Euclidean and Algebraic Geometry

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Combinatorial Geometry Problems
at the Algebraic Interface
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Overview

This lecture will give examples of how algebraic geometry can be applied to problems in Euclidean geometry.

Key points:

- Algebraic geometry can be a useful tool, but . . .
- There are limitations.

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Algebraic Geometry

In its simplest form, algebraic geometry is the study of geometric objects defined by algebraic equations, i.e., polynomials.

Algebraic geometry is good at:

- Counting (enumerative algebraic geometry)
- Giving Structure to Interesting Sets (moduli spaces)
- Understanding Structure (minimal model program)

The third bullet is beyond the scope of this talk.

Later I will give several examples of the second bullet. Let me begin with a classic example of the first bullet.

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Bezout's Theorem

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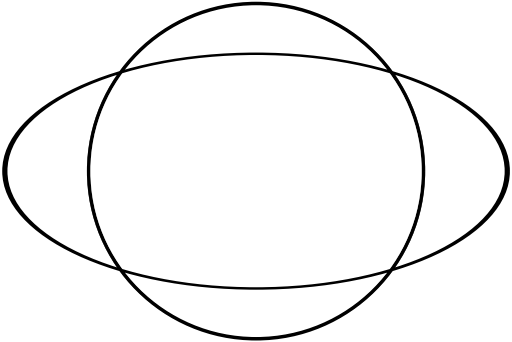
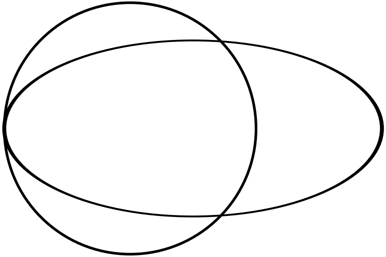
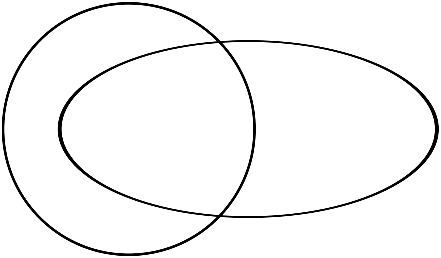


Fine Print, Part 1

Count with multiplicity:

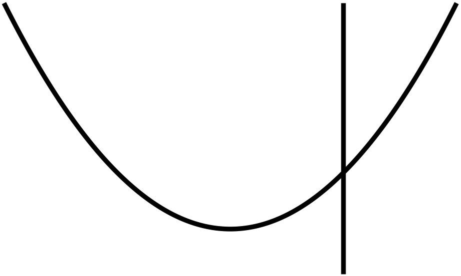


Work over $\mathbb{C}$:
(This will be
very important.)



Fine Print, Part 2

The curves $y = x^2$ and $x = 1$ should meet in $2 \cdot 1 = 2$ points.



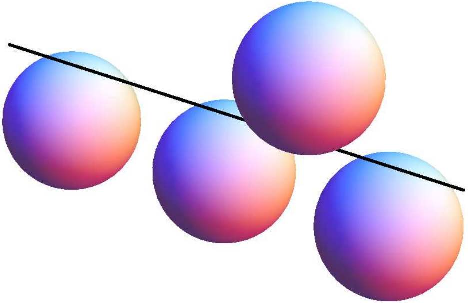
We need points at ∞! This is why projective space is so important in algebraic geometry.

Examples From Euclidean Geometry

1. Tangents to 4 Unit Spheres
2. A Combinatorial Moduli Space
3. Counting Balanced Subspaces
4. Two Quickies, If Time Permits

Tangents to 4 Unit Spheres

Given four unit spheres (= spheres of radius 1) in $\mathbb{R}^3$, how many lines can be simultaneously tangent to all three?



How many other tangent lines could there be?

Setup

Spheres: Assume the spheres have centers $0, c_1, c_2, c_3$ that are not coplanar.

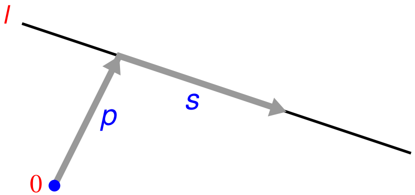
Lines: Write a line ℓ in $\mathbb{R}^3$ as:

$$\ell = \{p + \lambda s \mid \lambda \in \mathbb{R}\}, \text{ where}$$

p = offset vector (unique)

s = direction vector (unique up to constant multiple)

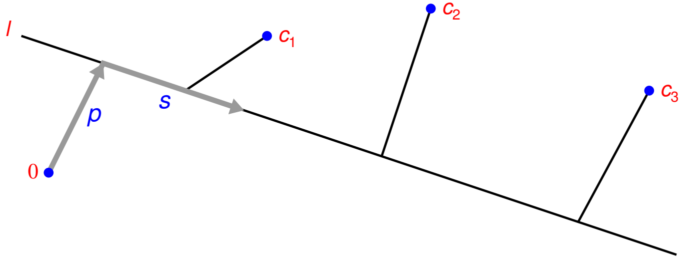
$p \cdot s = 0$.



ℓ is tangent to the 4 unit spheres $\iff$ ℓ has distance 1 to $0, c_1, c_2, c_3$.

A New Problem

When is a line ℓ equidistant from $0, c_1, c_2, c_3$?



- Distance from 0 to $\ell = \{p + \lambda s\}$ is $||p||$ since $p \cdot s = 0$.
- Distance from c_i to ℓ is $||p|| \iff$

$$c_i \cdot p = \frac{1}{2\,||s||^2}\,||c_i \times s||^2.$$

- Equidistant lines satisfy three linear equations!

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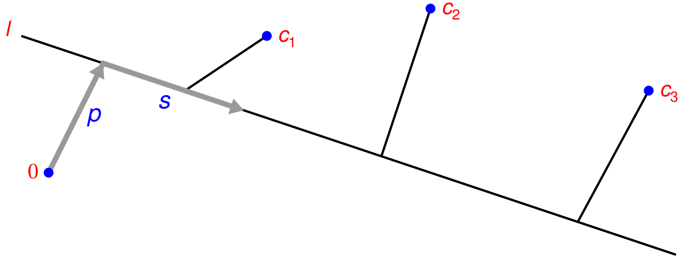

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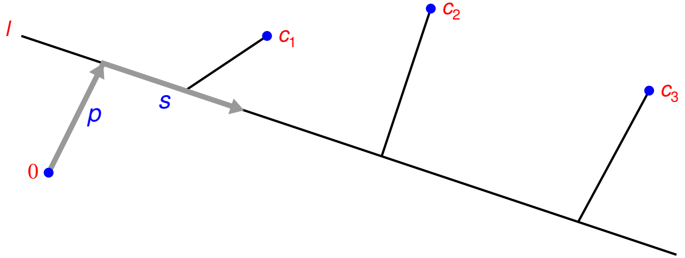

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- Equidistant lines satisfy three linear equations!

A Theorem

Solving these equations by Cramer's rule, we obtain:

Theorem

Suppose that the line $\ell = \{p + \lambda s\}$ is equidistant from the centers $0, c_1, c_2, c_3$. Define the vectors

$$\begin{aligned} n_1 &= c_2 \times c_3 \\ n_2 &= c_3 \times c_1 \\ n_3 &= c_1 \times c_2. \end{aligned}$$

Then

$$p = \frac{1}{2\,||s||^2 D} \sum_{i=1}^{3} ||c_i \times s||^2\, n_i,$$

where $D = \det(c_1, c_2, c_3) \neq 0$ (non-coplanar).

Key Point #1

If ℓ is equidistant from the centers, the offset vector is

$$p = \frac{1}{2\,||s||^2 D} \sum_{i=1}^{3} ||c_i \times s||^2\, n_i.$$

Then $p \cdot s = 0$ gives the equation in s:

$$\sum_{i=1}^{3} ||c_i \times s||^2\, s \cdot n_i = 0.$$

The direction vector s is unique up to multiplication by a nonzero scalar and hence gives a point in the projective plane. Thus we have proved:

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Key Point #2

When $\ell = \{p + \lambda s\}$ is equidistant, the distance is

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where $D = \det(c_1, c_2, c_3)$.

Hence the distance is 1 $\iff$

$$4\,||s||^4 D^2 = \left|\left| \sum_{i=1}^{3} ||c_i \times s||^2\, n_i \right|\right|^2$$

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At Most 12 Tangents!

Theorem (MacDonald-Pach-Theobald, 2001)

There are at most 12 lines tangent to 4 unit spheres when the centers are not collinear.

Proof for non-coplanar centers. If $\ell = \{p + \lambda s\}$ is tangent to unit spheres with centers $0, c_1, c_2, c_3$, then in the projective plane, the direction vector s lies in

$$\underbrace{\text{degree 3 curve}}_{\text{equidistant}} \cap \underbrace{\text{degree 4 curve}}_{\text{distance 1}}.$$

By Bezout, this intersection has 12 points over $\mathbb{C}$. QED

Note that $\{$direction vectors$\}$/rescaling is more than a just a set: it is the projective plane, which is an algebraic variety.

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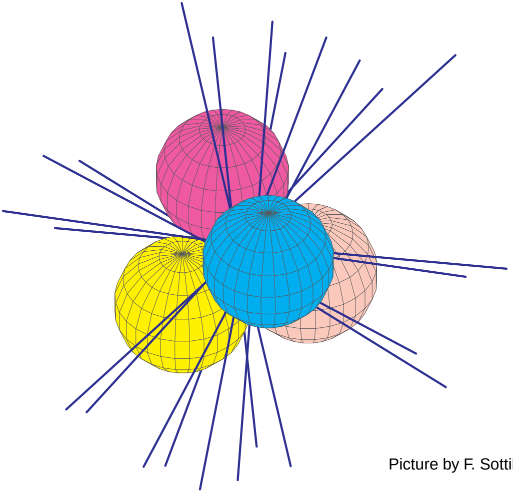
12 Real Tangents

For 12 lines tangent to four unit spheres, all 12 can be real:

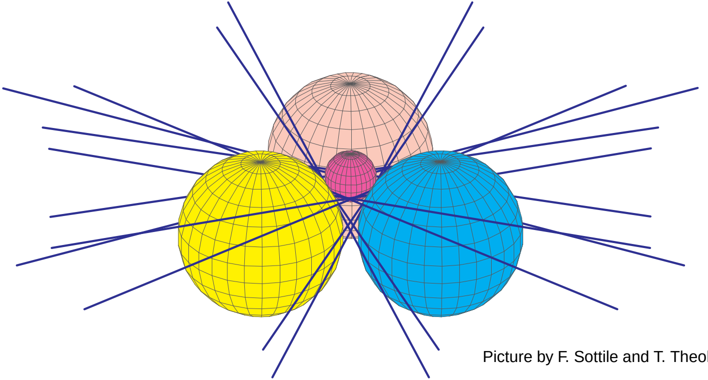


Picture by F. Sottile and T. Theobald

In this picture, the spheres are not disjoint.

An Open Question

If the spheres are allowed to have *unequal* radii, then we can have 12 real lines with disjoint spheres:



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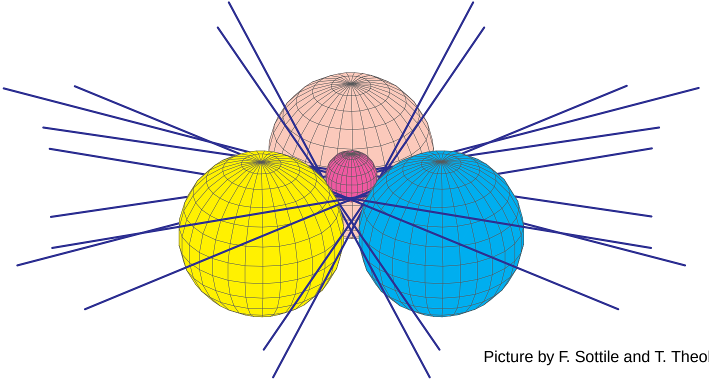

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A Combinatorial Moduli Space

A line arrangement $\mathscr{A}_n = \{L_1,\ldots,L_n\}$ in $\mathbb{P}^2$ is combinatorially equivalent to $\mathscr{B}_n$, written $\mathscr{A}_n \sim \mathscr{B}_n$, if there is an inclusion-preserving bijection between the their intersection lattices formed by all possible intersections of the lines.

The moduli space $M_{\mathscr{A}_n} = \{\mathscr{B}_n \mid \mathscr{B}_n \sim \mathscr{A}_n\}/\mathrm{PGL}(3)$ measures the difference between combinatorial and projective equivalence.

These moduli spaces are well understood for $n \leq 10$.

We will assume that $\mathscr{A}_n$ has only double and triple points and that every line of $\mathscr{A}_n$ contains at least three triple points.

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Moduli for $\mathscr{A}_{10}$

For $n = 10$, there are 18 cases where the moduli space is disconnected:

- 1 moduli space of dimension one.
- 10 moduli spaces of dimension zero that cannot be realized over $\mathbb{R}$.
- 7 moduli spaces of dimension zero realizable over $\mathbb{R}$, three of which have exactly two points.

For a moduli spaces of the form $\{\mathscr{A}_{10}, \mathscr{B}_{10}\}$, a natural question to study is whether the pairs

$$(\mathbb{P}^2, \mathscr{A}_{10}) \text{ and } (\mathbb{P}^2, \mathscr{B}_{10})$$

are homeomorphic. (They are.)

Combinatorial Symmetry of Line Arrangements and Applications, by Amram, Cohen, Sun, Teicher, Ye & Zarkh, arXiv:1310.0700

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Schubert Varieties

The set of d-dimensional subspaces of $\mathbb{C}^n$ has a natural structure as an algebraic variety, the Grassmannian $G(d,n)$. Inside of $G(d,n)$ are Schubert varieties, defined as follows. Given a flag of subspaces

$$W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k,$$

the corresponding Schubert variety is

$$\{V \in G(d,n) \mid \dim(V \cap W_i) \geq i,\ 1 \leq i \leq k\}.$$

Many problems in enumerative algebraic geometry can be formulated in terms of counting points in intersections of Schubert varieties. This led to the Schubert calculus.

I will discuss an example from *Real Solutions of a Problem in Enumerative Geometry* by Fehér and Matszangosz (arXiv:1401.4638). They use work of Vakil from 2006.

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The Result over $\mathbb{C}$

Consider four n-dimensional subspaces $V_1, \ldots, V_4$ is $\mathbb{C}^{2n}$ in general position. How many $2m$-dimensional subspaces $W \subseteq \mathbb{C}^{2n}$ satisfy

$$\dim(W \cap V_i) = m,\ 1 \leq i \leq 4?$$

We say that W is balanced with respect to $V_1, \ldots, V_4$.

This problem reduces to the intersection of four Schubert varieties. By the Schubert calculus, one gets the answer

$$\binom{n}{m}.$$

To do:

- Explain the combinatorics behind this answer.
- Ask the same question over $\mathbb{R}$.

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For $n = 7$, $m = 5$

The possibilities are $\binom{7}{5} = 21, \ 11, \ 5, \ 3.$

For General n, m

$$\sum_{i=0}^{c} \binom{c}{i} \binom{n-2c}{m-2i},$$

where c is an integer $0 \leq c \leq n/2$ depending on $V_1, \ldots, V_4$. Furthermore, all possible values of c can occur.

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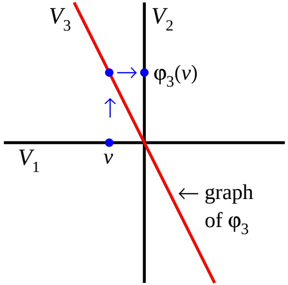
The Map $\varphi : V_1 \to V_1$

Given n-dimensional subspaces $V_1, \ldots, V_4 \subseteq \mathbb{C}^{2n}$, we will assume $V_i \cap V_j = \{0\}$ for all $i \neq j$. Then use V_3 to define an isomorphism

$$\varphi_3 : V_1 \to V_2$$

using



Similarly, V_4 gives an isomorphism $\varphi_4 : V_1 \to V_2$. Then we define $\varphi : V_1 \to V_1$ to be $\varphi = \varphi_3^{-1} \circ \varphi_4$.

The Key Theorem

General Position

$V_i \cap V_j = \{0\}$ for $i \neq j$ and $\varphi : V_1 \to V_1$ has distinct eigenvalues.

Theorem (Fehér & Matszangosz)

The map $W \mapsto W \cap V_1$ induces a bijection

$$\left\{ \begin{array}{c} 2m\text{-dimensional} \\ \text{balanced subspaces} \end{array} \right\} \simeq \left\{ \begin{array}{l} m\text{-dimensional subspaces} \\ \text{of } V_1 \text{ invariant under } \varphi \end{array} \right\}$$

Key Step of Proof.

For $W_1 \subseteq V_1$ m-dimensional and φ-invariant, show that

$$W = W_1 + \varphi_3(W_1) = W_1 + \varphi_3(\varphi(W_1)) = W_1 + \varphi_4(W_1)$$

is balanced of dimension $2m$. □

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$$\left\{ \begin{array}{c} 2m\text{-dimensional} \\ \text{balanced subspaces} \end{array} \right\} \simeq \left\{ \begin{array}{l} m\text{-dimensional subspaces} \\ \text{of } V_1 \text{ invariant under } \varphi \end{array} \right\}$$

Key Step of Proof.

For $W_1 \subseteq V_1$ m-dimensional and φ-invariant, show that

$$W = W_1 + \varphi_3(W_1) = W_1 + \varphi_3(\varphi(W_1)) = W_1 + \varphi_4(W_1)$$

is balanced of dimension $2m$. □

The Key Theorem

General Position

$V_i \cap V_j = \{0\}$ for $i \neq j$ and $\varphi : V_1 \to V_1$ has distinct eigenvalues.

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The Result over $\mathbb{C}$ and $\mathbb{R}$

We need to count m-dimensional φ-invariant subspaces.

Proof over $\mathbb{C}$.

V_1 is a direct sum of n one-dimensional eigenspaces of φ. An m-dimensional φ-invariant subspace is a direct sum of m of these. Hence there are $\binom{n}{m}$ such subspaces. □

Proof over $\mathbb{R}$.

Let $c = \#$ complex-conjugate pairs of eigenvalues of φ. Then V_1 is a direct sum of c two-dimensional invariant subspaces and $n - 2c$ one-dimensional eigenspaces. An m-dimensional φ-invariant subspace is built from i of the former and $m - 2i$ of the latter. This gives

$$\sum_{i=0}^{c} \binom{c}{i} \binom{n-2c}{m-2i}.$$

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A Moduli Space

The generic $V_1, \ldots, V_4$ form an open subset $U \subset G(n, 2n)^4$. Hence the moduli space is $U/\mathrm{GL}(2n)$. To understand this quotient, consider the map $U \to \mathbb{C}^n$ sending $(V_1, \ldots, V_4)$ to the coefficients of $\det(\varphi - xI)$.

Lemma (Fehér & Matszangosz)

The nonempty fibers of the map $U \to \mathbb{C}^n$ are precisely the orbits of GL$(2n)$ acting on U.

Note that 0 is not an eigenvalue since φ is an isomorphism; the same is true for 1 since $V_3 \cap V_4 = \{0\}$. If we write

$$\det(\varphi - xI) = (-1)^n x^n + a_1 x^{n-1} + \cdots + a_n,$$

then $U/\mathrm{GL}(2n)$ consists of all $(a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying

$$\mathrm{disc}(\det(\varphi - xI)) \neq 0, \quad a_n \neq 0, \quad a_1 + \cdots + a_n \neq (-1)^{n-1}.$$

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Two Examples

$n = 2,\ m = 1$

Here, $V_1, \ldots, V_4$ and W are planes through the origin in $\mathbb{C}^4$, which correspond to lines in projective 3-space $\mathbb{P}^3$. Since $\binom{2}{1} = 2$, we recover the classical fact that there are two lines meeting four given lines in $\mathbb{P}^3$ in general position.

$n = 1$

Here, $V_1, \ldots, V_4$ are lines through the origin in $\mathbb{C}^2$, which correspond to points in the projective line $\mathbb{P}^1$. Since $\varphi : V_1 \to V_1$ is multiplication by $a \neq 0, 1$, the map $(V_1, \ldots, V_4) \mapsto \varphi$ is (up to sign) the classical cross-ratio of four distinct points in $\mathbb{P}^2$. The moduli space is

$$U/\mathrm{GL}(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

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Quickie #1. The Polynomial Method

Let $\mathscr{L}$ be a collection of N lines in $\mathbb{R}^n$ (or F^n for any field F). A joint for $\mathscr{L}$ is a point p such that there are n lines in $\mathscr{L}$ which contain p and do not line in a hyperplane.

Theorem (Joints Conjecture)

The number of joints is at most $nN^{n/(n-1)}$.

This was proved independently in 2009/10 by Kaplan, Sharir, Shustin and also by Quilodrán.

The proof uses polynomials that vanish at the joints and a variant of Bezout's Theorem. All known proofs involve algebraic geometry in some way. Tao presents the proof in the context of what he calls the polynomial method.

Algebraic Combinatorial Geometry: The Polynomial Method in Arithmetic Combinatorics, Incidence Combinatorics, and Number Theory, by Tao, arXiv:1310.6482

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Quickie #2. Ellipses and Elliptic Curves

The next example is modest but amusing.

An ellipse centered at the origin is defined by an equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

of degree 2, while an elliptic curve in Weierstrass form is defined by an equation

$$y^2 = 4x^3 - g_2x - g_3$$

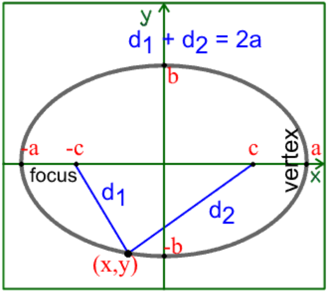
of degree 3. Clearly different – why such similar names?

Brief answer:

ellipse (Greeks) ⇒ arc length of ellipse (17th C) ⇒ elliptic integral (18th C) ⇒ elliptic function (19th C) ⇒ elliptic curve (20th C)

Ellipses in the Plane and Sphere

Here is another way to think about the ellipse:



people.richland.edu/james/lecture/m116/conics/elldef.html

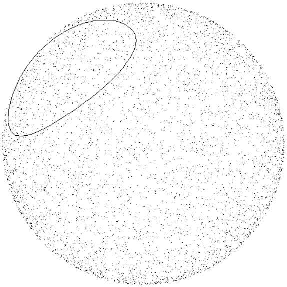
This construction works nicely in spherical geometry.

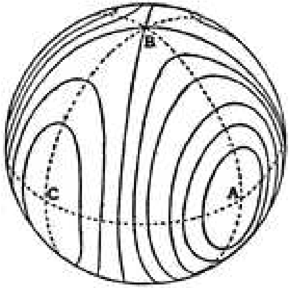
Given two points $A \neq B$ on the unit sphere S^2, a spherical ellipse is the locus of points $P \in S^2$ such that

$$\text{dist}(P,A) + \text{dist}(P,B) = 2a,$$

where "dist" is the great circle distance between the points.

Pictures of Spherical Ellipses





Images from:
`virtualmathmuseum.org//SpaceCurves/spherical_ellipse/spherical_ellipse.html`
`etc.usf.edu/clipart/galleries/275-spheres.`

Equation of a Spherical Ellipse

Fix $A \neq B$ in S^2. Then $P \in S^2$ satisfies

$$(1) \qquad \mathrm{dist}(P,A) + \mathrm{dist}(P,B) = 2a.$$

Recall that

$$\mathrm{dist}(P,A) = \cos^{-1}(P \cdot A) \quad \text{and} \quad \sin(\cos^{-1}(x)) = \sqrt{1-x^2}.$$

Applying cos to (1) and doing some algebra gives

$$\sin^2(2a) - (P\cdot A)^2 - (P\cdot B)^2 + 2\cos(2a)(P\cdot A)(P\cdot B) = 0.$$

Since $P \cdot P = 1$, we obtain the homogeneous equation

$$\sin^2(2a)(P\cdot P) - (P\cdot A)^2 - (P\cdot B)^2 + 2\cos(2a)(P\cdot A)(P\cdot B) = 0.$$

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Elliptic Ellipses are Elliptic!

It follows that our spherical ellipse in S^2 is the intersection

$$\underbrace{\text{quadric cone}}_{\text{degree 2}} \bigcap \underbrace{\text{sphere}}_{\text{degree 2}}.$$

However:

Theorem (Algebraic Geometry)

The intersection of two quadric surfaces in 3-dimensional space is an elliptic curve.

Spherical ellipses are elliptic curves. Spherical Geometry is also called Elliptic Geometry. Hence the title of the slide!

A small lie: a spherical ellipse is half of an elliptic curve, so

elliptic ellipses are half elliptic!

Reference: `virtualmathmuseum.org/SpaceCurves/spherical_ellipse/Spherical_Ellipse.pdf`.

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Conclusion

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In this talk, we have seen that algebraic geometry cannot claim to be unreasonably effective in Euclidean geometry – it has definite strengths and weaknesses.

Nevertheless, I hope I have convinced you that algebraic geometry is pretty darn good!

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