Lipschitz regularity of solutions of nonlinear elliptic integro-differential equations

joint work with Barles, Chasseigne, Ciomaga

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Nonlocal PDEs, Variational Problems and their Applications
Equations with composed/mixed ellipticity

A “composed” elliptic integro-differential equation

\[
\begin{cases}
\Lambda_1(x)(-\Delta)u + (1 - \Lambda_1(x))(-\Delta)^{\beta/2} u + f(x) = 0 \\
(-\Delta)^{\beta/2} = \text{fractional Laplacian} \\
0 \leq \Lambda_1(x) \leq 1, \text{Hölder continuous}
\end{cases}
\]

A “mixed” elliptic integro-differential equation

\[
\begin{cases}
(-\Delta_{x_1})u + (-\Delta_{x_2})^{\beta/2} u = f(x_1, x_2) \\
(-\Delta_{x_2})^{\beta/2} = \text{partial fractional Laplacian}
\end{cases}
\]

Question: are solutions Hölder/Lipschitz continuous in \( \mathbb{R}^d \)?
Equations with composed/mixed ellipticity

A “composed” elliptic integro-differential equation

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\Lambda_1(x)(-\Delta)u + (1 - \Lambda_1(x))(-\Delta)^{\frac{\beta}{2}}u + f(x) = 0 & \text{in } \mathbb{R}^d \\
(-\Delta)^{\frac{\beta}{2}} = \text{fractional Laplacian} \\
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Question: are solutions Hölder/Lipschitz continuous in \( \mathbb{R}^d \)?

Generalization to non-linear versions of these standing examples.
Outline of the talk

1. Motivations
2. Main results
3. The Ishii-Lions method
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1. Motivations
2. Main results
3. The Ishii-Lions method
Fractional Laplacian ($\beta \in (0, 2)$)

- Fourier multiplier: $(-\Delta)^{\frac{\beta}{2}} u = \mathcal{F}^{-1}(|\xi|^{\beta} \mathcal{F} u)$
- Singular integral: $(-\Delta)^{\frac{\beta}{2}} u = -c \int [u(x + z) - u(x)] \frac{dz}{|z|^{d+\beta}}$
- “Dirichlet-to-Neumann” formula

Properties

- Regularizing effect
- Positive max principle: $u(x) = \max u \Rightarrow (-\Delta)^{\frac{\beta}{2}} u(x) \geq 0$. 
“Nice” singular integral operators

Lévy measure $\mu$

\[
\int \min(1, |z|^2) \mu(dz) < +\infty
\]

Lévy operators

\[
L[u](x) = - \int [u(x + z) - u(x)] \mu_x(dz)
\]

with $\forall x, \mu_x = \text{Lévy measure}$

Lévy-Itô operators

\[
L_{LI}[u](x) = - \int [u(x + j(x, z)) - u(x)] \mu(dz)
\]

with $\mu = \text{Lévy measure}$
Lévy processes

Stochastic processes with stationary and independent increments

\[ X_t = \text{drift} + \text{diffusion} + \text{jumps} \]

Infinitesimal generators

\[ Lu = b \cdot Du + AD^2u + L[u] \]

\( \begin{align*}
\text{drift} & \\
\text{diffusion} & \\
\text{jumps} & 
\end{align*} \)
Stochastic control

Control of a SDE

\[ dX_t = b(X(t), \gamma(t))dt + \sigma(X(t), \gamma(t))dW_t + \int j(X(t), \gamma(t), z) \tilde{N}(dt, dz) \]

- \text{drift}
- \text{diffusion}
- \text{jumps}

Cost functional

\[ J(x, \gamma(\cdot)) = \mathbb{E} \left[ \int_0^\infty e^{-\lambda t} f(X(t), \gamma(t)) \, dt \right] . \]

Value function

\[ u(x) = \inf_{\gamma(\cdot)} J(x, \gamma(\cdot)) . \]
Bellman equation

General form

$$\sup_{\gamma} \left\{ L_{\gamma}^\gamma[u] - \text{Tr}(A_{\gamma} D^2 u) - b_{\gamma} \cdot Du - f_{\gamma} \right\} + \lambda u = 0$$

with $A_{\gamma} = \frac{1}{2} \sigma_{\gamma} \sigma_{\gamma}^T$

Example

$$(-\Delta)^{\frac{\beta}{2}} u + b(x)|Du|^{k+\tau} + |Du|^r + \lambda u = f(x)$$

Elliptic nonlinear PIDE
Partial Integro-Differential Equations (PIDE)

Wide range of applications

- Finance
- Dislocations
- Hydraulic fractures
- Combustion
- Fluid dynamics
- Life sciences
- Image
- Statistical Physics
- ...

(cf. Monneau’s talk)
(cf. Mellet’s talk)
(cf. Kiselev’s talk)
(cf. Bertozzi’s, Carillo’s, Gonzalez’s and Topaz’s talks)
(cf. Guidotti’s and Osher’s talks)
(cf. Lebowitz’s talk)
Regularity for PIDE

Linear equations (probability)
- Bass, Kassmann, Levin, Song, Vondracek...

Viscosity solutions
- Sayah, Jakobsen, Karlsen, CI, Barles, Chasseigne, Ciomaga, CI...

Fully non-linear elliptic equations
- Caffarelli, Silvestre ...

Quasi-geostrophic equation
- Caffarelli, Vasseur, Kiselev, Nazarov, Volberg, Silvestre, Dabkowski, Lemarié-Rieusset ...

Fractional Burgers
- Biler, Funaki, Karch, Woyczynski, Alibaud, Droniou, Vovelle, CI, Kiselev, Nazarov, Chan, Czubak, Silvestre, Du, Dong, Li ...

“Under divergence form”
- Komatsu, Kassmann, Barlow, Bass, Chen, Caffarelli, Chan, Vasseur...
Outline of the talk

1. Motivations
2. Main results
3. The Ishii-Lions method
Main result

Recall equations

**Theorem**

Assume \( \Lambda_1 \) is Hölder continuous. Solutions of Equations (1) and (2) are Hölder continuous if \( \beta \leq 1 \) and Lipschitz continuous if \( \beta > 1 \).

**Remarks**

- Non-linear equations (Bellman etc)
- Explicit Hölder exponent \( (\forall \alpha < \beta \leq 1) \)
- First order terms: “Ellipticity-Growth conditions”
- General Lévy measures
- General Lévy-Itô operators
Extension (I) : Lévy vs. Lévy-Itô

\[ - \int (u(x + z) - u(x)) \mu_x(dz) \quad \text{x-dependent Lévy measures} \]

\[ - \int (u(x + j(x, z)) - u(x)) \mu(dz) \quad \text{x-dependent jumps} \]

Comparison principle: ok? Comparison principle: ok!

Hölder continuity of coefficients

\[ \int_{B(0, \delta)} |z|^2 |\mu_x - \mu_y|(dz) \leq C \delta^{2-\beta} |x - y|^\gamma \]

\[ \int_{\mathbb{R}^d \setminus B(0, \delta)} |z| \|\mu_x - \mu_y\|(dz) \leq C \delta^{1-\beta} |x - y|^\gamma \quad (\beta \neq 1) \]
Extension (II): Composed Vs. Mixed

Recall equations

- Eq. (1): composed Barles-Chasseigne-Cl (JEMS, 2011)
- Eq. (2): mixed Barles-Chasseigne-Ciomaga-Cl (JDE)

General form of the equation

\[ F_0(u, Du, D^2 u, L_0[u]) + \sum_{i=1,2} F_i(x_i, D_{x_i} u, D_{x_i}^2 u, L[x_i, u]) = 0 \]

- \( F_0 \) is proper and Lipschitz w.r.t the last variable
- \( F_1 \) satisfies a Growth-Ellipticity condition
- \( F_2 \) satisfies a uniqueness-type condition

\( \Rightarrow \) partial regularity w.r.t. \( x_1 \) variables
Viscosity solutions for PIDE

- Soner (1986)
- Sayah, Lenhart
- Arisawa, Pham
- Ishii, Koike
- Alvarez, Tourin, Karlsen, Jakobsen
- Barles, Cl
- Barles, Chasseigne, Ciomaga
- ...

[252x252]
Definition of viscosity solutions

Test functions

Viscosity solutions

- (Subsolution) $\Phi^+$ touches $u$ from above at $x \Rightarrow F[\Phi^+](x) \leq 0$
- (Supersolution) $\Phi^-$ touches $u$ from below $\Rightarrow F[\Phi^-](x) \geq 0$

Examples

$$(-\Delta)^{\frac{\alpha}{2}} \Phi^+(x) - \text{Tr}(A(x)D^2\Phi^+(x))$$

$$- b(x) \cdot D\Phi^+(x) - f(x) + \lambda u(x) \leq 0$$

(Linear equation)
Definition of viscosity solutions

Test functions

Viscosity solutions

- (Subsolution) $\Phi^+$ touches $u$ from above at $x \Rightarrow F[\Phi^+](x) \leq 0$
- (Supersolution) $\Phi^-$ touches $u$ from below $\Rightarrow F[\Phi^-](x) \geq 0$

Examples

$$\inf_{\gamma_1} \sup_{\gamma_2} \left\{ (-\Delta)^{\alpha \over 2} \Phi^+(x) - \text{Tr}(A_{\gamma}(x)D^2\Phi^+(x)) \right. \left. - b_{\gamma}(x) \cdot D\Phi^+(x) - f_{\gamma}(x) \right\} + \lambda u(x) \leq 0$$

(Bellman equation)
Definition of viscosity solutions

Non-local equations $\Phi^\pm$ should be \textit{globally} above or below $u$

\[-\int [\Phi^+(x+z) - \Phi^+(x)]\mu(dz) \leq -\int [u(x+z) - u(x)]\mu(dz)\]

Equivalent definition split the integral

\[-\int_{B(0,r)} [\Phi^+(x+z) - \Phi^+(x)]\mu(dz) - \int_{\mathbb{R}^d \setminus B(0,r)} [u(x+z) - u(x)]\mu(dz) \leq -\int [u(x+z) - u(x)]\mu(dz)\]

References Sayah’91, CI’05, Barles-CI’08
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1. Motivations
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The Ishii-Lions method: local case (I)

Linear equation

\[-\Delta u - b \cdot Du - f + \lambda u = 0\]

What we want

\[u(x) - u(y) \leq L_1|x - y|^\alpha\]

Localization

\[u(x) - u(y) \leq L_1|x - y|^\alpha + L_2|x - x_0|^2\]

Proof by contradiction

Assume \[M = \sup_{x,y} u(x) - u(y) - \Phi(x - y) - \Gamma(x) > 0\]
for all \(\alpha \in (0, 1), L_1 > 0, L_2 > 0\)

In particular \(L_1|x - y|^\alpha \leq \|u\|_\infty + \|v\|_\infty\)
The Ishii-Lions method: local case (II)

Linear equation

\[-\Delta u - b \cdot Du - f + \lambda u = 0\]

Assume the solution \(u\) is smooth

- optimality condition:
  \[
  Du(x) = D\Phi(x - y) + D\Gamma(x) \\
  Du(y) = D\Phi(x - y)
  \]

- Second order optimality condition:
  \[
  \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}
  \]
  with \(X = D^2u(x) - D^2\Gamma(x)\), \(Y = D^2u(y)\), \(Z = D^2\Phi(x - y)\).

Use the equation twice and combine them

\[O(L_2) \leq \text{Tr}(X) - \text{Tr}(Y)\]
The Ishii-Lions method: local case (III)

Recall

\[ \Phi(z) = L_1 |z|^\alpha \] and \[ |x - y|^\alpha \leq \frac{1}{L_1} \]

\[ Z = L_1 D^2 |\cdot|^\alpha = L_1 |\cdot|^{\alpha-2} (I - (2 - \alpha)\widehat{x - y} \otimes \widehat{x - y}) \]

Use the matrix inequality

\[ \Rightarrow \quad \text{Tr}(X - Y) \leq -\frac{L_1 (1 - \alpha)}{|x - y|^{2-\alpha}} \]

\[ \Rightarrow \quad O(L_2) \leq -\frac{L_1 (1 - \alpha)}{|x - y|^{2-\alpha}} \]

If \( u \) is not smooth, use viscosity solution techniques

Jensen-Ishii’s lemma needed
Main idea in the previous proof
Use the concavity of $|\cdot|^\alpha$ to create a “large” negative term

For non-local case
Use the concavity “around” a given direction

Lipschitz regularity
Use $\Phi = L_1|\cdot| - \sigma|\cdot|^{1+\alpha}$. 
The Ishii-Lions method: Non-local case (combined - I)

Optimality condition ($\Gamma \equiv 0$)

\[ u(x + z) - u(y + z') - \Phi(x - y + z' - z) \leq u(x) - u(y) - \Phi(x - y) \]

This implies

\[ u(x + z) - u(x) \leq \Phi(x - y - z) - \Phi(x - y) \]
\[ u(y) - u(y + z) \leq \Phi(x - y + z') - \Phi(x - y) \]

From smooth to viscosity solutions

Jensen-Ishii’s lemma should be adapted
Joint work with G. Barles (Annales IHP, 2008)
\[ \int (\Phi(x - y + z) - \Phi(x - y)) \frac{dz}{|z|^{d+\alpha}} \]

\[ \| \int_{\mathbb{R}^d \setminus B}(\ldots) \frac{dz}{|z|^{d+\alpha}} \|_{\text{bounded}} \]

\[ + \int_{B \setminus C}(\ldots) \frac{dz}{|z|^{d+\alpha}} \|_{\text{controlled!}} \]

\[ + \int_{C}(\ldots) \frac{dz}{|z|^{d+\alpha}} \|_{\text{good!!}} \]
The Ishii-Lions method: Non-local case (mixed)

Prove

\[ u(x_1, x_2) - u(y_1, y_2) \leq L_1 |x_1 - y_1|^\alpha + \frac{|x_2 - y_2|^2}{2\varepsilon}. \]

Function \( \Phi \)

\[ \Phi(x_1, x_2) = L_1 |x_1|^\alpha + \frac{|x_2|^2}{2\varepsilon}. \]

Assumptions

- \( F_1 \) Growth-Ellipticity condition
- \( F_2 \) uniqueness-type condition
References


