Weak vorticity formulation for incompressible 2D Euler in domains with boundary

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OBS. Actually, iff.

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- Parallel pipe flow in a (periodic) pipe – let (x, r, ϕ) be cylindrical coordinates. Then $\mathbf{u} = u_\phi(r, t)\mathbf{e}_\phi + u_x(\phi, r, t)\mathbf{e_x}$, on the domain $\Omega = \{(x, y, z) \mid y^2 + z^2 < R,\ 0 < x < L\}$.

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Mechanism to generate small scales: ∂ layer + Kelvin-Helmholtz instability

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Seek **framework** for *weak solutions* of 2D Euler, in domains with (rigid) boundary, vortex sheet regularity, **which allow tracking vorticity dynamics**.

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$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p, & \text{in } \Omega \times (0, \infty) \\ \text{div } u = 0, & \text{in } \Omega \times [0, \infty) \\ u(x,t) \cdot \hat{n}(x) = 0, & \text{on } \partial\Omega \times [0, \infty) \\ u(x,0) = u_0(x), & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{1}$$

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$\mathcal{BM}(\Omega)$ are bounded Radon measures on Ω.

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If u is a weak solution then possible to recover pressure
$p \in L^\infty_{loc}((0,+\infty); \mathcal{D}'(\Omega))$.

Theorem (J.-M. Delort, JAMS, 1991)

Let $u_0 \in L^2_\sigma(\Omega)$ *be such that* $\omega_0 = \operatorname{curl} u_0 \in \mathcal{BM}_+(\Omega)$. *Then there exists (at least one!) weak solution*
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No tracking vortex dynamics or “conserved quantities".

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$$+ \int_{\mathbb{R}^2} \varphi(0,x)\omega_0(x)\,dx = 0,$$

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Term with $\int_0^\infty \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}$ comes from substituting $u = \nabla^\perp \Delta^{-1}\omega$ in nonlinear term and symmetrizing the kernel.

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Question: what survives in (6) for flows with vortex sheet regularity?

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Lemma

Let $u \in L^1_{loc}(\Omega)$ such that $\omega = \operatorname{curl} u \in \mathcal{BM}(\Omega)$, bounded measure. Then the circulation of u around $\partial\Omega$ is well-defined through the formula:

$$\int \varphi\,\omega + \int u \cdot \nabla^{\perp}\varphi = \gamma\varphi|_{\partial\Omega},$$

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Use Delort's argument with insight from "Schochet-proof". Ingredients:

- H_φ continuous off diagonal and bounded everywhere;

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Let $u \in L^2_\sigma(\Omega)$ be such that curl $u = \omega \in \mathcal{BM}(\Omega)$ and let γ be circulation of u around $\partial\Omega$. Then, if $\Phi = \nabla^\perp \varphi$ with $\varphi \in C^\infty(\Omega)$ and $\Phi \in C^\infty_c(\Omega)$, then

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- ω is the curl in the sense of *distributions*, hence no mass at boundary.

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Test functions for weak vorticity: φ such that $\nabla\varphi$ is compactly supported in space and time. I.e., φ constant *in neighborhood* of $\partial\Omega$.

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In Lopes-Filho-NL-Xin established existence of boundary coupled weak solution for half-plane. How? No mass going towards boundary (needed new *a priori* estimate).

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This *cannot* be controlled/excluded by *a priori* estimates!

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Similarly for torque:

$$\int_{\Gamma_j} p(x - \bar{x}_j)^\perp \cdot \hat{n}\, dS;$$

where $\bar{x}_j$ is the center of mass.

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