Arithmetic subgroups whose representations all map into $\mathrm{GL}_n(\mathbb{Z})$

Dave Witte Morris
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Suppose Γ is an arithmetic subgroup of a semisimple Lie group G. For any finite-dimensional representation $\rho \colon G \to \mathrm{GL}_n(\mathbb{R})$, a classical paper of J. Tits determines whether $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Combining this with a well-known surjectivity result in Galois cohomology provides a short proof of the known fact that every G has an arithmetic subgroup Γ, such that the containment is true for *every* representation ρ. We will not assume the audience is acquainted with Galois cohomology or the theorem of Tits.

G = linear semisimple Lie group (no compact factors)
$\quad \doteq \mathbf{G}(\mathbb{R})$ semisimple over $\mathbb{R}$ (no anisotropic factors)
Γ = arithmetic subgroup

Definition of arithmeticity
$\quad \Rightarrow \exists \rho \colon \widetilde{G} \dashrightarrow \mathrm{GL}_n(\mathbb{R}),\ \ \rho(\Gamma) \subseteq \mathrm{GL}_n(\mathbb{Z})$.

Definition

Γ **universally arithmetic** $\Leftrightarrow$ $\forall \rho \colon \widetilde{G} \to \mathrm{GL}_n(\mathbb{R})$,
$\quad \exists M \in \mathrm{GL}_n(\mathbb{R}),\ \ M\,\rho(\Gamma)\,M^{-1} \subseteq \mathrm{GL}_n(\mathbb{Z})$.

Eg. $\mathrm{SL}_n(\mathbb{Z})$ is universally arithmetic in $\mathrm{SL}_n(\mathbb{R})$.

Proposition (Morris [2004])

Every G has a universally arithmetic subgroup Γ.

Definition

Γ **universally arithmetic** $\Leftrightarrow$ $\forall \rho \colon \widetilde{G} \to \mathrm{GL}_n(\mathbb{R})$,
$\quad \exists M \in \mathrm{GL}_n(\mathbb{R}),\ \ M\,\rho(\Gamma)\,M^{-1} \subseteq \mathrm{GL}_n(\mathbb{Z})$.

Eg. Γ irred in $G_1 \times G_2$
$\quad \Rightarrow \pi_1(\Gamma)$ is dense in G_1, so $\not\subseteq\ (G_1)_{\mathbb{Z}}$
$\quad \Rightarrow \Gamma$ **not** universally arithmetic.

Γ universally arithmetic in $G_1 \times G_2$
$\quad \Rightarrow \Gamma \doteq \Gamma_1 \times \Gamma_2$ and Γ_i univ arith in G_i.

Warning. *Converse is not true.*

Prop. List (up to commens) all univ arith subgrps of simple G. (could also do semisimple)

Proposition (Morris [2004])

Every G has a universally arithmetic subgroup Γ.

Reps of G $\longleftrightarrow$ Reps of $\mathfrak{g}$ (characteristic 0)
So can restate (stronger) in terms of Lie algebras.

Definition

Semisimple Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ is **$\mathbb{R}$-universal**:
$\quad \forall$ homo $\rho \colon \mathfrak{g}_{\mathbb{Q}} \to \mathfrak{gl}_n(\mathbb{R})$ ($\mathbb{Q}$-linear),
$\quad\quad \exists M \in \mathrm{GL}_n(\mathbb{R}),\ \ M\,\rho(\mathfrak{g})\,M^{-1} \subseteq \mathfrak{gl}_n(\mathbb{Q})$.

Proposition (Raghunathan [1982], Morris [2004])

Every semisimple $\mathfrak{g}_{\mathbb{R}}$ has an $\mathbb{R}$-universal $\mathbb{Q}$-form:
$\quad \forall \mathfrak{g}_{\mathbb{R}},\ \exists$ *$\mathbb{R}$-universal* $\mathfrak{g}_{\mathbb{Q}},\ \ \mathfrak{g}_{\mathbb{R}} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$.

Proposition (Raghunathan [1982], Morris [2004])

Every semisimple $\mathfrak{g}_{\mathbb{R}}$ has an $\mathbb{R}$-universal $\mathbb{Q}$-form.

1. My original proof was tedious (and explicit).
2. Better pf (conceptual) by **Prasad-Rapinchuk** [2006].
3. Today: another proof (direct, natural).

Exercise

$\mathfrak{g}_{\mathbb{Q}}$ $\mathbb{R}$-univ $\Rightarrow$ irred reps over $\mathbb{Q}$ remain irred over $\mathbb{R}$
$\quad \Leftrightarrow$ "weakly $\mathbb{R}$-universal"

Converse often holds (split over quad ext, same $*$-action)

Eg. $\mathfrak{g}_{\mathbb{Q}}$ split $\Rightarrow$ irred rep is highest-weight module V_λ
$\quad \Rightarrow V_\lambda \otimes \mathbb{C}$ irred $\Rightarrow \mathfrak{g}_{\mathbb{Q}}$ weakly $\mathbb{R}$-universal.

Eg. $\mathfrak{g}_{\mathbb{Q}}$ split $\Rightarrow$ irred rep is highest-weight module V_λ
$\quad \Rightarrow V_\lambda \otimes \mathbb{C}$ irred $\Rightarrow \mathfrak{g}_{\mathbb{Q}}$ weakly $\mathbb{R}$-universal.

In general, $\rho \otimes \mathbb{C} = \sum_i V_{\lambda_i}$, $\{\lambda_i\} = \mathrm{Gal}(\overline{F}/F)$-orbit.

Today, assume $\mathrm{Aut}\,\mathfrak{g} = G$ (or $\mathfrak{g}$ is "inner form").
So all λ_i's **equal**: $\rho \otimes \mathbb{C} = mV_\lambda$. *Notation:* $\rho = \rho_\lambda$.

Schur's Lemma

$\quad \rho \colon \mathfrak{g}_F \to \mathfrak{gl}_n(F) \qquad C_L(\rho) = C_{\mathrm{Mat}_{n\times n}(L)}(\rho(\mathfrak{g}_F))$
ρ irreducible over L $\Leftrightarrow$ $C_L(\rho)$ is division algebra.

Cor. $\mathfrak{g}_{\mathbb{Q}}$ weakly $\mathbb{R}$-universal $\Leftrightarrow$
$\quad \forall$ $\mathbb{Q}$-irred ρ, $C_{\mathbb{Q}}(\rho) \otimes \mathbb{R} = C_{\mathbb{R}}(\rho)$ *is div alg.*

Tits: use **Galois cohomology** to calculate $C_F(\rho_\lambda)$.

How to use Galois cohomology

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ = complex semisimple Lie algebra
$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ = **split** $\mathbb{R}$-form
$\sigma\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ complex conjugation

$\mathfrak{g}_{\mathbb{R}}$ = any $\mathbb{R}$-form = $\mathbb{R}$-span of $\mathbb{C}$-basis, consts in $\mathbb{R}$.
So $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} + i\mathfrak{g}_{\mathbb{R}}$. Cplx conj w.r.t. $\mathfrak{g}_{\mathbb{R}}$ is $\sigma_{\mathbb{R}}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$.

σ and $\sigma_{\mathbb{R}}$ are conjugate-linear Lie alg autos of $\mathfrak{g}_{\mathbb{C}}$,
so $t_{\mathbb{R}} := \sigma_{\mathbb{R}}\,\sigma^{-1} \in \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}}) = G_{\mathbb{C}}$. **Note:** $t_{\mathbb{R}}\,{}^{\sigma}t_{\mathbb{R}} = 1$.
$\{\mathbb{R}\text{-forms of } \mathfrak{g}\} \to G_{\mathbb{C}}$ $\quad t_{\mathbb{R}} \in H^1(\operatorname{Gal}(\mathbb{C}/\mathbb{R}); G_{\mathbb{C}})$

$\exists \alpha_{\mathbb{R}}\colon \mathfrak{g}_{\mathbb{R}} \xrightarrow{\cong} \mathfrak{g}'_{\mathbb{R}} \iff \exists \alpha \in \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}}),\ \sigma'_{\mathbb{R}} = \alpha\,\sigma_{\mathbb{R}}\,\alpha^{-1}$
$\iff \exists x \in G_{\mathbb{C}},\ t'_{\mathbb{R}} = \sigma'_{\mathbb{R}}\sigma^{-1} = x\,t_{\mathbb{R}}\sigma\,x^{-1}\sigma^{-1} = x\,t_{\mathbb{R}}\,\overline{x}^{-1}$
$t'_{\mathbb{R}}$ is "*cohomologous*" to $t_{\mathbb{R}}$

$\mathfrak{g}_{\mathbb{R}}$ = $\mathbb{R}$-form of $\mathfrak{g}_{\mathbb{C}} \rightsquigarrow t_{\mathbb{R}} \in G_{\mathbb{C}}$ $\qquad t_{\mathbb{R}}\,\overline{t_{\mathbb{R}}} = 1$

Tits calculated $C_{\mathbb{R}}(\rho_\lambda)$ from λ [1971]

$z_{\mathbb{R}} := t_{\mathbb{R}}\,\overline{t_{\mathbb{R}}} \in Z := Z(\widetilde{G}_{\mathbb{C}})$ in $\widetilde{G}_{\mathbb{C}}$.
In fact, $\overline{z_{\mathbb{R}}} = z_{\mathbb{R}}$, so $z_{\mathbb{R}} \in Z_{\mathbb{R}}$. So $\lambda(z_{\mathbb{R}}) \in \{\pm 1\}$.

$$C_{\mathbb{R}}(\rho) \doteq \left(\frac{-1, \lambda(z_{\mathbb{R}})}{\mathbb{R}}\right) \doteq \begin{cases} \mathbb{R} & \text{if } \lambda(z_{\mathbb{R}}) = 1; \\ \mathbb{H} & \text{if } \lambda(z_{\mathbb{R}}) = -1. \end{cases}$$

$\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}} \rightsquigarrow t_{\mathbb{Q}} \in G_L$ if $\mathfrak{g}_{\mathbb{Q}}$ splits over $L = \mathbb{Q}[i]$.

$$C_{\mathbb{Q}}(\rho) \doteq \left(\frac{-1, \lambda(z_{\mathbb{Q}})}{\mathbb{Q}}\right) \doteq \begin{cases} \mathbb{Q} & \text{if } \lambda(z_{\mathbb{Q}}) = 1; \\ \mathbb{H}_{\mathbb{Q}} & \text{if } \lambda(z_{\mathbb{Q}}) = -1. \end{cases}$$

Cor. $\mathfrak{g}_{\mathbb{Q}}$ weakly $\mathbb{R}$-universal (split over quadratic extension)
$\iff \forall$ $\mathbb{Q}$-irred ρ, $C_{\mathbb{Q}}(\rho) \otimes \mathbb{R} = C_{\mathbb{R}}(\rho)$ is div alg
$\iff \forall \lambda, \quad \lambda(z_{\mathbb{Q}}) \neq 1 \Rightarrow \lambda(z_{\mathbb{R}}) \neq 1$.

Cor. $\mathfrak{g}_{\mathbb{Q}}$ $\mathbb{R}$-univ $\iff \forall \lambda,\ \lambda(z_{\mathbb{R}}) = 1 \Rightarrow \lambda(z_{\mathbb{Q}}) = 1$.

Proposition (Raghunathan [1982], Morris [2004])

Every semisimple $\mathfrak{g}_{\mathbb{R}}$ has an $\mathbb{R}$-universal $\mathbb{Q}$-form.

Outline of proof (for inner forms).

$t_{\mathbb{R}} \in G_{\mathbb{C}}$ = adjoint grp, $\quad z_{\mathbb{R}} := t_{\mathbb{R}}\,\overline{t_{\mathbb{R}}} \in Z(\widetilde{G}_{\mathbb{C}}) =: Z_{\mathbb{C}}$.
$\overline{z_{\mathbb{R}}} = z_{\mathbb{R}} \Rightarrow z \in Z_{\mathbb{R}} = Z_{\mathbb{Q}}$. (inner form)
So $\widehat{G}_{\mathbb{C}} := \widetilde{G}_{\mathbb{C}}/\langle z \rangle$ is a $\mathbb{Q}$-group.
$t_{\mathbb{R}}\,\overline{t_{\mathbb{R}}} = z = 1$ in $\widehat{G}_{\mathbb{C}}$ $\quad \Rightarrow t_{\mathbb{R}}$ is coho to $t_{\mathbb{Q}} \in \widehat{G}_{\mathbb{Q}[i]}$.
$H^1(\mathbb{Q}[i]/\mathbb{Q}; \widehat{G}_{\mathbb{Q}[i]}) \twoheadrightarrow H^1(\mathbb{C}/\mathbb{R}; \widehat{G}_{\mathbb{C}})$ [~Kneser]
Let $\mathfrak{g}_{\mathbb{Q}}$ be $\mathbb{Q}$-form corresponding to $t_{\mathbb{Q}}$
= fixed points of $t_{\mathbb{Q}}\,\sigma$ in $\mathfrak{g}_{\mathbb{Q}[i]}$
$\lambda(z_{\mathbb{R}}) = 1 \Rightarrow \lambda(\langle z_{\mathbb{R}} \rangle) = 1 \Rightarrow \lambda(z_{\mathbb{Q}}) = 1$. □

A list of references is in the bibliography of:

Dave Witte Morris:
A cohomological proof that real representations of semisimple Lie algebras have $\mathbb{Q}$-forms (preprint).
arxiv:1410.2339