Monodromy and Arithmetic Groups

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Thin Groups and Arithmetic Groups

Definition

(P.Sarnak) A subgroup $\Gamma \subset SL_N(\mathbb{Z})$ is said to be **thin** if it has infinite index in the integer points of its Zariski closure $\mathcal{G} \subset SL_N$.
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Using Tits' ping-pong construction, it is easy to produce many examples of thin groups; these thin groups are free products of smaller groups.
It is not so easy to exhibit thin groups which are not free products.

Monodromy Groups

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It is then of interest to know if these monodromy groups are thin or not. In a sizeable number of cases, the monodromy is indeed thin (and many where the monodromy is arithmetic i.e. not thin).

Elliptic Curves

Consider the Legendre family of elliptic curves E_λ given by

$$y^2 = x(x-1)(x-\lambda),$$

where $\lambda \in S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. We then get the space X of elliptic curves fibering over S. The fundamental group of S operates on $H^1 = \mathbb{Z}^2$ of the generic elliptic curve, and we get a representation

$$F_2 \to SL_2(\mathbb{Z})$$

which realises the free group on two generators as the group generated by the matrices

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In this case, the monodromy is indeed arithmetic, since the above subgroup has finite index in $SL_2(\mathbb{Z})$.

Product of Elliptic Curves

Fix $c \in S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Let $\lambda \in S$ be such that $\lambda \neq c^{-1}$. Then, for each such λ we have the product $A_\lambda = E_\lambda \times E_{c\lambda}$ and hence a family of Abelian surfaces fibering over $\lambda \in S \setminus \{c^{-1}\}$. The fundamental group of the latter space is the free group F_3 on three generators.

Theorem

(Nori) The image of the monodromy representation of F_3 on the first homology of the product A_λ is Zariski dense in $SL_2 \times SL_2$ but has infinite index in $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$; in fact it is not finitely presented.

Hyperelliptic Case

If $d = 2$, then the above theorem on families of elliptic curves is true for all n, and is due to A'Campo (1979). The monodromy is then a subgroup of finite index in $Sp_{2g}(\mathbb{Z})$ where g is the genus of the hyperelliptic curve whose affine part is given by the equation

$$y^2 = (x - a_1) \cdots (x - a_{n+1}).$$

Given $d \geq 2$ and $f \in S$ where S the family of monic polynomials of degree n with distinct roots, the curve whose affine part is given by

$$y^d = f(x),$$

is called the *generalised hyperellpitic curve*.
In this case, Mcmullen proved that if $n \leq 2d$ then the monodromy group is arithmetic, using Deligne -Mostow theory.
He also raised the question whether the monodromy group is arithmetic in these cases.

Theorem

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Monodromy and Differential Equations

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$$\frac{d^n u}{dz^n} + a_{n-1}(z)\frac{d^{n-1} u}{dz^{n-1}} + \cdots + a_1(z)\frac{du}{dz} + a_0(z)u = 0.$$

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If now U is any connected open set in $\mathbb{C}$, then using this theorem of Cauchy, we can analytically continue the solutions along closed loops in U and thus there is an action of $\pi_1(U)$ on the space of solutions of the foregoing differential equation. This representation is called the monodromy representation of the differential equation.

Hypergeometric Equation

We take for $\mathcal{U}$ the complex plane punctured at $0, 1$ i.e. $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. The fundamental group of $\mathcal{U}$ may be viewed as the free group on three generators h_0, h_1, h_∞ modulo the relation $h_0 h_1 h_\infty = 1$.

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Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ be such that α_i, β_j are rational numbers which lie in the closed open interval $[0, 1)$. Assume $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_n$ and that $\alpha_j \neq \beta_k$ for any j, k.

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Let $z \in \mathcal{U}$ and set $\theta = z\frac{d}{dz}$.
Consider the one variable *hypergeometric* differential equation

$$[(\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n)]u = 0.$$

Monodromy of the Hypergeometric Equation

Put $f(t) = \prod_{j=1}^{n}(t - e^{2\pi i\alpha_j})$ and $g(t) = \prod_{j=1}^{n}(t - e^{2\pi i\beta_k})$; under our assumtions, f, g are coprime polynomials. Let A, B be the companion matrices of f, g respectively. The following theorem of Levelt completely describes the monodromy of the hypergeometric equation.

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Using this result, Beukers and Heckman determined the Zariski closure of the monodromy group.

Zariski Closure of Monodromy

We now make the simplifying assumtion that f, g are products of cyclotomic polynomials. Then Levelt's Theorem ensures that after a conjugation, $A, B \in GL_n(\mathbb{Z})$. Let G be the Zariski closure of the monodromy group $\Gamma(f, g) =< A, B >$ in GL_n.

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Theorem

(Beukers-Heckman) The Zariski closure G is either finite, or the symplectic group Sp_n of a nondegenerate symplectic form in n variables (in which case, $f(0) = g(0) = 1$), or the orthogonal group $O(f)$ of a nondegenerate quadratic form in n variables (if n is odd, then this implies $f(0) = -g(0) = \pm 1$).

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Beukers and Heckman also determine the signature of the quadratic form over the reals.

Hyperbolic Orthogonal monodromy

We first consider the case when G is the orthogonal group and the signature of the quadratic form over $\mathbb{R}$ is *hyperbolic* i.e. of type $(n-1,1)$. In this case, most monodromy groups seem to be **thin**, thanks to the following result of Fuchs, Meiri and Sarnak:

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Theorem

(Fuchs, Meiri and Sarnak) There exist infinitely many n for which the Zariski closure is $O(n-1,1)$ *and the monodromy* $\Gamma(f,g)$ *is* **thin**.

Nonhyperbolic Orthogonal Case

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If we assume $n \geq 5$ and $G = O(p, q)$ with $p, q \geq 2$, there are no examples known (to me!) of thin monodromy. On the other hand, one has:

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For example, take $f = x^5 - 1$ and $g = (x^3 + 1)(x^2 - x + 1)$. Then $\Gamma = \Gamma(f, g)$ can be shown to be an arithmetic subgroup of $G = O(2, 3)$. Using this, one can show:

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Using this, one can show:
If $f = (x^5 - 1)(1 + x^{12})^m$ and $g = (x^3 + 1)(x^2 - x + 1)(1 + x^6 + x^{12})^m$, then $\Gamma(f, g)$ is an arithmetic subgroup of $G = O(p, q)$ with $p + q = 12m + 5$, and $p, q \geq 2$.

Symplectic Hypergeometric Monodromy

Suppose now that f, g are products of cyclotomic polynomials of degree n with no common root, and $f(0) = g(0) = 1$. Then it may be shown that $\Gamma(f, g)$ is contained in the symplectic group. Write

$$h = f - g = cX^r + \cdots + c_1,$$

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Using this criterion, one can verify that out of 111 hypergeometric groups in $Sp_4(\mathbb{Z})$, 65 are arithmetic. For example, $\Gamma = \Gamma(f, g)$ has finite index in $Sp_4(\mathbb{Z})$ if $f = X^4 + 1, \quad g = X^4 + X^2 + 1$. The proof uses the fact that that under the assumptions, the monodromy group contains an arithmetic subgroup of the unipotent radical of a parabolic subgroup of Sp_n (and is Zariski dense, by the result of Beukers and Heckman).

A Theorem of Tits

Tits proved that if Γ is a finite index subgroup of $SL_n(\mathbb{Z})$ or $Sp_{2g}(\mathbb{Z})$, and $U^{\pm}$ denote the upper and lower triangular unipotent matrices in Γ, then the subgroup generated by $U^{\pm}$ is also of finite index in Γ.

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Corollary

If $\Gamma \subset Sp_{2g}(\mathbb{Z})$ is a Zariski dense subgroup, and V is the unipotent radical of any parabolic $\mathbb{Q}$ subgroup of Sp_{2g}, and $\Gamma \cap V$ has finite index in $V(\mathbb{Z})$ then Γ has finite index in $Sp_{2g}(\mathbb{Z})$.

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We use this corollary to show that our monodromy group $\Gamma \subset Sp_{2g}(\mathbb{Z})$ has finite index.

The Maximally Unipotent Case

Take $n = 4$, and assume $f = (x-1)^4$. Then the monodromy h_∞ at infinity is maximally unipotent in $Sp_4(\mathbb{Z})$. Choose g satisfying the conditions above. There are exactly 14 such g. The corresponding $\Gamma(f, g)$ arise as monodromy of (14) families of CY threefolds fibering over the thrice punctured projective line.

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Example: $f = (x-1)^4, \quad g = (x^2 - x + 1)^2$ and
$f - g = -2x^3 + 4x^2 - 2x$ has leading coefficient ≤ 2 in absolute value.

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For example, the theorem of A'Campo may be interpreted as saying that *for every even integer* $n \geq 2$, the monodromy group $\Gamma(f, g) \subset Sp_n(\mathbb{Z})$ is arithmetic, where $f = \frac{X^{n+1}+1}{X+1}$ and $g = (X-1)(\frac{X^n+1}{X+1})$.

Unitary Groups

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Deligne and Mostow used monodromy of cyclic coverings of the projective line to exhibit (what are now called) thin groups; they also showed that some of these monodromy groups are non-arithmetic lattices in $U(2,1)$ and $U(3,1)$.

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A degree $d \geq 2$ cyclic covering of the projective line is of the form

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where $d, (k_i)_{1 \leq i \leq n+1}$ can be assumed to be co-prime, with $1 \leq k_i \leq d-1$. Here a_i are $n+1$ distinct complex numbers.

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where $d, (k_i)_{1 \leq i \leq n+1}$ can be assumed to be co-prime, with $1 \leq k_i \leq d - 1$. Here a_i are $n + 1$ distinct complex numbers.

We can now fix d, n and the k_i and vary the numbers a_i under the constraint that they are all distinct. To each point in the space S of $n + 1$-tuples of complex numbers a_i with all a_i distinct we get a d-fold cyclic covering of the projective line and hence a compact Riemann surface $X^*_{a,k}$ whose affine part is given by the above equation. We have thus a fibration $X \to S$ where X is the collection of the $X^*_{a,k}$.

Each of the curves $X^*_{a,k}$ have the same genus, call it g. The monodromy representation is now a homomorphism $\pi_1(S)$ into $GL_{2g}(\mathbb{Z})$. Each of these curves comes equipped with an action of the cyclic group $C_d = \mathbb{Z}/d\mathbb{Z}$ of order d, by fixing a primitive d-th root ω of unity and sending a generator T of $\mathbb{Z}/d\mathbb{Z}$ to the automorphism $y \mapsto \omega y$ for (x, y) in the affine part of $X^*_{a,k}$. Moreover, the monodromy action commutes with this action of the cyclic group C_d.

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The monodromy group is easily shown to preserve a hermitian form h on H^1 of $X^*_{a,k}$: the "intersection form" $\alpha \wedge \overline{\beta}$ for $\alpha, \beta \in H^1(X^*_{a,k}, \mathbb{C})$. Moreover, it preserves the eigenspaces for C_d. Hence the monodromy preserves the part of H^1 where the generator $T \in C_d$ acts by the scalar ω^f for some $f \in \mathbb{Z}/d\mathbb{Z}$. The restriction of the intersection form h to this ω^f-part has signature (p_f, q_f) say. We thus get a homomorphism $\pi_1(S) \to U(p_f, q_f)$.

Deligne-Mostow

If $x \in \mathbb{R}$, write $\{x\}$ for its fractional part. For each i write $\mu_i = \{\frac{k_i f}{d}\}$. Write $\mu_\infty = 2 - \sum_{i=1}^{n+1} \mu_i$. Suppose the μ_i (including μ_∞) satisfy the following conditions.

$$0 < \mu_\infty < 1, \ \textit{if} \ \mu_i \neq \mu_j, \ \textit{then} \ (1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z}$$

$$\textit{If} \ \ \mu_i = \mu_j \quad \textit{then} \ (1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2}\mathbb{Z}.$$

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Theorem

(Deligne-Mostow) Under the above assumptions, the group $U(p_f, q_f)$ is $U(n-1, 1)$, Moreover, the image of the monodromy group in $U(n-1, 1)$ is a lattice. In general, it is a non-arithmetic lattice. In particular, the monodromy group is thin.

An Example

Take the equation

$$y^{18} = (x - a_1)(x - a_2)(x - a_3)(x - a_4).$$

The monodromy, as the distinct a_i vary, on the first homology of these curves is such that if we take the projection to the f-th factor, with $f = 7$ is not arithmetic, but discrete: $\mu_i = \frac{7}{18}$ and hence $(1 - \mu_i - \mu_j)^{-1} = (1 - 14/18)^{-1} = 9/2$ is a half integer. $\mu_\infty = 2 - 4\frac{7}{18} = 8/18$ and $(1 - \mu_\infty - \mu_i)^{-1} = (1 - 15/18)^{-1} = 6$ is an integer. Hence by the Deligne Mostow criterion, the projection to f-th factor is discrete and is a lattice in $U(2, 1)$. The projection is not arithmetic.

$n \geq 2d$

A necessary condition for non-arithmetic lattices as above is that the relevant unitary group $U(p, q)$ must be $U(n-1, 1)$. This translates into the condition that $2 - \sum_{i=1}^{n+1} \mu_i = \mu_\infty \geq 0$. That is

$$2 \geq \sum_{i=1}^{n+1} \{\frac{k_i f}{d}\} \geq \frac{n+1}{d}.$$

Therefore, $n \leq 2d - 1$. Hence if $n \geq 2d$ we do not have rank one factors. In this situation, we have the

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Theorem

If $d \geq 3$ and $n \geq 2d$ and each k_i is coprime to d, then the monodromy group acting on $H^1(X_{a,k}^*, \mathbb{Z})$ is an arithmetic group in a product G_∞ of unitary groups: $G_\infty = \prod U(p_f, q_f)$.

When $k_i = 1$ for all i

We describe the proof in the simpler case when all the k_i are 1; thus we are looking at the family $\mathcal{F}$ given by

$$y^d = (x - a_1) \cdots (x - a_{n+1}),$$

as the a_i vary so that $a_i \neq a_j$ if $i \neq j$; we are thus looking at the family (as P varies) of equations

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The space $\mathcal{P}$ of such polynomials may be shown to have fundamental group B_{n+1}, the braid group on $n + 1$ generators. The braid group has generators $s_1, \cdots, s_n$ with the "braid relations"

$$s_i s_j = s_j s_i \quad (| i - j | \geq 2), \quad s_i s_j s_i = s_j s_i s_j \quad (| i - j | = 1).$$

The Reduced Burau representation

The braid group B_{n+1} has a representation into GL_n over the ring $R = \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials in one variable q. Let $M = R^n$ be the free module of rank n, with standard basis $e_1, e_2, \cdots, e_n$. Then the braid group B_{n+1} acts on R^n as follows.

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$$s_i(e_{i-1}) = e_{i-1} + qe_i \quad s_i(e_{i+1}) = e_{i+1} + e_i.$$

This representation is called the **reduced Burau representation.**

Hermitian form on Burau representation

Consider the reduced Burau representation R^n. The ring $R = \mathbb{Z}[q, q^{-1}]$ has an involution, denoted $f \mapsto \bar{f}$ induced by $q \mapsto q^{-1}$. With respect to this involution, we define a skew hermitian form h on R^n given by

$$h(e_i, e_i) = q - q^{-1}, \quad h(e_i, e_{i+1}) = -\frac{q-1}{q}.$$

It is easy to see that B_{n+1} preserves this (skew) hermitian form and hence the reduced Burau representation maps B_{n+1} into $U(h)$ the unitary group of this skew hermitian form h.

Burau Representation at d-th roots of unity

Given $d \geq 2$, consider the quotient homomorphism of rings $R \to R/(1 + q + \cdots + q^{d-1}) = R_d$. This induces a homomorphism of algebraic groups $U(h, R) \to U(h, R_d)$. We then get the composite homomorphism $B_{n+1} \to U(h, R_d) \subset GL_n(R/(1 + q + \cdots + q^{d-1})$, and the latter representation, denoted $\rho_n(d)$, is called the reduced Burau representation evaluated at all non-trivial d-th roots of unity.

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If ω is a fixed primitive d-th root of unity and $f \in \mathbb{Z}/d\mathbb{Z}$, then we can specialise q to ω^f and get a homomorphism $B_{n+1} \to U(h, \mathbb{Z}[\omega^f])$, the ring of integers in a cyclotomic extension of $\mathbb{Q}$. The signature of the Hermitian form h is (p_f, q_f) say. We thus get a map

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$$\rho_n(d) : B_{n+1} \to \prod_f U(p_f, q_f),$$

the latter a product of unitary groups.

Burau and Monodromy

The monodromy action of B_{n+1} on the homology of the d-fold cover X_a^* whose affine part is given by the equation

$$y^d = (x - a_1)\cdots(x - a_{n+1}),$$

is closely related to the reduced Burau representation at d-th roots of unity.

Theorem

If $n+1$ and d are coprime, then the above monodromy action is isomorphic to the reduced Burau representation $\rho_n(d)$.
In general, the monodromy representation is a quotient of the reduced Burau representation.

In particular, the arithmeticity of monodromy follows from the arithmeticity of the image of the Burau representation $\rho_n(d)$.

Image of the Burau

Thus the proof of the theorem amounts to proving

Theorem

If $d \geq 3$ and $n \geq 2d$, then the image of the Burau representation $\rho_n(d) : B_{n+1} \to U(h, R_d)$ is an arithmetic subgroup of the latter unitary group.

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The proof is by induction on $n \geq 2d$, and it can be proved directly for $n = 2d$. For $n = 2d$ one checks directly that the image contains an arithmetic subgroup of the unipotent radical of a maximal parabolic subgroup of the unitary group $U(h, R_d)$.

A criterion for Arithmeticity

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Suppose that G is a linear algebraic group defined over a number field K; denote by O_K the ring of integers in K. Suppose G is such that

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and $K - rank(G) \geq 1$. Suppose that $\Gamma \subset G(O_K)$ is a Zariski dense subgroup in G, such that the intersection of Γ with the integer points $U(O_K)$ has finite index in $U(O_K)$ where U is the unipotent radical of a maximal parabolic subgroup of G defined over K. Then Γ is arithmetic, i.e. Γ has finite index in $G(O_K)$.

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We can use this in the case $n = 2d$ to conclude that the image of B_{n+1} is arithmetic: it can be shown that the image U_0 of the group generated by the commutators $[z, B_n]$, where z lies in the centre of B_{n-1}, is an arithmetic subgroup of a suitable unipotent radical as above.

Thank you for your attention.