L^p-Improving Estimates and Restriction in Intermediate Dimensions

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Prologue: What is Combinatorics?

- I will pretend that a problem is a combinatorics problem whenever the inputs are without loss of generality nonnegative and the outputs are monotone functions of the input.
- There are more serious connections between what I will discuss today and discrete analysis, but I will leave those discussions to others more qualified.

Two Related Problems

Suppose that μ is any positive, finite measure on $\mathbb{R}^d$ (presumably mutually singular with respect to Lebesgue measure):

L^p-Improving Estimates

For which pairs of p and q in $[1, \infty]$ do we have

$$||f * \mu||_{L^q(\mathbb{R}^d)} \lesssim ||f||_{L^p(\mathbb{R}^d)} \ \forall f \in L^p(\mathbb{R}^d)?$$

Necessary: $q \geq p$. Trivial: $q = p$. Interesting: $q > p$.

Fourier Restriction

For which pairs p and q in $[1, \infty]$ do we have

$$||\hat{f}||_{L^q(d\mu)} \lesssim ||f||_{L^p(\mathbb{R}^d)} \ \forall f \in L^p(\mathbb{R}^d)?$$

Necessary: $p \in [1, 2)$. Trivial: $p = q = 1$. Interesting: $p > 1$.

In both cases, $\text{supp}(\mu)$ should not be too near to a hyperplane.

L^p-improving / Fourier Restriction

- Both problems have a mathematical lineage that goes way back (e.g., C. Fefferman (1970) and Zygmund (1974)). Geometric versions (when μ is of smooth density on a submanifold) arose the most naturally when, e.g., studying the Radon transform and the Bochner-Riesz multipliers.
- Both problems have a natural affine-invariance (if an affine transformation is applied to μ, the norms of both operators change by a power of the determinant).
- Both problems have a natural monotonicity: if $\mu_1 \leq \mu_2$, and estimates are satisfied for μ_2, then the same estimates will be satisfied for μ_1 with possibly smaller constant.
- Techniques for approaching either problem often have analogues when approaching the other problem. E.g., similar multilinearization techniques and van der Monde estimates can be found for Fourier restriction to curves *a la* Drury (1985) or Drury and Marshall (1985, 1987)) and L^p-improving estimates for curves *a la* Christ (1998).

Dimensions of Measures on $\mathbb{R}^d$

Dimension:

We say that μ has dimension at least a when

$$\mu(B) \lesssim |B|^{\frac{a}{d}}$$

for all Euclidean balls B in $\mathbb{R}^d$, where $|B|$ is Lebesgue measure.

Fourier Dimension:

We say that μ has Fourier dimension at least b when

$$|\hat{\mu}(\xi)| \lesssim |\xi|^{-\frac{b}{2}} \ \forall \xi \in \mathbb{R}^d.$$

A standard case to consider: when μ is a measure of smooth density on a hypersurface in $\mathbb{R}^d$ with nonvanishing Gaussian curvature, dimension = Fourier dimension = $d - 1$.

Connections to Dimension and Fourier Dimension

- When μ satisfies either an L^p-improving estimate or a Fourier restriction estimate, it will necessarily have nonzero dimension via a Knapp-type argument. In fact, there will be an estimate relating μ-measures of convex sets to their Euclidean volumes. Oberlin (2000), Bak, Oberlin, Seeger (2008), and others successfully exploit this condition in several cases.
- Greenleaf (1981) showed that Fourier decay for submanifolds of Euclidean space implies L^p–L^2 Fourier restriction. Iosevich (1999) shows the converse is true for smooth hypersurfaces of finite polygonal type (surface caps are essentially comparable to polygons of bounded number of sides.) G. Mockenhaupt (2000), Mitsis (2002) and Bak and Seeger (2011) generalize and strengthen earlier restriction estimates to abstract measures.
- Note: Neither condition is affine invariant, but that's not necessarily bad.

Today's Main Points

1. Nontrivial estimates of the dimension and Fourier dimension of μ imply L^p-improving and Fourier restriction estimates in very natural ways.
2. These methods work quite well in low codimension. With a certain outlook on life, though, it may be said that these methods rarely give sharp estimates in either the L^p-improving or the Fourier case outside of the hypersurface case.
3. In cases when dimensionality arguments alone fall short, there are alternate, combinatorial, methods which allow for sharp estimates anyway. These methods also respect the natural positivity and affine-invariance of the estimates. The main situation to be discussed today is the case when μ is a measure of smooth density on appropriate n-dimensional submanifolds of $\mathbb{R}^{2n}$.
4. Ultimately these problems seem to beg for a fully general combinatorial solution, but it's still not clear how that might work.

Estimates from Dimension Data

Restriction: Bak and Seeger (2011)

If μ has dimension at least a and Fourier dimension at least b, then

$$||\hat{f}||_{L^2(d\mu)} \lesssim ||f||_{L^{\frac{4(d-a)+2b}{4(d-a)+b},2}(\mathbb{R}^d)} \quad \forall f.$$

L^p-Improving Estimates

If μ has dimension at least a and Fourier dimension at least b, then

$$||f * \mu||_{L^{\frac{2(d-a)+b}{d-a}}(\mathbb{R}^d)} \lesssim ||f||_{L^{\frac{2(d-a)+b}{(d-a)+b}}(\mathbb{R}^d)} \quad \forall f.$$

In dimension d with dimension and Fourier dimension at least $d-1$, these correspond to the familiar Thomas-Stein and $L^{\frac{d+1}{d}} \to L^{d+1}$ estimate, both of which are, in some sense, sharp.

Side Note: The Fourier Restriction estimate automatically implies an L^p-improving estimate via Hausdorff-Young (which comes out strictly worse than the stated one), but not vice-versa.

Rough Sketch: Fourier Restriction

The bulk of the Bak-Seeger proof is in moving from the restricted type estimate to the enhanced Lorentz space $L^{p,2}$. The restricted inequality is nice and short: First, use the method of TT^*:

$$\int |\hat{f}(\xi)|^2 d\mu = \int_{\mathbb{R}^d} \overline{f(x)} \ (\widehat{\mu} * f)(x) dx.$$

Next fix some integer j and write $\hat{\mu} = k_j + k^j$ where k_j is supported on a ball at the origin of radius $\approx 2^j$ and k^j is supported away from a ball of radius $\approx 2^j$. There are two basic estimates:

$$||k^j||_\infty \lesssim 2^{-\frac{jb}{2}} \Rightarrow ||f * k^j||_\infty \lesssim 2^{-\frac{jb}{2}} ||f||_1,$$

$$||k_j * f||_2 \lesssim \left[\sup_\xi |\mu * \widehat{\chi_{B_{2^j}}}(\xi)| \right] ||f||_2 \lesssim 2^{(d-a)j} ||f||_2.$$

Rough Sketch: Fourier Restriction, Part II

Test on characteristic functions and Sum the two basic estimates:

$$\int |\widehat{\chi_E}(\xi)|^2 d\mu \leq \int_E |(k_j * \chi_E)(x)| dx + \int_E |(k^j * \chi_E)(x)| dx$$
$$\lesssim 2^{-\frac{jb}{2}} |E|^2 + 2^{(d-a)j} |E|.$$

To finish, you optimize over j: $2^j \approx |E|^{2/(2(d-a)+b)}$:

$$\int |\widehat{\chi_E}(\xi)|^2 d\mu \lesssim |E|^{\frac{4(d-a)+b}{2(d-a)+b}}.$$

Taking the square root gives exactly the restricted analogue of the Bak-Seeger inequality.

Rough Sketch: L^p-Improving via Dimensional Information

Let P_j be a Littlewood-Paley-type projection onto frequencies of size $\lesssim 2^j$. The two basic estimates are:

$$||P_j\mu||_\infty \lesssim 2^{(d-a)j} \Rightarrow ||\mu * (P_j f)||_\infty \lesssim 2^{(d-a)j}||f||_1,$$

$$||(I - P_j)\mu * f||_2 \lesssim 2^{-\frac{jb}{2}}||f||_2.$$

Optimize the following sum over j:

$$\int \mu * P_j \chi_E(x)\chi_F(x)dx + \int \mu * (I - P_j)\chi_E(x)\chi_F(x)dx.$$

When $2^j \approx (|E||F|)^{-1/(2(d-a)+b)}$:

$$\int \mu * \chi_E(x)\chi_F(x)dx \lesssim (|E||F|)^{\frac{(d-a)+b}{2(d-a)+b}}.$$

This argument can be powered-up via analytic interpolation similar to Oberlin and Stein (1982).

Abstract Measures of Intermediate Dimension in $\mathbb{R}^{2n}$

Corollary

If μ is a positive, finite measure on $\mathbb{R}^{2n}$ with dimension at least n & Fourier dimension at least n, then

$$||f * \mu||_{L^3(\mathbb{R}^{2n})} \lesssim ||f||_{L^{\frac{3}{2}}(\mathbb{R}^{2n})} \; \forall f,$$

$$||\hat{f}||_{L^2(d\mu)} \lesssim ||f||_{L^{\frac{6}{5},2}(\mathbb{R}^{2n})} \; \forall f.$$

- Canonical Model: The parabola $(t, t^2) \subset \mathbb{R}^2$.
- Another Example: The 2-surface $(t_1, t_2, t_1^2 - t_2^2, 2t_1 t_2) \subset \mathbb{R}^4$.
- Also note: the hypotheses are stable under the submanifold assumption (more about this soon).

Bigger Question

How does one identify what the "least flat" k-dimensional submanifolds of $\mathbb{R}^d$ look like?

When are Dimension-Only Estimates Sharp?

X. Chen (2012) showed that there exist measures on $\mathbb{R}$ for which the Bak-Seeger inequality is not sharp. He uses a probabilistic argument of Körner (2008) that there are measures supported on sets of Hausdorff dimension $\frac{1}{2}$ whose double convolution is a continuous function and establishes the following abstract result:

Theorem (X. Chen 2012)

Let μ be a Borel probability measure on $\mathbb{R}^d$, let $1 \leq r \leq \infty$, and assume that $\mu^{*n} \in L^r(\mathbb{R}^d)$. Let $1 \leq p \leq \frac{2n}{2n-1}$ (if $r \geq 2$) and $1 \leq p \leq \frac{nr'}{nr'-1}$ (if $1 \leq r \leq 2$) and let $1 \leq q \leq \frac{q'}{nr'}$. Then

$$||\hat{f}||_{L^q(d\mu)} \lesssim ||f||_{L^p(\mathbb{R}^d)} \; \forall f \in \mathcal{S}(\mathbb{R}^d).$$

For these measures he obtains an $L^{\frac{4}{3}} \to L^2$ Fourier restriction estimate which beats the Bak-Seeger estimate (because a and b cannot exceed $\frac{1}{2}$, so the exponent is no higher than $\frac{6}{5}$).

The Case of Submanifolds of Intermediate Dimension

Theorem (G. 2013, 2014)

If μ is positive, compactly supported, and bounded density on a "typical" quadratic n-dimensional submanifold of $\mathbb{R}^{2n}$ then

$$||f * \mu||_{L^{3\pm}(\mathbb{R}^{2n})} \lesssim ||f||_{L^{\frac{3}{2}\pm}(\mathbb{R}^{2n})} \; \forall f,$$
$$||\hat{f}||_{L^{\frac{4}{3}\pm}(d\mu)} \lesssim ||f||_{L^{\frac{4}{3}\pm}(\mathbb{R}^{2n})} \; \forall f,$$

where $\pm$ indicates any estimate attainable by strict interpolation of the stated exponents and one of the official "trivial" estimates.

These results improve on the dimension-only estimates because **the Fourier dimension is not typically** n.

Comments on the Theorems

- Both endpoint estimates (which are barely missed) are sharp estimates in the sense that standard Knapp-type examples show that they are nontrivial vertices of the Riesz diagram.
- In particular, these estimates miss the generalized Thomas-Stein estimate by an ϵ.
- This restriction theorem falls just short of establishing the estimate

  $$||\chi_E \mu * \chi_E \mu||_2 \lesssim |E|.$$

  Chen's result would give $L^{4/3}$ to L^1, but this is to be expected because his hypothesis is somewhat weaker (namely, that the right-hand side is $|E|^0$ rather than $|E|^1$).
- It's not clear if there's something here that's good for general measures or if it's the structure of the submanifold which really pushes things.

Submanifolds of Intermediate Dimension

Suppose $U \subset \mathbb{R}^n$ is bounded & open, and $\varphi : U \to \mathbb{R}^n$ is smooth. Suppose μ is a measure of smooth density supported on $\Sigma \subset \mathbb{R}^{2n}$ parametrized by $\gamma(t) := (t, \varphi(t))$. The dimension of μ equals n.

Phong-Stein Rotational Curvature Condition

We say Σ has nonvanishing Phong-Stein Rotational Curvature exactly when, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, the projected submainfold $(t, \xi \cdot \varphi(t)) \subset \mathbb{R}^{n+1}$ has nonvanishing Gaussian curvature.

With Phong-Stein rotational curvature, all critical points of $\xi' \cdot t + \xi \cdot \varphi(t)$ will be nondegenerate, and so standard oscillatory integral estimates will give

$$\left| \int_U e^{2\pi i (\xi' \cdot t + \xi \cdot \varphi(t))} w(t) dt \right| \lesssim \left(|\xi| + |\xi'| \right)^{-\frac{n}{2}},$$

i.e., Fourier dimension of μ is n. Note rotational curvature is necessary and sufficient for this conclusion. Also it's stable with respect to small perturbations of Σ.

When is Rotational Curvature Nondegenerate?

Let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$. Fix a point $t = t_0$. We'd need

$$\det \left(\sum_{j=1}^{n} \xi_j \nabla^2 \varphi_j(t_0) \right) \neq 0 \text{ for } \xi \neq 0.$$

(∇^2 is the Hessian.) We can define a "product" $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\xi \cdot \nu := \left(\sum_{j=1}^{n} \xi_j \nabla^2 \varphi_j(t_0) \right) \nu$$

Check: For any nonzero b and any a, there are unique x and y such that $a = bx = yb$. (Regarding y: Non-uniqueness implies non-invertibility of our matrix; non-existence implies that the image of b over all matrices is dimension $n-1$ which also implies some matrix must send b to zero.) Thus we have constructed a real division algebra.

$$\Rightarrow n = 1, 2, 4, \text{ or } 8.$$

Rotational Curvature: It Gets Worse

Also notice $(t_1, \ldots, t_n, t_1^2, \ldots, t_n^2)$ has vanishing rotational curvature at every point for $n \geq 2$, as do all small perturbations of this example. Thus **in half dimension, rotational curvature is "generically" nonvanishing only for curves in the plane.**

In general:

Theorem

Consider a standard d-dimensional ensemble of n-dimensional submanifolds in $\mathbb{R}^d$, and let $n' = d - n$ be codimension. Factor $n = 2^{4q+r}s$ for integers q, r, s, such that s is odd and $0 \leq r \leq 3$. Then the ensemble must have vanishing rotational curvature at every point when $n' > 8q + 2^r$.

Corollary: Codimension grows at most logarithmically with the dimension. For odd dimensions, codimension is simply 1.

Proof: Adams, Lax, and Phillips (1965) or Friedland, Robbin, and Sylvester (1984).

Rotational Curvature: Sensitive Dependence on Dimension

The Radon-Hurwitz function is the mapping $\rho(2^{4q+r}s) := 2^r + 8q$ from the previous slide. The theorem is that if $M_1, \ldots, M_k$ are $n \times n$ matrices and $k > \rho(n)$, then there must be a nontrivial linear combination of these matrices which is singular. Moreover, if the matrices M_j are symmetric, then the same conclusion holds when $k > \rho(n/2) + 1$. Thus, in the convolution case we're looking at today, the examples (t, t^2) and $(t_1, t_2, t_1^2 - t_2^2, 2t_1t_2)$ are essentially the only examples (no higher-dimensional examples exist).

The appearance of this kind of sensitive, number-theoretic dependence on dimension is rare but not unheard of. Christ's thesis (1982) identifies a difference between restriction problems in odd and even dimensions. Bourgain and Guth's restriction theorem (2011) depends on dimension mod 3.

An Alternate Approach

Recall μ is a measure of smooth density supported on the manifold Σ parametrized by $\gamma(t) := (t, \varphi(t))$. We study the incidence maps:

$$(t^{(1)}, t^{(2)}) \in U \times U \mapsto \gamma(t^{(1)}) \pm \gamma(t^{(2)}).$$

The absolute value of the Jacobian determinant of the maps equal:

$$J(t^{(1)}, t^{(2)}) := \left| \det \left[d\varphi(t^{(1)}) - d\varphi(t^{(2)}) \right] \right|.$$

At t we write a trilinear form:

$$Q_t(u, v, w) := \sum_{i,j,k=1}^{n} \frac{\partial^2 \varphi_i}{\partial t_j \partial t_k}(t) u_i v_j w_k.$$

We have $J(t^{(1)}, t^{(2)}) = |\det Q_{t^{(1)}}(\cdot, t^{(1)} - t^{(2)}, \cdot)| +$ higher order terms. Rotational curvature instead says $|\det Q_{t^{(1)}}(\xi, \cdot, \cdot)| \neq 0$ for $\xi \neq 0$.

Connections: Incidence Jacobian and Rotational Curvature

Proposition

$\det Q(\cdot, t, \cdot) \neq 0 \ \forall t \in \mathbb{R}^n \setminus \{0\} \Leftrightarrow \det Q(\xi, \cdot, \cdot) \neq 0 \ \forall \xi \in \mathbb{R}^n \setminus \{0\}$.

Proof of $\Leftarrow$: Degeneracy of $Q(\xi, \cdot, \cdot)$ means that there must be a nonzero $t \in \mathbb{R}^n$ such that $Q(\xi, t, \cdot)$ is the zero functional, which implies that $\det Q(\cdot, t, \cdot) = 0$ for that t.

This means that nonvanishing Rotational curvature is equivalent to an estimate of the form

$$J(t^{(1)}, t^{(2)}) \approx |t^{(1)} - t^{(2)}|^n$$

near the diagonal. We conclude that such an estimate for J is possible when $n = 1, 2, 4,$ or 8, but rarely true. We need a weaker condition on J which still guarantees the boundedness properties we're looking for.

L^p-Improving: An Inflation-Type Argument

First expand:

$$\int |\chi_E * \mu(x)|^3 dx = \int \prod_{j=1}^{3} \chi_E(x + \gamma(t^{(j)})) dt^{(1)} \cdots dt^{(3)} dx.$$

By symmetry (up to a factor of 3), we may assume that $J(t^{(1)}, t^{(2)})$ and $J(t^{(1)}, t^{(3)})$ are no greater than $J(t^{(2)}, t^{(3)})$. If we could establish a sublevel set estimate of the form

$$|\{t \in \mathbb{R}^n \mid \max\{J(t,a), J(t,b)\} \leq J(a,b)\}| \lesssim J(a,b) \ \forall a, b,$$

we would have

$$\int |\chi_E * \mu(x)|^3 dx \lesssim \int \prod_{j=2}^{3} \chi_E(x + \gamma(t^{(j)})) J(t^{(2)}, t^{(3)}) dt^{(2)} dt^{(3)} dx$$
$$\lesssim |E|^2$$

(assuming boundedly many nondegenerate solutions to the equation $y = \gamma(t^{(1)}) - \gamma(t^{(2)})$).

Fourier Extension: A Convolution-Type Argument

We observe by Hausdorff-Young that for $q \geq 4$,

$$||\widehat{\chi_E \mu}||_q = ||(\widehat{\chi_E \mu})^2||_{q/2}^{1/2} \leq ||\chi_E \mu * \chi_E \mu||^{1/2}_{L^{\frac{q}{q-2}}(\mathbb{R}^{2n})}.$$

Next you expand the convolution and use duality:

$$\int \chi_E \mu * \chi_E \mu G dx = \int \chi_E(t^{(1)}) \chi_E(t^{(2)}) G(\gamma(t^{(1)}) + \gamma(t^{(2)})) dt^{(1)} dt^{(2)}.$$

We already know that

$$\int G(\gamma(t^{(1)}) + \gamma(t^{(2)})) J(t^{(1)}, t^{(2)}) dt^{(1)} dt^{(2)} \lesssim ||G||_1,$$

so if we knew that

$$\int \chi_E(t^{(1)}) \chi_E(t^{(2)}) \frac{dt^{(1)} dt^{(2)}}{J(t^{(1)}, t^{(2)})} \lesssim \mu(E),$$

we could interpolate and conclude that

$$||\widehat{\chi_E \mu}||_{L^4(\mathbb{R}^d)} \lesssim \mu(E)^{\frac{1}{4}} \text{ or } ||\hat{f}||_{L^{\frac{4}{3},\infty}(d\mu)} \lesssim ||f||_{L^{\frac{4}{3}}(\mathbb{R}^{2n})}.$$

What's Really Needed From the Incidence Jacobian

The two estimates for the Incidence Jacobian

$$|\{t \in \mathbb{R}^n \mid \max\{J(t,a), J(t,b)\} \leq J(a,b)\}| \lesssim J(a,b) \; \forall a, b,$$

$$\int \chi_E(t^{(1)}) \chi_E(t^{(2)}) \frac{dt^{(1)} dt^{(2)}}{J(t^{(1)}, t^{(2)})} \lesssim |E|,$$

are somewhat difficult (the latter one is probably almost always false), but up to an ϵ loss in various exponents, they can both be shown to hold generically when $\varphi(t)$ is, for example, purely quadratic as a function of t. Both arguments on the previous two slides can be easily adapted so that ϵ losses in the power of J (appearing on the right-hand side of the first condition and the integrand of the second condition) yield estimates with only ϵ losses as well.

Showing the Conditions (with ϵ loss) Typically Hold

Lemma (Rearrangement)

Suppose F is nonnegative and decreasing on $[0, \infty)$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be smooth and have the property that $\frac{\partial^2}{\partial u_j^2} \varphi(u) = 0$ for all $j = 1, \ldots, n$ and all $u \in \mathbb{R}^n$. Then

$$\int_B F(|\varphi(u)|) du \leq \int_B F\left(\left| u_1 \cdots u_n \frac{\partial^n \varphi}{\partial u_1 \cdots \partial u_n}(0) \right| \right) du$$

when B is any n-fold product of intervals centered at the origin.

Proof is by induction on n. To apply the lemma, we perturb a trilinear form along the diagonal

$$Q_s(u, v, w) := \sum_{i=1}^n s_i u_i v_i w_i + \sum_{i,j,k=1} c_{ijk} u_i v_j w_k$$

and set

$$\varphi_v(s) := \det Q_s(\cdot, v, \cdot)$$

Showing the Conditions (with ϵ loss) Typically Hold

- We integrate $|\varphi_v(s)|^{-1+\epsilon}$ on the unit box in v and s. Integrating over s first reduces the integral (by the lemma) to an integral of $\prod_j |v_j|^{-1+\epsilon}$, which is clearly finite. By Fubini, this means that the integral over v only is finite for almost every choice of s.
- By taking a countable sequence of ϵ's tending to zero, we can, in fact, show that for almost every s, we have integrability for any power strictly greater than -1.
- The argument works whether or not we assume that Q is symmetric in the second and third places.
- This gives us the sublevel set estimate we need for L^p-improving, and Minkowski's inequality gives the desired estimate for Fourier restriction.
- The result would perhaps be a little nicer if we had a dense open set rather than the complement of a null set.

Where Do We Go From Here?

- A natural (but infinitesimal) improvement to the restriction result would come from being able to establish estimates of the form

$$\left|\left|f * |p|^{-\theta}\right|\right|_{\frac{2}{\theta}} \lesssim ||f||_{\frac{2}{2-\theta}}$$

  for $\theta \in [0, 1)$, where p is an appropriate homogeneous polynomial of degree n on $\mathbb{R}^n$. What is the least restrictive condition on p needed? Is there a way to do it that doesn't use analytic interpolation?

- Similarly, we could get the endpoint $L^{\frac{3}{2}} \to L^3$ estimate if we used the full power of the sublevel set

$$|\{t \in \mathbb{R}^n \ | \ \max\{J(t-a), J(t-b)\} \leq J(a-b)\}| .$$

- The proofs rely heavily on the submanifold structure of the measures. Is it necessary?

The End

Thank you.

Epilogue: The Oberlin Condition

Basic Inequalities Implying an Oberlin Condition

$$\forall f \quad ||\hat{f}||_{L^q(d\mu)} \lesssim ||f||_{L^p(\mathbb{R}^d)} \Rightarrow \forall B \quad \mu(B) \lesssim (\text{vol} B)^{\frac{q}{p'}}$$

$$\forall f \quad ||f * \mu||_{L^q(\mathbb{R}^d)} \lesssim ||f||_{L^p(\mathbb{R}^d)} \Rightarrow \forall B \quad \mu(B) \lesssim (\text{vol} B)^{\frac{1}{p} - \frac{1}{q}}$$

Sufficiency Results

Oberlin (2000): If μ is supported on a smooth hypersurface in $\mathbb{R}^d$, μ satisfies Oberlin's condition with $\alpha = \frac{d-1}{d+1}$ iff $||\mu * f||_{d+1} \lesssim ||f||_{\frac{d+1}{d},1}$ for all f (modulo a bounded nondegenerate multiplicity condition).
Bak, Oberlin, Seeger (2008): If μ is supported on a curve in $\mathbb{R}^d$ satisfying monotonicity conditions, $||\hat{f}||_{L^{1+\alpha}(\mu)} \lesssim ||f||_{1+\alpha,1}$ when μ satisfies Oberlin's inequality with exponent α.

BUT Adapting work of Graham, Hare, and Ritter (1989) allows one to show that both sufficiency results are false if the geometric structure is removed.