Distinct distances on algebraic curves

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IPAM Workshop I:
Combinatorial geometry problems at the algebraic interface

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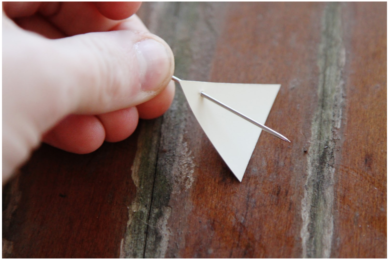
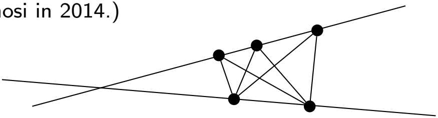


- **SSS** - Micha Sharir, Adam Sheffer, and József Solymosi, *Distinct distances on two lines*, JCTA 2013 / arXiv:1302.3081.
- **PZ** - János Pach and Frank de Zeeuw, *Distinct distances on algebraic curves in the plane*, SoCG 2014 / arXiv:1308.0177.
- **VZ** - Claudiu Valculescu and Frank de Zeeuw, *Distinct pinned triangle areas on algebraic curves*, arXiv:1403.3867.

Background

- **Elekes-Rónyai** (2000): Given two sets of n points on two lines in $\mathbb{R}^2$, the number of distinct distances between the sets is superlinear, unless the lines are parallel or orthogonal.

  (Corollary of a more general theorem about polynomials, improved by Raz-Sharir-Solymosi in 2014.)



- **Elekes** (1999): The number of distances is $\Omega(n^{5/4})$, unless the two lines are parallel or orthogonal.

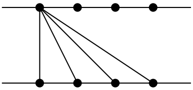
- **Conjecture**: $\Omega(n^{2-\epsilon})$

Distances between two algebraic curves

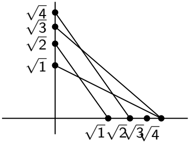
Theorem (Sharir-Sheffer-Solymosi, 2013)

Given two n-point sets on two lines in $\mathbb{R}^2$, the number of distances between them is $\Omega(n^{4/3})$, unless the lines are parallel or orthogonal.

cn distances on parallel lines:



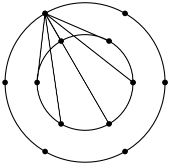
cn distances on orthogonal lines:



Theorem (Pach-De Zeeuw, 2013)

Given two n-point sets on two irreducible algebraic curves of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{4/3})$ distances between them, unless the curves are parallel lines, orthogonal lines, or concentric circles.

cn on concentric circles:



Distances on a single algebraic curve

Theorem (Charalambides, July 2013)

Given n points on an irreducible algebraic curve of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{5/4})$ distinct distances, unless it is a line or a circle.

Charalambides used the approach of Elekes's 1999 paper, with some algebraic geometry, analysis, and rigidity theory.

Theorem (PZ, August 2013)

Given n points on an irreducible algebraic curve of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{4/3})$ distinct distances, unless it is a line or a circle.

Charalambides proved his bound in any dimension, with the exceptional curves being "algebraic helices" like

$$\gamma(t) = (\cos(\lambda_1 t), \sin(\lambda_1 t), \cos(\lambda_2 t), \sin(\lambda_2 t)), \quad \lambda_i \in \mathbb{Q} \cdot \pi.$$

Motivation

Theorem (PZ)

Given n points on an ~~irreducible~~ algebraic curve of degree d in $\mathbb{R}^2$, there are $\geq c_d n^{4/3}$ distances, unless it ~~is~~ **contains** *a line or circle.*

Interpolation: Given any n points in $\mathbb{R}^2$, there is a curve of degree $c\sqrt{n}$ passing through them.

Suppose we lived in a fantasy world where we could prove the above with $c_d = d^{-2/3}$. Then it would follow that a set with $o(n)$ distinct distances would have to have many points on lines or circles. That sounds like a grid...

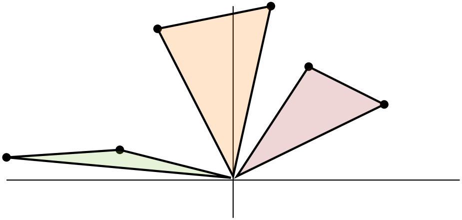
Back to the real world. In the current proof we have $c_d = d^{-11}$. It seems that the best we could do with this setup is $d^{-4/3}$.

Main open problems

- Improve the **exponent** $4/3$.
- Extend to curves in **higher dimensions**.
- Extend to **general polynomials** $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:
  For $S \subset C$ we should have
  $$|F(S,S)| = \Omega_{\deg(C),\deg(F)}\left(|S|^{4/3}\right),$$
  unless C is special or F is special.
- Extend to functions with **more variables**, like distinct areas of triangles determined by triples of points on a curve.
- Extend to "**implicit functions**": e.g., show that if n points on a curve span $\Omega(n^{2-\alpha})$ triple lines, then it must be a cubic curve (done for small α by Elekes-Szabó).
  Or with unit area triangles.

Distinct pinned triangle areas

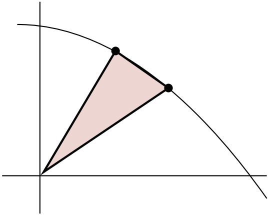
For $p, q \in \mathbb{R}^2$, let $F(p,q) = y_p x_q - x_p y_q$. Then $|F(p,q)|/2$ is the area of the triangle spanned by p, q, and the origin.



Theorem (Iosevich-{Roche-Newton}-Rudnev, 2011)

A set of n points in $\mathbb{R}^2$ determines $\Omega(n/\log(n))$ distinct values of F, unless the points lie on a line (through the origin).

Pinned triangle areas on curves

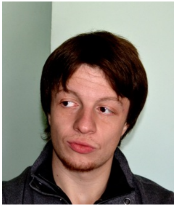


Theorem (Charalambides, 2013)

Given n points on an irreducible algebraic curve of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{5/4})$ distinct values of F, unless the curve is a line, ellipse centred at the origin, or hyperbola centred at the origin.

Theorem (Valculescu-De Zeeuw, 2014)

Given n points on an irreducible algebraic curve of degree d in $\mathbb{R}^2$, there are $\Omega_d(n^{4/3})$ distinct values of F, unless the curve is a line, ellipse centred at the origin, or hyperbola centred at the origin.



Proof of SSS

Recall:

Theorem (Sharir-Sheffer-Solymosi, 2013)

Given two lines in $\mathbb{R}^2$ with n points each, the number of distances is $|D| = \Omega(n^{4/3})$, unless the two lines are parallel or orthogonal.

We have $S_1 \subset l_1$, $S_2 \subset l_2$, $|S_1| = |S_2| = n$.
We count the quadruples

$$Q = \left\{(p, p', q, q') \in S_1^2 \times S_2^2 : d(p,q) = d(p',q')\right\}.$$

Then

$$\frac{n^4}{|D|} \leq |Q| \textcircled{\leq} cn^{8/3} \quad \Rightarrow \quad |D| = \Omega(n^{4/3}).$$

The lower bound for $|Q|$ is easy with Cauchy-Schwarz; the upper bound is all the work. The main tool is Pach-Sharir.

Proof of SSS

For $p_i, p_j \in S_1$ define an algebraic curve in $\mathbb{R}^4$ by

$$\gamma_{ij} = \left\{(q, q') \in l_2 \times l_2 : d(p_i, q) = d(p_j, q')\right\},$$

which is actually a hyperbola on a fixed plane.
We have n^2 curves in

$$\Gamma = \{\gamma_{ij} : p_i, p_j \in S_1\}$$

and n^2 points in

$$P = \{(q_s, q_t) : q_s, q_t \in S_2\}.$$

Then

$$|Q| = I(P, \Gamma) = |\{(p, \gamma) \in P \times \Gamma : p \in \gamma\}|.$$

Proof of SSS

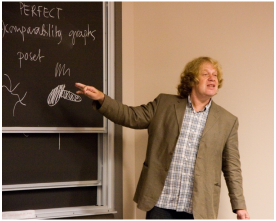
Theorem (Pach-Sharir, 1992)

Given $P \subset \mathbb{R}^D$ and Γ a set of algebraic curves in $\mathbb{R}^D$ with <u>two degrees of freedom</u>:

- *any $\gamma, \gamma' \in \Gamma$ intersect in at most s points of P,*
- *any $p, p' \in P$ belong to at most s curves of Γ.*

Then

$$I(P, \Gamma) = O_s\left(|P|^{2/3}|\Gamma|^{2/3} + |P| + |\Gamma|\right).$$



Proof of SSS

Our P, Γ have **two degrees of freedom**:

• Because the lines are not parallel or orthogonal, the curves are distinct and irreducible, so $|\gamma_{ij} \cap \gamma_{kl}| \leq 4$ by Bézout.
(Write out the equation: reducible $\Rightarrow$ parallel, non-distinct $\Rightarrow$ orthogonal)

• Define "**dual**" curves for $q_s, q_t \in S_2$:

$$\widetilde{\gamma}_{st} = \left\{(p, p') \in l_1 \times l_1 : d(p, q_s) = d(p', q_t)\right\}.$$

Then $|\widetilde{\gamma}_{st} \cap \widetilde{\gamma}_{uv}| \leq 4$ means $(q_s, q_t), (q_u, q_v)$ belong to ≤ 4 $\gamma_{ij} \in \Gamma$.

So:

$$I(P, \Gamma) = O\left((n^2)^{2/3}(n^2)^{2/3}\right) = O\left(n^{8/3}\right).$$

Done.

Proof of VZ

Recall $F(p,q) = y_p x_q - x_p y_q$.

Theorem (VZ, 2014)

If S is contained in an irreducible algebraic curve C of degree d in $\mathbb{R}^2$, then $|F(S,S)| = \Omega_d(|S|^{4/3})$, unless C is a line, ellipse centred at the origin, or hyperbola centred at the origin.

Preparation: We first prepare S so that it contains at most one point on any line through the origin.
This is possible by removing at most $d-1$ points of S per line, leaving $\geq |S|/d$ points. This does not affect the bound.
The reason is that now, for distinct p_i, p_k,

$$F(p_i, q) = 0, \quad F(p_k, q) = 0$$

are independent linear equations.

Proof of VZ

We surprise everyone by bounding the quadruples

$$Q = \{(p, p', q, q') \in S^4 : F(p,q) = F(p', q')\}$$

by

$$\frac{n^4}{|F(S,S)|} \le |Q| = I(P,\Gamma) \le cn^{8/3} \quad \Rightarrow \quad |F(S,S)| = \Omega(n^{4/3});$$

for the upper bound on $|Q|$ we will define points P and curves Γ, and show that they have two degrees of freedom.

Proof of VZ

For $p_i, p_j \in S$ define an algebraic curve in $\mathbb{R}^4$:

$$C_{ij} = \left\{(q, q') \in C \times C : F(p_i, q) = F(p_j, q')\right\}.$$

We have n^2 curves in $\Gamma = \{\gamma_{ij} : p_i, p_j \in S\}$ and n^2 points in $P = \{(q_s, q_t) : q_s, q_t \in S\}$. Define "dual" curves

$$\widetilde{C}_{st} = \left\{(p, p') \in C \times C : F(p, q_s) = F(p', q_t)\right\}.$$

Finally, define "bad sets"

$$\Gamma_0 = \{C_{ij} \in \Gamma : \exists C_{kl} \text{ such that } |C_{ij} \cap C_{kl}| = \infty\},$$
$$P_0 = \{(q_s, q_t) \in P : \exists \widetilde{C}_{uv} \text{ such that } |\widetilde{C}_{st} \cap \widetilde{C}_{uv}| = \infty\},$$

and $\Gamma_1 = \Gamma \backslash \Gamma_0$, $P_1 = P \backslash P_0$.

Proof of VZ

Lemma

If $C_{ij}, C_{kl} \in \Gamma_1$, then $|C_{ij} \cap C_{kl}| = O_d(1)$.
Any two points in P_1 belong to $O_d(1)$ curves in Γ.

Proof.

We want to bound the number of real solutions (q, q') of

$$f(q) = 0, \quad f(q') = 0, \quad F(p_i, q) = F(p_j, q'), \quad F(p_k, q) = F(p_l, q'),$$

where f is the polynomial defining C. Pick your method:

- Oleinik-Petrovski-Milnor-Thom;
- Move to $\mathbb{C}$ and use complex Bézout;
- Last two equations define a plane, apply real planar Bézout there.

Do the same for the dual curves. □

So P_1, Γ_1 have two degrees of freedom and $I(P_1, \Gamma_1) = O_d(n^{8/3})$.

Proof of VZ

Now the bad sets Γ_0, P_0.
For linear $T : \mathbb{R}^2 \to \mathbb{R}^2$, set $G_T = \{(q, q') \in C \times C : T(q) = q'\}$.

Lemma

For any $C_{ij}, C_{kl} \in \Gamma$ we have $C_{ij} \cap C_{kl} = G_T$ for some linear T.
If $|G_T| = |C_{ij} \cap C_{kl}| = \infty$, then T is an automorphism of C.

Proof.

If $(q, q') \in C_{ij} \cap C_{kl}$ then since $F(p_i, q) = y_{p_i} x_q - x_{p_i} y_q$ we have

$$M_{ik} q = M_{jl} q' \quad \text{with} \quad M_{ik} = \begin{pmatrix} y_{p_i} & -x_{p_i} \\ y_{p_k} & -x_{p_k} \end{pmatrix}, \quad M_{jl} = \begin{pmatrix} y_{p_j} & -x_{p_j} \\ y_{p_l} & -x_{p_l} \end{pmatrix}.$$

The matrices are invertible thanks to the preparation of S.
So $q' = M_{jl}^{-1} M_{ik} q =: T(q)$. Also vice versa.

If $|G_T| = \infty$, then $|T(C) \cap C| = \infty$, so $T(C) = C$. □

Proof of VZ

Lemma

$$I(P,\Gamma_0) = O_d(n^2), \qquad I(P_0,\Gamma) = O_d(n^2).$$

Proof.

By the Automorphism Lemma below, C has $\leq 4d$ linear automorphisms, unless it is a special curve.
It is not hard to see that each automorphism occurs $\leq n$ times.
So $|\Gamma_0| \leq 4dn$ and the lemma follows easily. □

This finishes the proof:

$$I(P,\Gamma) \leq I(P_0,\Gamma) + I(P,\Gamma_0) + I(P_1,\Gamma_1) = O_d\left(n^{8/3}\right).$$

Automorphism Lemma

Lemma

An irreducible algebraic curve of degree d has $\leq 4d$ linear automorphisms, unless it is a line or linearly equivalent to one of:

Ellipses :	$x^2 + y^2 = 1$	$\begin{pmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix}$
Hyperbolas :	$xy = 1$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$
Parabolas :	$x = y^2$	$\begin{pmatrix} a^2 & 0 \\ 0 & a \end{pmatrix}$
"**Pseudohyperbolas**" :	$x^p y^q = 1$	$\begin{pmatrix} a^q & 0 \\ 0 & a^{-p} \end{pmatrix}$
"**Pseudocusps**" :	$x^p = y^q$	$\begin{pmatrix} a^q & 0 \\ 0 & a^p \end{pmatrix}$

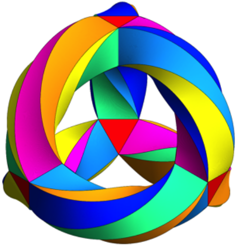
Calculation shows that the last three cannot occur for our F. $\square$

About the Automorphism Lemma

Of course, something much stronger has long been known:

Theorem (Hurwitz, 1893)

A nonsingular curve of genus $g \geq 2$ has at most $84(g-1)$ polynomial automorphisms.



But this does not give the detailed information that we need. In particular, it does not apply to singular curves, which is a problem if we want to do interpolation.

About the Automorphism Lemma

Idea of our proof of the Automorphism Lemma:
Most algebraic curves cannot contain an infinite orbit $\{T^{(k)}(p)\}$.

E.g., let $C : f(x,y) = \sum a_{ij}x^i y^j$ and $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}_{>0}$.
Then, if $p = (x_0, y_0)$ and $T^{(k)}(p) \in C$, we would have

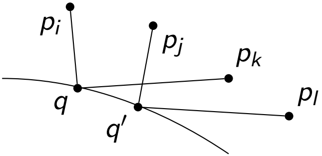
$$0 = f(\lambda^k x_0, \mu^k y_0) = \sum \left(a_{ij} x_0^i y_0^j \right) e^{(\ln(|\lambda|)i + \ln(|\mu|)j)k} = \sum b_{ij} e^{c_{ij}k}.$$

Such a function has only finitely many roots k (unless...).

We do this for each Jordan form, and we get exactly the exceptions in the lemma.

Proof sketch of PZ

As above but with $F(p, q) = (x_p - x_q)^2 + (y_p - y_q)^2$.
If $(q, q') \in C_{ij} \cap C_{kl}$, then



Suppose $F(p_i, p_k) = F(p_j, p_l)$ (other case is annoying...).
$\Rightarrow \exists$ isometry T so that $T(p_i) = p_k$, $T(p_j) = p_l \Rightarrow T(q) = q'$ (...)
Then $|C_{ij} \cap C_{kl}| = \infty \ \Rightarrow \ |T(C) \cap C| = \infty \ \Rightarrow \ T(C) = C$.

Lemma (Isometry Lemma)

An irreducible algebraic curve of degree d has at most $4d$ isometries, unless it is a line or a circle.

Main references (in order of appearance)

- György Elekes and Lajos Rónyai, *A combinatorial problem on polynomials and rational functions*, JCTA, 89 (2000), 1–20.
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