Bounds and inequalities for Littlewood–Richardson coefficients

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Joint with Igor Pak and Greta Panova

LR coefficients $c_{\mu\nu}^{\lambda}$

e.g. via Schur polynomials $s_\lambda(x_1, \ldots, x_n) := \det[x_i^{\lambda_j + n - j}] / \prod_{i<j}(x_i - x_j)$

$$s_\mu \cdot s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} \, s_\lambda \qquad |\lambda| = |\mu| + |\nu|$$

Many interpretations: combinatorial, geometric, representation-theoretic.

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We're interested in **large** $c^{\lambda}_{\mu\nu}$

Dimensions

$$f^\lambda = \dim \mathbb{S}^\lambda = \#\text{SYT shape } \lambda \text{ i.e. chains } \varnothing \to \lambda \text{ in Young's lattice}$$

$$= \frac{n!}{\prod_{\square \in \lambda} \text{hook}_\square} \quad \text{(hook-length formula)}$$

$$f^{321} = \frac{6!}{1{\cdot}1{\cdot}1{\cdot}3{\cdot}3{\cdot}5} = 8$$

1	3	6
2	4	
5		

(vague) meta message

Large LR $c_{\mu\nu}^{\lambda}$ <—∼—∼—∼—∼—∼—> **Large dim** f^{λ}

Max LR and dim

$$C(n,k) = \max_{\lambda \vdash n \ \mu \vdash k \ \nu \vdash n-k} c^{\lambda}_{\mu\nu} \qquad C(n) = \max_{k} C(n,k)$$

$$D(n) = \max_{\lambda \vdash n} f^{\lambda}$$

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k	1	2	3	4	5	6	7	8	9	10	11
$D(k)$	1	1	2	3	6	16	35	90	216	768	2310

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Theorem (Pak–Panova-Y. 2019)

- *stability:* $C(n,k) = D(k)$ *for* $n \geq \binom{k+1}{2}$
- *monotonicity:* $C(n,k) \leq C(n+1,k)$ *and* $C(n) \leq C(n+1)$

quick plan

1) asymptotics of $D(n)$

2) asymtotitcs of $C(n)$

Largest dimensions

(Old) Problem: The asymptotics of $D(n)$

Bivins–Metropolis–Stein–Wells '54, Baer–Brock '68, McKay '76, Rasala '77

Largest dimension

Burnside identity:

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n! \implies \frac{\sqrt{n!}}{p(n)} \leq D(n) < \sqrt{n!}$$

$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} = \#$ partitions of n

early wrong conjectures: $D(n) \geq \sqrt{n!}/poly(n)$

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$$\sqrt{n!}\ e^{-1.29\sqrt{n}} \le D(n) \le \sqrt{n!}\ e^{-0.11\sqrt{n}}$$

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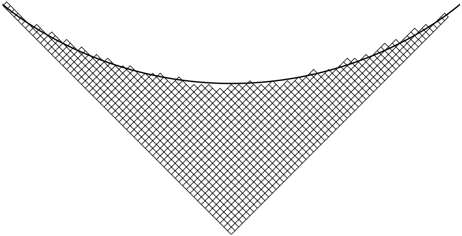
(Q) What partitions attain max dimensions?

Partitions for largest & typical dimensions

Vershik-Kerov–Logan-Shepp (VKLS) limit shape[1]

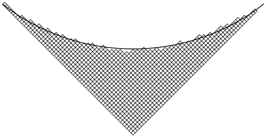


$$[\lambda] \to \Omega(x) = \frac{2}{\pi}\left(x \arcsin(x/2) + \sqrt{4 - x^2}\right) \quad x \in [-2, 2]$$

$$\sup_t |[\lambda](t)/\sqrt{n} - \Omega(t)| < c/n^{1/6}$$

[1] pic from Romik's book; partition sampled from the Plancherel measure $\frac{(f^\lambda)^2}{n!}$

Partitions attaining largest dimensions



Partitions sequence $\lambda^{(n)} \vdash n$ is **Plancherel** if

$$f^{\lambda^{(n)}} \geq \sqrt{n!}\ e^{-c\sqrt{n}}$$

Theorem (Logan-Shepp 1977, Vershik-Kerov 1985)

Every Plancherel sequence has VKLS limit shape.

note: $f^{\lambda} = \sqrt{n!}e^{o(n)}$ is enough for VKLS shape

related: solution to Ulam's problem on *longest increasing subsequences*, $\lambda_1 \sim 2\sqrt{n}$.

Stanley's problem

Theorem (Stanley 2015)

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Problem (Stanley)

What partitions λ, μ, ν attain the maximum?

Max LR

Theorem (Pak-Panova-Y 2019)

$$\binom{n}{k}^{1/2} e^{-d\sqrt{n}} \leq C(n,k) \leq \binom{n}{k}^{1/2}$$

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In fact,

$$\sum_{\lambda \vdash n} (c^{\lambda}_{\mu\nu})^2 \leq \binom{n}{k} \qquad \sum_{\mu \vdash k, \nu \vdash n-k} (c^{\lambda}_{\mu\nu})^2 \leq \binom{n}{k}$$

$$\sum_{\lambda \vdash n, \mu \vdash k, \nu \vdash n-k} (c^{\lambda}_{\mu\nu})^2 \geq \binom{n}{k}$$

Asympt. largest LR attained on Plancherel seq

Theorem (Pak-Panova-Y 2019)

(i) $\forall$ *Plancherel* $\lambda \vdash n$ $\exists$ *Plancherel* $\mu \vdash k = n\theta$*,* $\nu \vdash n(1-\theta)$*,* $\theta \in (0,1)$*:*

$$c^{\lambda}_{\mu\nu} = \binom{n}{k}^{1/2} e^{-O(\sqrt{n})}$$

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(iii) $\forall$ *Plancherel* λ, μ $\exists$ ν *with VKLS limit shape:*

$$f^{\nu} = \sqrt{(n-k)!}\; e^{-O(n^{2/3}\log n)} \qquad c^{\lambda}_{\mu\nu} = \binom{n}{k}^{1/2} e^{-O(n^{2/3}\log n)}$$

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Proof ideas: Estimates from the identities

$$\sum_{\lambda \vdash n} c^{\lambda}_{\mu\nu} f^{\lambda} = \binom{n}{k} f^{\mu} f^{\nu} \qquad \sum_{\mu \vdash k, \nu \vdash n-k} c^{\lambda}_{\mu\nu} f^{\mu} f^{\nu} = f^{\lambda}$$

For (iii), skew SYT $f^{\mu/\nu}$ new bounds + properties of VKLS shape.

Large dim doesn't imply large LR

Theorem (Pak-Panova-Y 2019)

$\mu, \nu \vdash n/2$ *Plancherel* $\exists\, \lambda$ *with VKLS limit shape*

$$f^\lambda = \sqrt{n!}\; e^{O(\sqrt{n}\log n)} \quad \& \quad c^\lambda_{\mu\nu} = 0.$$

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Conjecture. $\exists$ Plancherel λ, μ, ν

$$\frac{1}{\sqrt{n}}\left(\frac{n}{2} - \log_2 c^\lambda_{\mu\nu}\right) \to \infty$$

Large LR implies (relatively) large dim

Theorem (Pak-Panova-Y 2019)

Let $\lambda \vdash n, \ \mu, \nu \vdash n/2$ *with*

$$c^{\lambda}_{\mu\nu} = \binom{n}{n/2}^{1/2} e^{-O(n/\log n)}$$

$$\Longrightarrow \ f^{\lambda} = \sqrt{n!}\ e^{-O(n)}, \quad f^{\mu}, f^{\nu} = \sqrt{(n/2)!}\ e^{-O(n)}$$

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$$\Longrightarrow \;\; f^{\lambda} = \sqrt{n!}\; e^{-O(n)}, \quad f^{\mu}, f^{\nu} = \sqrt{(n/2)!}\; e^{-O(n)}$$

Proof ideas:

$$\sum_{\lambda \vdash n} (c^{\lambda}_{\mu\nu})^2 = \sum_{\alpha,\beta,\gamma,\delta} c^{\mu}_{\alpha\gamma} c^{\mu}_{\alpha\delta} c^{\nu}_{\beta\gamma} c^{\nu}_{\beta\delta} \quad \text{(from skew Cauchy)}$$

$$\max_{\lambda} c^{\lambda}_{\mu\nu} \le e^{a\sqrt{n}} \max_{\alpha,\beta} c^{\mu}_{\alpha\beta} \max_{\alpha,\beta} c^{\nu}_{\alpha\beta} \qquad\qquad f^{\lambda} \ge e^{-un} (c^{\lambda}_{\mu\nu})^{\log_2 n}$$

Max LR with few rows

$$C_\ell(n) := \max_{\lambda \vdash n,\ \ell(\lambda)=\ell,\ \mu,\nu} c^\lambda_{\mu\nu}$$

Theorem (Pak-Panova-Y. 2019)

$$n^{\ell^2/2 - a\ell} e^{-b\ell^2 \log \ell} \le C_\ell(n) \le (n+1)^{\ell^2/2}$$

Proof ideas: Knutson-Tao interpretations, Schur polynomials bounds.

Corollary

$$\log C_\ell(n) \sim \frac{1}{2}\ell^2 \log n, \qquad \ell = O(\sqrt{n}/\log n)$$

Containment of max LR

Theorem (Lam-Postnikov-Pylyavskyy 2007)

$$c^{\lambda}_{\mu\nu} \leq c^{\lambda}_{\mu\cup\nu,\mu\cap\nu}$$

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$$\forall n \quad \exists\, \mu \subseteq \nu \subseteq \lambda \quad c^{\lambda}_{\mu\nu} = C(n)$$

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$$\forall n \quad \exists\, \mu \subseteq \nu \subseteq \lambda \quad c^{\lambda}_{\mu\nu} = C(n)$$

Conjecture [PPY]

$$c^{\lambda}_{\mu\nu} = C(n) \implies \mu \subseteq \nu \subseteq \lambda$$

Remark: $C(n,k)$ for $k = 1, \ldots, n$ is symmetric but *not* unimodal, otherwise $\mu = \nu$

Some conjectures, questions

- $C(n) < 2^{n/2} e^{-a\sqrt{n}}$ or even $C(n, \theta n) < \binom{n}{\theta n} e^{-a\sqrt{n}}$

$$C(20, 7) = 11 < \sqrt{\binom{20}{7}} \approx 278.42$$

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- stronger version: $\lambda \vdash n$, $\mu, \nu \vdash n/2$

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- LR bounds for other limit shapes

Rahmet!

Thank you!