Interval complexes, linear resolutions, and spaces of homomorphisms

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The setup - Ideals and Resolutions

Fix a field $\mathbb{K}$ and let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ denote the polynomial ring. Let $I \subset R$ be a homogenous ideal with quotient ring R/I.

- Recall that a *free resolution* of I is a long exact sequence

$$\mathcal{F} = 0 \leftarrow I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0.$$

  where each $F_i = \oplus_j R(-j)^{\beta_{i,j}}$

- The resolution is *minimal* if the $\beta_{i,j}$ are minimal among all resolutions, in which case $\beta_{i,j} = \operatorname{Tor}_i^R(I, \mathbb{K})_j$ are called the *(graded) Betti numbers* of I.

- Replace last map with $0 \leftarrow R/I \leftarrow R \leftarrow F_0$ to get a resolution of R/I.

Constructing a resolution

A free resolution of I:

$$\mathcal{F} : 0 \leftarrow I \xleftarrow[\partial_0]{} F_0 \xleftarrow[\partial_1]{} F_1 \leftarrow \cdots \leftarrow F_d \leftarrow 0.$$

where each $F_i = \oplus_j R(-j)^{\beta_{i,j}}$

How to construct $\mathcal{F}$:

- If $I = \langle f_1, f_2, \ldots, f_j \rangle$, say all of degree d, let $F_0 = R(-d)^j$ and a map $\partial_0 : F_0 \to I$.
- The module $\ker(\partial_0)$ is the (first) syzygy module, say it has generators $\{g_1, g_2, \ldots, g_k\}$. Let $F_1 = R^k$ and map $\partial_1 : F_1 \to F_0$.
- Continue this process. The *Hilbert Syzygy Theorem* says that eventually (at most by the nth step) we'll get an injective map. That's your resolution!

Combinatorial invariants

- The Betti numbers of I can be collected in its *Betti table*, (j, i)-entry given by $\beta_{i,i+j}$.

$$0 \leftarrow I \leftarrow R[-2]^8 \leftarrow R[-3]^{13} \oplus R[-4] \leftarrow R[-5]^7 \oplus R[-6]^2 \leftarrow R[-5] \oplus R[-6] \leftarrow 0$$

```
            0 1  2 3 4
total:      1 8 14 9 2
    0:      1 .  . . .
    1:      . 8 13 7 1
    2:      . .  1 2 1
```

- The *projective dimension* and *(Castelnuovo–Mumford) regularity* are:

$$\operatorname{pdim}(I) = \max\{i : \beta_{i,i+j} \neq 0 \text{ for some } j\}$$

$$\operatorname{reg}(I) = \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}$$

- The ideal I has a *linear resolution* if all entries in the matrices representing the boundary maps are linear forms.

Resolutions - who cares?

The minimal free resolution of a graded R-module M is a way to extract algebraic/geometric information about M.

- A way to study the *Hilbert function* of M, dimension of the dth graded component.
- Coefficents of the Hilbert polynomial are determined by the graded Betti numbers $\beta_{i,j}$.
- Minimal free resolutions detect *depth*, Cohen-Macaulay properties, etc. (via Auslander-Buchsbaum):
$$\operatorname{pdim}(M) + \operatorname{depth}(M) = n$$
- A measure of the *complexity* of the ideal.

Monomial is good enough

Gröbner theory and the monotonicity theorem allow us to focus on *monomial* ideals.

- By example: If $I = \langle x^2 - y, x^3 - x \rangle$,
  $G = \langle x^2 - y, x^3 - x, y^2 - y \rangle$,
  $\mathrm{in}_< I = \langle x^2, x^3, y^2 \rangle$.
- Monotonicity of Betti numbers:

  $$\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\mathrm{in}_< I)$$

  with equality in many interesting cases!
- Gröbner degeneration also detects desirable algebraic properties (eg Cohen-Macaulayness).
- If I has a square-free initial ideal $\mathrm{in}(I)$, then the *extremal* Betti numbers of I and $\mathrm{in}(I)$ are the same [Conca, Varbaro].

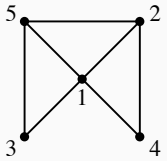
Facet ideals

We will consider monomials defined by (pure) simplicial complexes.

- Suppose Δ is a pure d-dimensional simplicial complex on vertex set $[n] = 1, 2, \ldots, n$. From this we construct the *facet ideal* in $\mathbb{K}$ generated by degree d monomials:

$$I_\Delta = \langle x_{i_0} x_{i_1} \cdot \cdots \cdot x_{i_d} : (i_0, i_1, \ldots, i_d) \in \Delta \rangle$$

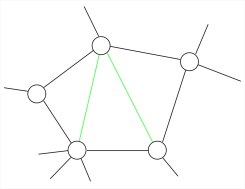
- Example ($d = 1$, $n = 5$)



$$I_\Delta = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5 \rangle.$$

Edge ideals and linear resolutions

- For $d = 1$ we have that Δ is a graph and these are often called *edge ideals*.
- Recall that a graph G is *chordal* if G has no induced cycles of length ≥ 4.



- For a graph G on vertex set $[n]$, the complement is the graph $\overline{G}$ with edges given by all missing 2-sets.

Theorem (Fröberg)

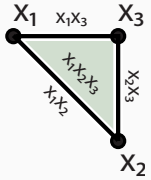
The edge ideal I_G has a linear resolution if and only if $\overline{G}$ is a chordal graph.

- Considerable efforts to generalize to d-dimensional simplicial complexes.

Cellular resolutions

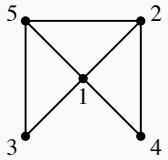
- Explicit minimal resolutions are known for some classes of ideals (Eliahou-Kervaire, Eagon-Northcott), but they are often cumbersome to work with, or at least to 'picture'.
- Often we can encode syzygy modules and differential maps in face structure of polyhedral (CW-) complex, via the chain complex that computes cellular (co)homology.
- The resulting *cellular resolutions* were introduced by Bayer and Sturmfels in their study of the *Scarf complex* of a generic ideal.
- For example the simplex supports a resolution of the ideal $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$.

$$0 \leftarrow \mathfrak{m} \leftarrow R[-1]^3 \leftarrow R[-2]^3 \leftarrow R[-3] \leftarrow 0.$$



Resolutions of facet ideals

- Suppose Δ is a d-dimensional simplicial complex on vertex set $[n]$ and let I_Δ denote its facet idea (equivalently, I_Δ is squarefree monomial and equigenerated in degree $d+1$).
- Example (again): Let $d = 1$, $n = 5$, and consider the graph G:



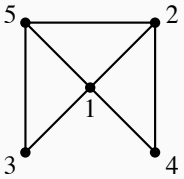
$$I_G = \langle x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5 \rangle.$$

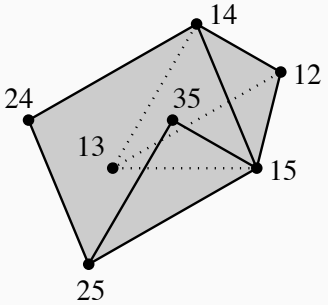
- In what follows we write a facet of Δ as $F = (v_0 v_1 \cdots v_d)$ where $v_0 < v_1 < \cdots < v_d$.
- The facets of Δ are generators of an ideal, we look for a 'space of (directed) facets' to describe syzygies of this ideal.

A space of facets

Suppose Δ is a d-dim complex on vertex set $[n]$. Then X_Δ is the polyhedral complex (subcomplex of $\prod_{i=1}^{d} \Delta_{n-1}$) satisfying:

- The vertices of X_Δ are $(v_0 v_1 \cdots v_d)$, where $(v_0 v_1 \cdots v_d) \in \Delta$.
- The higher dimensional cells are of the form $\sigma_0 \times \sigma_1 \times \cdots \times \sigma_d$, where $\sigma_i \subset [n]$ and $\sigma_0 < \sigma_1 < \cdots < \sigma_d$, satisfying ...

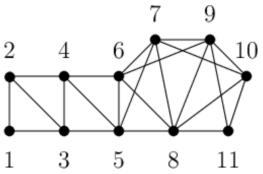


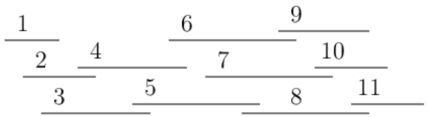


- Label each face by the LCM of the monomials labels of the vertices it contains.

Interval graphs

- The construction of X_Δ construction makes sense for any (pure) simplicial complex Δ. When does it support a resolution of I_Δ?
- A graph G is *interval* if its vertices can be represented as intervals in real line $I_1, \ldots, I_n$ such that $i \sim j$ if and only if $I_1 \cap I_2 \neq \emptyset$.





- Interval graphs are chordal.
- Label the left endpoint of each interval moving L to R, then G is interval $\Leftrightarrow$

  If $e = ab$ is an edge (for $a < b$), we have ac is an edge for all $a < c < b$.

Interval complexes

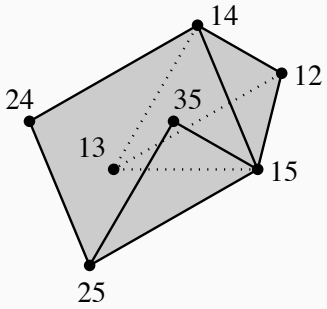
This inspires the following definition. Suppose Δ is a pure d-dimensional simplicial complex on vertex set $[n]$.

- Δ is *interval* if there exists a labeling of it vertex set such that for any facet $F = a_0 a_1 \cdots a_d$, we have that $a_0 i_1 i_2 \cdots i_d \in \Delta$, for all $i_1 \leq a_1, i_2 \leq a_2, \ldots, i_d \leq a_d$.
- For example $\Delta = \{1234, 1235, 2345, 2346, 2356, 2456\}$ are the facets of a 3-dim interval complex on vertex set $[6]$.
- Strictly includes the class of *shifted* complexes.
- Such complexes were used (and defined this way) by Benedetti, Seccia, and Varbaro: the determinantal facet ideals of such complexes have square-free initial ideals.

A minimal resolution

Theorem (D-, Engström, 2012)

Suppose Δ is a d-dimensional interval complex, and let $\overline{\Delta}$ denote its complement. Then the complex $X_{\overline{\Delta}}$ supports a minimal resolution of the facet ideal $I_{\overline{\Delta}}$.



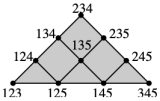
- Betti numbers of I_G are given by the f-vector of $X_{\overline{\Delta}}$.

$$0 \leftarrow I \leftarrow R[-2]^7 \leftarrow R[-3]^{11} \leftarrow R[-4]^6 \leftarrow R[-5] \leftarrow 0$$

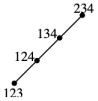
Linear resolutions and chordal complexes

- Our construction generalizes the *box of complexes* resolution of Nagel and Reiner (they considered shifted complexes).
- Since the dimension of a face of X_Γ can be read off from the monomial label, we conclude that I_Γ has a *linear* resolution when Γ is the complement of an interval complex.
- Recall that interval graphs are chordal, and hence this provides a generalization/specialization of Fröberg's Theorem.
- There exist chordal graphs G such that $\Delta_{\overline{G}}$ does *not* support a resolution of $I_{\overline{G}}$.
- Question: How do interval complexes relate to other notions of *chordal* simplicial complexes?

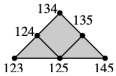
Examples $d = 2$



Graph 2



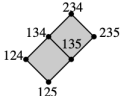
Graph 3



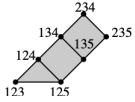
Graph 4

Graph 5

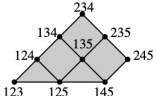
Graph 6



Graph 7

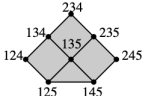


Graph 8



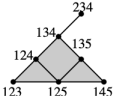
Graph 9

Graph 10



Graph 11

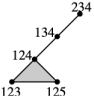
Graph 12



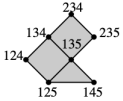
Graph 13

Graph 14

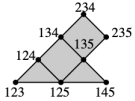
Graph 15



Graph 16



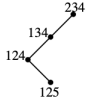
Graph 17



Graph 18

Graph 19

Graph 20

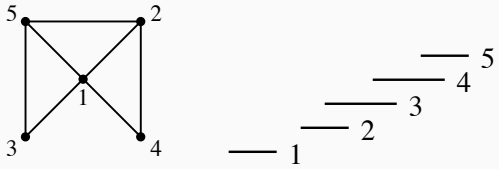


Graph 21

Cointerval complexes

In our old paper, we gave a definition of a *cointerval* complex.

- The definition is recursive, we define d-dim cointerval complexes in terms of $(d-1)$-dim cointerval complexes. Helpful for our proofs.
- Base case of $d = 1$ corresponds to the *complements of interval* graphs: vertices given by intervals, adjacent if *non*-overlapping.



Proposition (D-, Goeckner, Pavelka)

A complex Δ is interval if and only if $\overline{\Delta}$ is cointerval.

Tools - topology of posets

How do we prove that the labeled complex supports a resolution?

- To prove that $X = X_{\overline{\Delta}}$ supports a minimal resolution of the edge ideal I_G we must study the topology of certain induced subcomplexes $X_{\leq \alpha}$.
- Here $X_{\leq \alpha}$ denotes the sub complex of X generated by all vertices (=monomials) which divide α.
- In the case of $X_{\cdot}$ we can utilize tools from *poset topology* and Quillen type fiber lemmas.

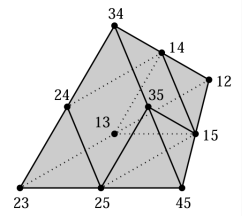
A nice embedding

Recall that X_Δ is defined as a subcomplex of $\prod_{i=1}^{d} \Delta_{n-1}$ - that's big!

Turns out we can do better.

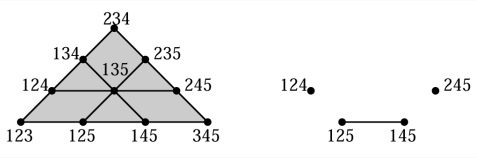
- Let $\Delta_{d,n}$ denote the d-skeleton of the simplex on n vertices.

Proposition (D-, Engström)
The ideal $I_{\Delta_{d,n}}$ has a minimal resolution supported on a certain fine mixed subdivision of the dilated simplex $(d+1)\Delta_{n-d}$.



A nice embedding

- As a corollary, for any cointerval complex Δ, the ideal I_Δ has minimal cellular resolution supported on a subcomplex of $(d+1)\Delta_{n-d}$.
- We use the mixed subdivision corresponding to the *staircase triangulation* of $\Delta_d \times \Delta_{n-d}$.
- Turns out not all regular mixed subdivisions work!



- Side note: *any* regular mixed subdivision (of certain classes of generalized permutohedra) can be used to produce cellular resolutions (of the ideal generated by the 0-cells, eg $\mathfrak{m}^k$).

Related work

Can think of our space of facets as moduli space of homomorphisms $E \to \Delta$. Other classes of ideals are defined similarly:

- Ideals defined by nondegenerate simplicial maps [Braun, Browder, Klee]
- Ideals of poset homomorphisms [Flöystad, Greve, Herzog, Juhnke-Kubitzke, Katthän, Madani]
- Notion of *path ideals* of a graph, as a generalization of an edge ideal (cellular resolutions for certain classes described by [Chau, Kara, Wang])

Homomorphism complexes of digraphs studied more recently with Anurag Singh:

- If T_n is the acyclic tournament, then $\mathrm{Hom}(G, T_n)$ is contractible for *any* digraph G.

Further thoughts

- Does X_Δ support resolutions for a larger class of complexes? Can we characterize the class that works?
- Can use an interval decomposition of a graph/complex G to produce (non-minimal) cellular resolutions of I_G?
- Is the Alexander dual (appropriately defined) of an interval complex shellable? Vertex decomposable?
- Can we get cellular resolutions by considering other *homomorphism* complexes?
- How do interval complexes relate to other notions of *chordal* simplicial complexes?
- Interval graphs (and chordal graphs) can be recognized in polynomial time. Is the same true for interval complexes?

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Thanks for listening!