpoint $p = p_{\text{cr}}$ use Bourgain's argument.

$$\Big\| \sum_{j=0}^{\infty} \widehat{\mu}_j * f \Big\|_{L^{p'_{\text{cr}},\infty}} \lesssim \|f\|_{L^{p_{\text{cr}},1}} .$$

Hence we get at least $\mathcal{F} : L^{p_{\text{cr}},1} \to L^2(d\mu)$.

• $\mathcal{F} : L^{p_{\rm cr},1} \to L^2(d\mu)$ implies

$$\|f * \widehat{\mu}_j\|_2 \lesssim 2^{j(d-\alpha)/2}\|f\|_{L^{p_{\rm cr},1}} \qquad (1)$$

Use

$$\|f * \widehat{\mu}_j\|_2 = \|\widehat{f}\,\mu_j\|_2 \lesssim 2^{j(d-\alpha)/2}\Big(\int |\widehat{f}|^2|\mu_j|d\xi\Big)^{1/2}$$

and $(\int |\widehat{f}|^2|\mu_j|d\xi)^{1/2} \lesssim \|f\|_{L^{p_{\rm cr},1}}$. Thus (1).

• Similar bound on line connecting $(1/p_{\rm cr}, 1/2)$ and $(1/2, 1/p'_{\rm cr})$.

• Interpolate these again with the $L^1 \to L^\infty$ bound for $\widehat{\mu}_j *$, using Bourgain's trick.

• This gives

$$\|\widehat{\mu} * f\|_{L^{q,\infty}} \lesssim \|f\|_{L^{p,1}}$$

for $(1/p, 1/q)$ on line segment parallel to the diagonal with midpoint $(1/p_{\rm cr}, 1/p'_{\rm cr})$.

• Marcinkiewicz gives $\|\widehat{\mu} * f\|_{L^{p'_{\rm cr},r}} \lesssim \|f\|_{L^{p_{\rm cr},r}}$ which we need for $r = 2$.

• Original question in the Bak-S. project: Given Hörmander's class of oscillatory integral operators

$$T_\lambda f(x) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x,y)}\chi(x,y)f(y)dy; \quad x \in \mathbb{R}^{d-1}$$

$$\text{rank } (\Phi''_{yx}) = d-1$$

$$\Phi''_{yx}u = 0 \implies \det\Big(\langle \Phi_y, u\rangle''_{xx}\Big) \neq 0$$

• Stein: $T_\lambda : L^{\frac{2(d+1)}{d+3}}(\mathbb{R}^d) \to L^2(\mathbb{R}^{d-1})$ with $\|T_\lambda\| = O(\lambda^{-d/p'})$.

• Bourgain: in this generality no such $L^p \to L^q$ result with for $p > \frac{2(d+1)}{d+3}$.

Q: Can $L^{\frac{2(d+1)}{d+3}}$ be replaced by $L^{\frac{2(d+1)}{d+3},2}$? Yes.

Similar results for

• Estimate for eigenfunctions for eigenfunctions of the Laplacian on compact manifolds (Sogge [88]).

$$-\Delta e_\lambda = \lambda^2 e_\lambda, \quad \|e_\lambda\|_2 = 1$$

$$\|e_\lambda\|_{L^{q,2}} \lesssim \lambda^{d(1/2-1/q)-1/2}, \quad q = \frac{2(d+1)}{d-1}$$

• Lorentz improvements in

Estimates for Fourier integral operators with (one-sided) fold singularities (Greenleaf-S [94]).

Restrictions of eigenfunctions to submanifolds (Burq-Gérard-Tzvetkov [07], Hu [09]).

Higher rank Stein-Tomas theorem

• Removing logarithms in Simon Marshall's work [11] on estimates for eigenfunctions of the algebra of invariant operators on locally symmetric spaces (and symmetric spaces of compact type) of *higher rank*.

• Also yields endpoint results to work of Mockenhaupt [1991] on K-orbits in symmetric spaces of Euclidean type (quotients of Cartan motion groups by maximal compact subgroups).

An example in the Euclidean case

Let $M(r,n) \equiv \mathbb{R}^d$ be the Euclidean space of $r \times n$ real matrices, $r \leq n$, $d = rn$.

Compact group $K = SO(r) \times SO(n)$ acts on $r \times n$ matrices X by

$$k.X \equiv Ad(k)X = k_1 X k_2^{-1}$$

Let $\mathfrak{a}$ (or $\mathfrak{a}^*$) be the subspace of diagonal matrices

$$a = \begin{pmatrix} a_1 & \dots & 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & 0 & \dots & \dots & 0 \\ 0 & \dots & a_r & 0 & \dots & \dots & 0 \end{pmatrix}$$

Let $\mathfrak{a}_+^\circ = \{a \in \mathfrak{a} : 0 < a_1 < \dots < a_r\}$. For $a \in \mathfrak{a}_+^\circ$ consider the *regular* K-orbit

$$\Sigma_a = \{k_1 a k_2^{-1} : \, k_1 \in SO_r, k_2 \in SO_n\},$$

which is r-co-dimensional in $M(r,n)$.

• When $r = 1$, Σ_{a_1} is sphere of radius a_1.

• Endpoint version of Mockenhaupt's 1991 result (Marshall, S.):

$$\iint_{SO_r \times SO_n} \left|\widehat{f}(Ad(k)a)\right|^2 dk \lesssim \|f\|^2_{L^{p,2}(\mathbb{R}^d)},$$

$$p \le p_{\mathrm{cr}} = \frac{2(d+r)}{d+3r}.$$

This estimate is uniform as a varies over compact subsets of $\mathfrak{a}_+^\circ$.

Open problems:

Problem: Characterizations of polyradial M_p^2 multipliers

Well known characterization for rank one: Let $1 \le p \le \frac{2(d+1)}{d+3}$, $m(\xi) = h(|\xi|)$. Then

$$\|\mathcal{F}^{-1}[h(|\cdot|)\widehat{f}]\|_2 \le C\|f\|_p$$

for all $f \in \mathcal{S}$ if and only if

$$\sup_{R>0} R^{d(\frac{1}{p}-\frac{1}{2})}\Big(\int_R^{2R} |h(a)|^2 \frac{da}{a}\Big)^{1/2} < \infty\,.$$

This is a corollary of the Stein-Tomas restriction theorem and Littlewood-Paley.

Question: Is there a similar higher rank generalization for polyradial multipliers, i.e. K-invariant multipliers on $M(r,n)$?

Related: Prove the sharp dependence on a in the L^2 restriction theorem theorem for K-orbits Σ_a when a tends to the boundary of $\mathfrak{a}_+^\circ$.

Problem: Is there an $L^p \to L^1(\Sigma_a)$ estimate for some $p > \frac{2(d+r)}{d+3r}$?

Problem: Characterization for K-invariant M_p^p multipliers?

In particular, if m is K-invariant and compactly supported in $\mathbb{R}^d \setminus \{0\}$, when is the necessary condition $\mathcal{F}^{-1}[m] \in L^p$ also sufficient for $m \in M_p^p$?

Is it true for some $p > 1$? It can be conjectured for $1 < p < \frac{2d}{d+r}$.

We know it for $1 < p < \frac{2(d-1)}{d+1}$ in the rank one case (Heo, Nazarov, S.)

Back to Fourier restriction for measures.
Recall: if μ satisfies

$$(*) \quad |\widehat{\mu}(\xi)| = O(|\xi|^{-\beta/2}), \quad \mu(B(x,r)) = O(r^{\alpha})$$

then $\mathcal{F} : L^p \to L^2(d\mu)$ for $p \leq p_{\text{cr}} = \frac{4(d-\alpha)+2\beta}{4(d-\alpha)+\beta}$.

Questions (Mockenhaupt [2000])

• Is $\mathcal{F} : L^p \to L^1(d\mu)$ for some $p > p_{\text{cr}}$?

• Could there even be a better range for Stein-Tomas for Salem sets? Are there probability measures μ satisfying (*) with $\beta = \alpha$ so that dim(supp $(\mu)) = \alpha$ and $\mathcal{F} : L^p \to L^2(d\mu)$ for some $p > p_{\text{cr}}$?

• Analogues for $\beta < \alpha$ well known (e.g. Christ [85] for curves in $\mathbb{R}^d$, $d \geq 3$, ...).

If $\beta = \alpha$ the possible improvement range could be $p_{\text{cr}} < p \leq \frac{2d}{2d-\alpha}$.

Sharpness of p_{cr}: *Standard Knapp examples*

• Arithmetic progression Knapp examples by K.Hambrook and I. Łaba (GAFA 13), for Salem measures, $d = 1$:

Given integers M, N,

$$1 < M < N,\ \alpha = \frac{\log M}{\log N},\ p > p_{\text{cr}} = \frac{4-2\alpha}{4-3\alpha},$$

there is an α*-upper regular measure supported on an* α*-dimensional subset of* $\mathbb{R}$*, for which* $\widehat{\mu}(\xi) = O(|\xi|^{\varepsilon-\alpha/2})$ $\forall\varepsilon > 0$ *and for which* $\mathcal{F}$ *does not map* L^p *to* $L^2(d\mu)$.

• Used random Cantor set construction containing suitable arithmetic progressions at every stage.

• Further *questions* about sharpness: For $\beta \le \alpha$ are there α-upper-regular measures, for which for which $\widehat{\mu}(\xi) = O(|\xi|^{-\beta/2})$ such that the p_{cr}-result is sharp?

Answered affirmatively by Xianghong Chen.

Theorem (X. Chen, to appear in TAMS)

(i) Given $0 < \alpha < 1$ there is a probability measure μ supported on $E \subset [0,1]$, $\dim(E) = a$:

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\alpha/2}$$
$$\mu(I) \approx \frac{|I|^{\alpha}}{\log(1/|I|)}$$

for all intervals of length $< 1/2$ centered in E.

(ii) Given $0 < \beta \leq \alpha < 1$, $\exists$ probability measure μ supported on $E \subset [0,1]$, $\dim(E) = \alpha$:

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\beta/2},$$
$$\frac{|I|^{\alpha}}{\log(1/|I|)^{1+\varepsilon}} \lesssim \mu(I) \lesssim \frac{|I|^{\alpha}}{\log(1/|I|)},$$

when $|I| < 1/2$, I centered in E, so that $\mathcal{F}$ does not map L^p to $L^2(d\mu)$ for $p > p_{\text{cr}}(a,\beta)$.

Nonsharpness:

• T. Körner: "On a theorem of Saeki concerning convolution squares of singular measures" (08) showed the existence of probability measures supported on subsets of $\mathbb{R}$, of Hausdorff dimension $\alpha \geq 1/2$, for which

$$\mu * \mu \in C^{\frac{2\alpha-1}{2}}.$$

This gives the estimate

$$\|\widehat{f d\mu}\|_{L^4(\mathbb{R})} \lesssim \|f\|_{L^2(d\mu)}$$

and thus $\mathcal{F} : L^{4/3} \to L^2(d\mu)$.

(W. Rudin, A. Iosevich-S.Roudenko, X. Chen).

Körner's arguments use randomness and Baire category. Defines a metric space of pairs (μ, E) with μ supported in the compact set E, $\mu * \mu \in C^{(2\alpha-1)/2}$, and shows that the sets of pairs (μ, E) with $\dim E = \alpha$ is generic.

Theorem: (X. Chen, S.) Given $0 < \alpha < d$, there exists a probability measure μ supported on a compact set of Hausdorff dimension α, so that

$$\begin{gathered} \widehat{\mu}(\xi) = O(|\xi|^{-\alpha/2}) \\ \mu(B(x,r)) \lesssim r^{\alpha} \end{gathered} \qquad (*)$$

$\forall x, r$, and, for higher convolution powers, $\ell = 1, 2, \ldots$

$$\sup_x \mu^{*\ell}(B(x,r)) \lesssim r^{\alpha\ell} \text{ if } \alpha\ell < d$$

and

$$\mu^{*\ell} \in C^{\frac{\alpha\ell - d}{2}} \text{ if } \alpha\ell \geq d.$$

• If we let $\alpha = d/n$ for some $n \in \mathbb{N}$ then we get a Salem measure μ, with $\mu^{*n} \in C_c(\mathbb{R}^d)$, for which assumptions (*) are valid, and

$$\mathcal{F} : L^p \to L^2(d\mu) \text{ for } 1 \leq p \leq \frac{2d}{2d - \alpha}.$$

This is the optimal range.

Remark: These measures are not Ahlfors-David α-regular

• Improvement for *some* $p > p_{\text{cr}}$ if $\alpha < 2d/3$.

• *Application:* Multipliers of “Bochner-Riesz” type.

• Let $\alpha = d/n$, $1 \le p \le \frac{2d}{d-\alpha}$ and let μ be as constructed above. Let $\chi \in C_c^\infty$, $\lambda > 0$. Set

$$m_\lambda(\xi) = \int_{\mathbb{R}^d} \chi(\xi - \eta)|\xi - \eta|^{\lambda-\alpha} d\mu(\eta).$$

Then for $p \le q \le 2$

$$\|\mathcal{F}^{-1}[m_\lambda \widehat{f}]\|_q \lesssim \|f\|_p, \quad \lambda > d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}.$$

One can also construct measures as before with

$$\int_0^1 \left[r^{-\alpha} \sup_x \mu(B(x,r))\right]^{1/2} \frac{dr}{r} < \infty$$

and then the conclusion holds for

$$\lambda \ge d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}.$$