Arctic Boundaries for Ice Models

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Alternating Sign Matrices

An **alternating sign matrix** (ASM) is a square matrix, with each entry in $\{-1, 0, 1\}$, satisfying the following two properties.

- The nonzero entries in each row and column alternate between -1 and 1
- The entries in each row and column sum to 1

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question

How does a large, uniformly random alternating sign matrix "look?"

Frozen Region of an ASM

Given an $N \times N$ alternating sign matrix $\mathbf{M}$, we say that $(i, j) \in [1, N] \times [1, N]$ is in the **frozen region** of $\mathbf{M}$ if at least one of the following holds.

- Each entry in $\mathbf{M}$ (weakly) northwest of (i, j) is equal to 0.
- Each entry in $\mathbf{M}$ (weakly) southwest of (i, j) is equal to 0.
- Each entry in $\mathbf{M}$ (weakly) northeast of (i, j) is equal to 0.
- Each entry in $\mathbf{M}$ (weakly) southeast of (i, j) is equal to 0.

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & -1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & -1 & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Question

What is the boundary of the frozen region of a large, uniformly random ASM?

Boundary Parameterization

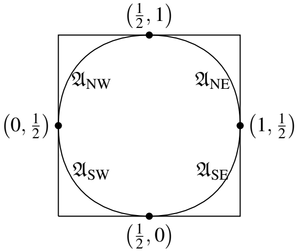
Define the portion of an ellipse

$$\mathfrak{A}_{\mathrm{SE}} = \left\{(x,y) \in \mathbb{R}^2 : (2x-1)^2 + (2y-1)^2 - 4(1-x)y = 1\right\} \cap \left(\left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]\right),$$

and its reflections

$$\mathfrak{A}_{\mathrm{SW}} = \left\{(x,y) \in \mathbb{R}^2 : (1-x, y) \in \mathfrak{A}_{\mathrm{SE}}\right\}; \qquad \mathfrak{A}_{\mathrm{NE}} = \left\{(x,y) \in \mathbb{R}^2 : (x, 1-y) \in \mathfrak{A}_{\mathrm{SE}}\right\};$$

$$\mathfrak{A}_{\mathrm{NW}} = \left\{(x,y) \in \mathbb{R}^2 : (1-x, 1-y) \in \mathfrak{A}_{\mathrm{SE}}\right\}.$$



- Let $\mathfrak{A} = \mathfrak{A}_{\mathrm{SE}} \cup \mathfrak{A}_{\mathrm{SW}} \cup \mathfrak{A}_{\mathrm{NE}} \cup \mathfrak{A}_{\mathrm{NW}}$.
- Then $\mathfrak{A}$ is **not smooth** at its four tangency points with $[0,1] \times [0,1]$.

Arctic Curves for ASMs

- Let $N \in \mathbb{Z}_{>0}$ be a large integer.
- Let $\mathbf{M}$ denote an $N \times N$ alternating sign matrix, chosen uniformly at random.
- Let $(i, j) \in [1, N] \times [1, N]$ be an integer pair, and set $z = \left(\frac{i}{N}, \frac{j}{N} \right) \in [0, 1] \times [0, 1]$.
- Fix a real number $\varepsilon > 0$, and assume that $\operatorname{dist}(z, \mathfrak{A}) > \varepsilon$.

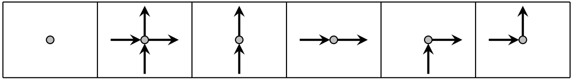
Theorem (A., 2018)

There exists $\delta = \delta(\varepsilon) > 0$ such that, with probability at least $1 - e^{-\delta N}$, (i, j) is in the frozen region of $\boldsymbol{M}$ if and only if z is outside of $\mathfrak{A}$.

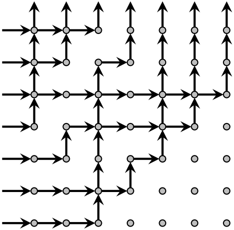
- Eloranta (1999), Sylijuåsen–Zvonarev (2004), Allison–Reshetikhin (2005): Predicted existence of arctic boundary
- Colomo–Pronko (2010): Predicted above form of arctic boundary
- Colomo–Sportiello (in preparation): Different proof of the above result

Six-Vertex Ensembles and Ice Models

Let $\Lambda \subset \mathbb{Z}^2$ be finite, and assign each vertex in Λ one of the following six edge configurations, such that neighboring configurations are consistent.



- **Ice model**: The weight of each vertex is equal to 1.
- **Domain-wall boundary conditions** arise when $\Lambda = [1, N] \times [1, N]$, and arrows enter from the left boundary and exit through the top.



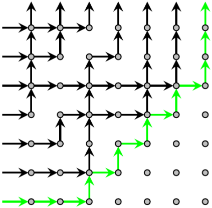
Six-vertex ensembles are collections of non-crossing directed (up-right) paths.

Bijection Between ASMs and Six-Vertex Ensembles

There is a bijection between $N \times N$ alternating sign matrices and domain-wall six-vertex ensembles on $[1, N] \times [1, N]$.

ASM	Six-Vertex
1	
-1	

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & \mathbf{0} \\ 0 & 1 & 0 & -1 & 1 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$



The southeast part of the boundary of the frozen region of an ASM coincides with the bottommost path of the associated domain-wall six-vertex ensemble.

Trajectory of the Bottom Path of the Ice Model

- Let $N > 0$ be a large integer.
- Let $\mathcal{E}$ denote a sample of the ice model on $[1, N] \times [1, N]$.
- Denote the non-crossing paths in $\mathcal{E}$, from bottom to top, by $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$.
- Define $I_1 = \left[0, \frac{1}{2}\right] \times \{0\}$ and $I_2 = \{1\} \times \left[\frac{1}{2}, 1\right]$, and let $\mathfrak{P} = I_1 \cup \mathfrak{A}_{\text{SE}} \cup I_2$.

By symmetry and the above bijection, we must show the following theorem.

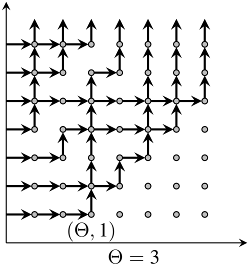
Theorem (A., 2018)

For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\operatorname{dist}\left(N^{-1}\boldsymbol{p}_1, \mathfrak{P}\right) < \varepsilon$ *holds with probability at least* $1 - e^{-\delta N}$.

- Proof based on a justification, in the ice model case, of the **(geometric) tangent method**, a general heuristic introduced by Colomo–Sportiello (2016) for deriving arctic boundaries of statistical mechanical models
- Our proof is not very model-dependent and also should apply to other families statistical mechanical systems

Refined Partition Function

- Domain-wall six-vertex ensemble $\mathcal{E}$ with paths $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$
- Let $\Theta = \Theta(\mathcal{E}) \in [1, N]$ be such that $\mathbf{p}_1$ exits the bottom row at $(\Theta, 1)$



- The **partition function** Z_N counts domain-wall six-vertex ensembles $\mathcal{E}$.
- The **refined partition function** $Z_N(k)$ counts those with $\Theta(\mathcal{E}) = k$.
- Define the k**-refined correlation function** $H_N(k)$ by

$$H_N(k) = \mathbb{P}\big[\Theta(\mathcal{E}) = k\big] = \frac{Z_N(k)}{Z_N}.$$

Refined Enumeration

- Zeilberger (1996): $H_N(k) = \binom{N+k-2}{N-1}\binom{2N-k-1}{N-1}\binom{3N-2}{N-1}^{-1}$
- Thus, for fixed $\kappa > 0$, we have for large N that

$$H_N(\kappa N) = \exp\Big(- \big(\mathfrak{h}(\kappa) + o(1)\big)N\Big),$$

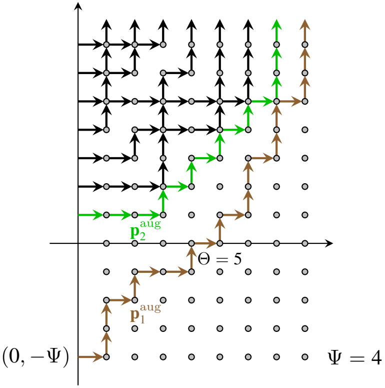
for an **explicit** $\mathfrak{h}(\kappa)$ given by

$$\mathfrak{h}(\kappa) = (1+\kappa)\log(1+\kappa) + (2-\kappa)\log(2-\kappa) - \kappa\log\kappa \\ - (1-\kappa)\log(1-\kappa) - 3\log 3 + 2\log 2$$

- **Tangency point**: $\mathfrak{h}(\kappa)$ minimized at $\kappa = \frac{1}{2}$, so we likely have $\Theta \approx \frac{N}{2}$
- Colomo–Sportiello (2016): Use the function $\mathfrak{h}$ to predict a parameterization for the limiting trajectory of $\mathbf{p}_1$

Augmented Domains and Ensembles

For $\Psi \in \mathbb{Z}_{\geq 0}$, a Ψ**-augmented ensemble** is a domain-wall six-vertex ensemble on $[1, N] \times [1, N]$, with an additional path entering at $(0, -\Psi)$ and exiting at $(N+1, N)$.

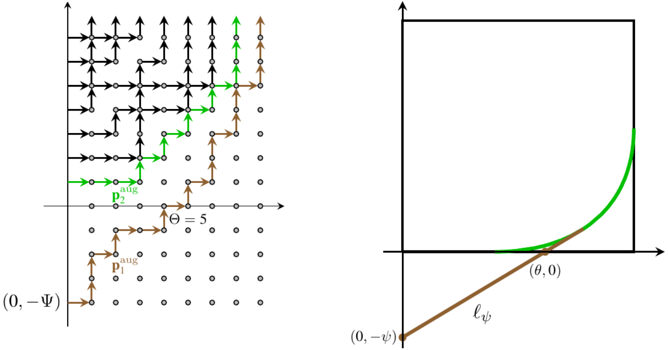


- Denote the paths in this ensemble, from bottom to top, by $\mathbf{p}_1^{\mathrm{aug}}, \mathbf{p}_2^{\mathrm{aug}}, \ldots, \mathbf{p}_{N+1}^{\mathrm{aug}}$.
- Let Θ denote be such that $\mathbf{p}_1^{\mathrm{aug}}$ exits the x-axis at $(\Theta, 0)$

Tangency Assumption

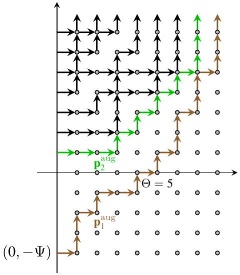
- Fix $\psi > 0$, and let $\Psi \approx \psi N$
- Select a Ψ-augmented ensemble $\mathcal{E}_\Psi$ uniformly at random
- With high probability, we will have $\Theta = \Theta(\mathcal{E}_\Psi) \approx \theta N$, for some $\theta = \theta(\psi) > 0$

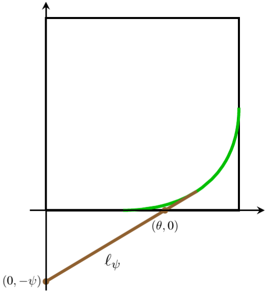
Belief: As N tends to ∞, $\mathbf{p}_1^{\text{aug}}$ first approximates a line ℓ_ψ tangent to the arctic curve of the domain-wall ice model and then merges with it.



Determining the Arctic Boundary

- If we could determine $\theta = \theta(\psi)$ for each $\psi > 0$, then we would determine ℓ_ψ.
- By varying over ψ, this would give $\mathfrak{A}_{\text{SE}}$.





- The number of augmented ensembles $\mathcal{E}_\Psi$ with $\Theta(\mathcal{E}_\Psi) = \Phi \approx \varphi N$ is proportional to

$$H_{N+1}(\Phi) \binom{\Phi + \Psi - 1}{\Psi} = \exp\Big(\big(g_\psi(\varphi) + o(1)\big)N\Big),$$

where $g_\psi(\varphi) = (\varphi + \psi)\log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - \mathfrak{h}(\varphi)$.

- The maximizer $\varphi = \theta$ of $g_\psi(\varphi)$ determines $\theta = \theta(\psi) > 0$.

Tangent Method Heuristic

The tangent method (Colomo–Sportiello, 2016)

- Using exact asymptotics for $H_N(k)$, find explicit $\mathfrak{h} : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$H_N(\kappa N) = \exp\Big(- \big(\mathfrak{h}(\kappa) + o(1)\big) N\Big)$$

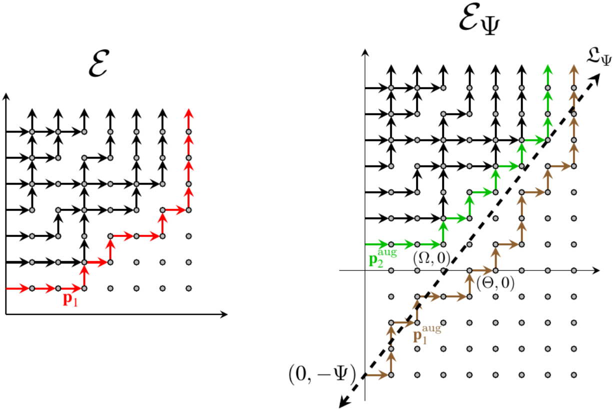
- Define $g_\psi(\varphi) = (\varphi + \psi) \log(\varphi + \psi) - \varphi \log \varphi - \psi \log \psi - \mathfrak{h}(\varphi)$
- Let $\theta = \theta(\psi)$ denote the maximizer of g_ψ
- For each ψ, let ℓ_ψ denote the line through $(0, -\psi)$ and $(0, \theta)$
- Then the arctic boundary is the convex envelope formed by the ℓ_ψ after varying over ψ, which is $\mathfrak{A}_{\mathrm{SE}}$

Issues

- Must justify the tangency assumption
- It is not transparent that arctic curve exists (namely, that $\mathbf{p}_1$ in the original model or $\mathbf{p}_2^{\mathrm{aug}}$ in the augmented model have limiting trajectories)
- The introduction of the new path $\mathbf{p}_1^{\mathrm{aug}}$ in the augmented model might change the trajectory of $\mathbf{p}_1$ in the original model

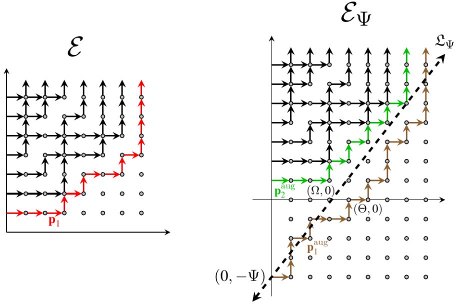
Notation

- Let $\mathcal{E}$ and $\mathcal{E}_\Psi$ be a domain-wall six-vertex ensemble and a Ψ-augmented ensemble, respectively, both chosen uniformly at random.
- Let $\mathfrak{L} = \mathfrak{L}_\Psi$ be the tangent line to $\mathbf{p}_2^{\mathrm{aug}}$ through $(0, -\Psi)$.
- Let $(\Omega, 0) = \mathfrak{L}_\Psi \cap \{y = 0\}$, and let $\mathbf{p}_1^{\mathrm{aug}}$ exit the x-axis at $(\Theta, 0)$.



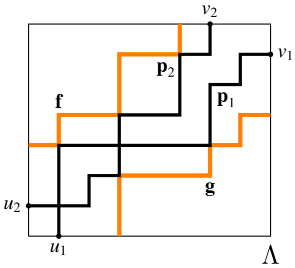
Proof Outline

1. *Tangency*: $\mathbb{P}\big[|\Omega - \Theta| < \varepsilon N\big] > 1 - C\exp(-c\varepsilon^2 N)$
2. *Concentration Estimate*: $\mathbb{P}\big[|\Theta - \theta N| < \varepsilon N\big] > 1 - C\exp(-c\varepsilon^2 N)$
3. *Comparing $\mathbf{p}_1$ and $\mathbf{p}_2^{\text{aug}}$*: Stochastically bound $\mathbf{p}_1$ approximately above and approximately below by $\mathbf{p}_2^{\text{aug}}$
    1. Couple $\mathcal{E}$ and $\mathcal{E}_\Psi$ in two ways, such that $\mathbf{p}_1$ is (weakly) below $\mathbf{p}_2^{\text{aug}}$ under the first and $\mathbf{p}_2$ is (weakly) above $\mathbf{p}_2^{\text{aug}}$ under the second
    2. $\mathbb{P}\big[\operatorname{dist}(\mathbf{p}_1, \mathbf{p}_2) < \varepsilon N\big] > 1 - C\exp(-c\varepsilon^2 N)$



Boundary Data

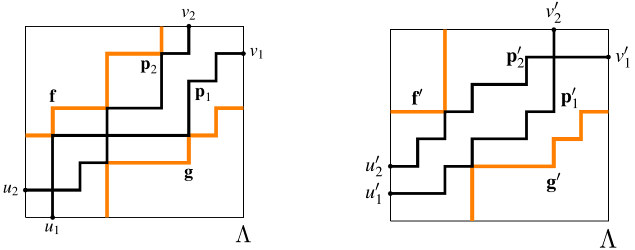
If X and Y are noncrossing paths/vertices, with X northwest of Y, we say $X \leq Y$.



- Rectangle Λ
- "Barrier paths" $\mathbf{f}$ and $\mathbf{g}$ with $\mathbf{f} \leq \mathbf{g}$
- "Entrance vertices" $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ with $u_1 \geq u_2 \geq \cdots \geq u_m$
- "Exit vertices" $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, with $v_1 \geq v_2 \geq \cdots \geq v_m$
- Let $\mathfrak{E}^{\mathbf{u};\mathbf{w}}_{\mathbf{f};\mathbf{g}}$ denote set of six-vertex ensembles on Λ whose paths $\mathbf{p}_1 \geq \mathbf{p}_2 \geq \cdots \mathbf{p}_m$ satisfy $\mathbf{f} \leq \mathbf{p}_i \leq \mathbf{g}$, such that $\mathbf{p}_i$ enters Λ through u_i and exits Λ through v_i

Monotone Couplings

- Assume boundary data $(\mathbf{f}, \mathbf{g}; \mathbf{u}, \mathbf{v})$ and $(\mathbf{f}', \mathbf{g}'; \mathbf{u}', \mathbf{v}')$ satisfy $\mathbf{f} \geq \mathbf{f}'$, $\mathbf{g} \geq \mathbf{g}'$, $\mathbf{u} \geq \mathbf{u}'$, $\mathbf{v} \geq \mathbf{v}'$
- Uniformly random ensembles $\mathcal{E}$ and $\mathcal{E}'$ in $\mathfrak{E} = \mathfrak{E}_{\mathbf{f};\mathbf{g}}^{\mathbf{u};\mathbf{w}}$ and $\mathfrak{E}' = \mathfrak{E}_{\mathbf{f}';\mathbf{g}'}^{\mathbf{u}';\mathbf{w}'}$, respectively
- Paths of $\mathcal{E}$ and $\mathcal{E}'$ are $\mathbf{p}_1 \geq \mathbf{p}_2 \geq \cdots \mathbf{p}_m$ and $\mathbf{p}'_1 \geq \mathbf{p}'_2 \geq \cdots \mathbf{p}'_m$, respectively



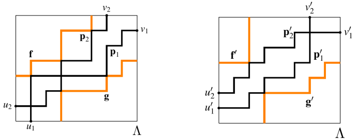
Lemma

The laws of $\mathcal{E}$ and $\mathcal{E}'$ can be coupled so that each $\mathbf{p}_i \geq \mathbf{p}'_i$, almost surely.

- Allows $\mathbf{f}, \mathbf{f}' = -\infty$ and/or $\mathbf{g}, \mathbf{g}' = \infty$
- For $\mathbf{f}, \mathbf{f}' = -\infty$ and $\mathbf{g}, \mathbf{g}' = \infty$, above lemma is equivalent to monotonicity for height function
- Proof uses monotone preserving property of Glauber dynamics (similar idea used by Corwin–Hammond, 2014)

Proof Outline for Monotonicity

There exist $\mathcal{E}(0) \in \mathfrak{E}$ and $\mathcal{E}'(0) \in \mathfrak{E}'$ with paths $\mathbf{p}_i(0)$ and $\mathbf{p}'_i(0)$, respectively, so that $\mathbf{p}_i(0) \geq \mathbf{p}'_i(0)$.



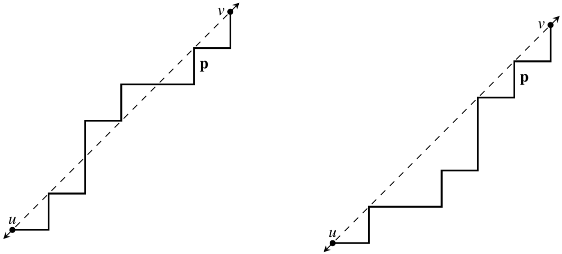
Run the **Glauber dynamics** on $\big(\mathcal{E}(0), \mathcal{E}'(0)\big)$

- Select a face F of Λ uniformly at random
- With probability $\frac{1}{2}$, perform the below "up-flip" (if possible) in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at F
- Otherwise perform the below "down-flip" in $\mathcal{E}(0)$ and $\mathcal{E}'(0)$ at F
- This produces new (random, coupled) six-vertex ensembles $\mathcal{E}(1) \in \mathfrak{E}$ and $\mathcal{E}'(1) \in \mathfrak{E}'$
- Repeating this, we obtain random, coupled $\mathcal{E}(1), \mathcal{E}(2), \ldots \in \mathfrak{E}$ and $\mathcal{E}'(1), \mathcal{E}'(2), \ldots \in \mathfrak{E}'$

- **Monotone preserving property**: If each $\mathbf{p}_i(t) \geq \mathbf{p}'_i(t)$, then each $\mathbf{p}_i(t+1) \geq \mathbf{p}'_i(t+1)$
- Then $\mathcal{E}(\infty) = \lim_{t\to\infty} \mathcal{E}(t)$ and $\mathcal{E}'(\infty) = \lim_{t\to\infty} \mathcal{E}'(t)$ are uniform on $\mathfrak{E}$ and $\mathfrak{E}'$, respectively, since the Glauber dynamics are stationary with respect to these uniform measures, and each $\mathbf{p}_i(\infty) \geq \mathbf{p}'_i(\infty)$ almost surely

Linearity Estimates

- Let $u, v \in \mathbb{Z}^2$, with v northeast of u, and set $\mathrm{dist}(u, v) = M$.
- Let $\ell = \ell(u, v)$ denote the line through u and v.



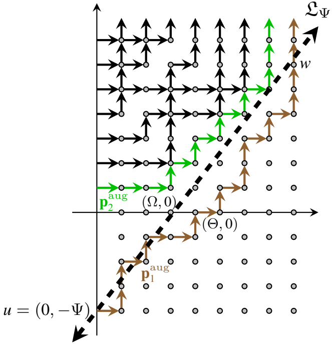
The following standard estimates state (possibly conditioned) random walks are nearly linear.

Lemma

1. *For a uniformly random path* $\boldsymbol{p}$ *from* u *to* v, $\mathbb{P}\big[\,\mathrm{dist}(\boldsymbol{p}, \ell) < \varepsilon M\,\big] > 1 - C\exp(-c\varepsilon^2 M)$.
2. *For a uniformly random path* $\boldsymbol{p}$ *from* u *to* v *conditioned to lie weakly below (or above)* ℓ, $\mathbb{P}\big[\,\mathrm{dist}(\boldsymbol{p}, \ell) < \varepsilon M\,\big] > 1 - C\exp(-c\varepsilon^2 M)$.

Proof of $\Theta \approx \Omega$

Set $u = (0, -\Psi)$, and let w be the first vertex in $\mathbf{p}_1^{\text{aug}}$ above the x-axis such that w is (weakly) below $\mathfrak{L}_\Psi$ but the next vertex in $\mathbf{p}_1^{\text{aug}}$ is not.



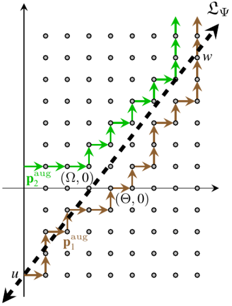
We condition on the following.

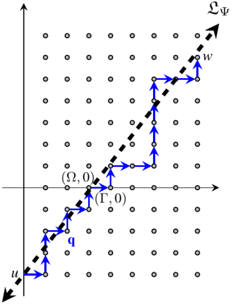
- The paths $\mathbf{p}_2^{\text{aug}}, \mathbf{p}_3^{\text{aug}}, \dots, \mathbf{p}_{N+1}^{\text{aug}}$
- The event that $\mathbf{p}_1^{\text{aug}}$ passes through w, and the part of $\mathbf{p}_1^{\text{aug}}$ northeast of w

Gibbs property: The law of $\mathbf{p}_1^{\text{aug}}$ southwest of w is given by a uniformly random path from u to w, conditioned to remain weakly below $\mathbf{p}_2^{\text{aug}}$.

Proof of $\Theta \approx \Omega$

Gibbs property: The law of $\mathbf{p}_1^{\text{aug}}$ is given by a uniformly random path in $\mathfrak{E}^{u;w}_{\mathbf{p}_2^{\text{aug}};\infty}$.

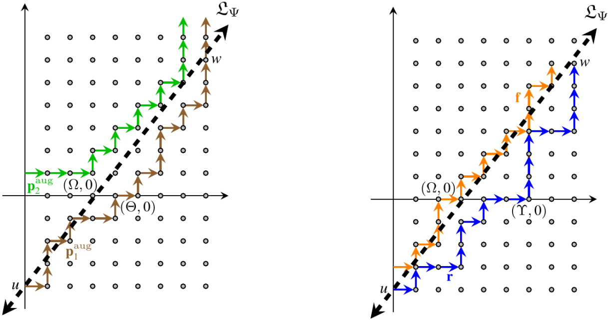




- Let $\mathbf{q}$ be a uniformly random path in $\mathfrak{E}^{u;w}_{-\infty,\infty}$ (from u to w without barriers)
- By the linearity estimate, $\mathbf{q}$ is εN-linear with probability $1 - C\exp(-c\varepsilon^2 N)$
- So, if $\mathbf{q}$ exits the x-axis at $(\Gamma, 0)$, then $\mathbb{P}\big[|\Gamma - \Omega| < \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$
- By monotonicity, we may couple $\mathbf{p}_1^{\text{aug}}$ and $\mathbf{q}$ so that $\mathbf{p}_1^{\text{aug}} \geq \mathbf{q}$ almost surely
- Thus, $\mathbb{P}\big[\Theta \geq \Omega - \varepsilon N\big] \geq \mathbb{P}\big[\Gamma \geq \Omega - \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$

Proof of $\Theta \approx \Omega$

Gibbs property: The law of $\mathbf{p}_1^{\text{aug}}$ is given by a uniformly random path in $\mathfrak{E}^{u;w}_{\mathbf{p}_2^{\text{aug}};\infty}$.



- Let $\mathbf{r}$ be a uniformly random path from u to v, conditioned to lie weakly below $\mathfrak{L}_\Psi$ (so it is uniform on $\mathfrak{E}^{u;w}_{\mathbf{f},\infty}$, for some $\mathbf{f} \geq \mathbf{p}_2^{\text{aug}}$)
- By the linearity estimate, $\mathbf{r}$ is εN-linear with probability $1 - C\exp(-c\varepsilon^2 N)$
- So, if $\mathbf{r}$ exits the x-axis at $(\Upsilon, 0)$, then $\mathbb{P}\big[|\Upsilon - \Omega| < \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$
- By monotonicity, we may couple $\mathbf{p}_1^{\text{aug}}$ and $\mathbf{r}$ so that $\mathbf{p}_1^{\text{aug}} \leq \mathbf{r}$ almost surely
- Thus, $\mathbb{P}\big[\Theta \leq \Omega + \varepsilon N\big] \geq \mathbb{P}\big[\Upsilon \leq \Omega + \varepsilon N\big] \geq 1 - C\exp(-c\varepsilon^2 N)$