Arithmetic subgroups whose representations all map into $\text{GL}_n(\mathbb{Z})$

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Suppose $\Gamma$ is an arithmetic subgroup of a semisimple Lie group $G$. For any finite-dimensional representation $\rho: G \to \text{GL}_n(\mathbb{R})$, a classical paper of J. Tits determines whether $\rho(\Gamma)$ is conjugate to a subgroup of $\text{GL}_n(\mathbb{Z})$. Combining this with a well-known surjectivity result in Galois cohomology provides a short proof of the known fact that every $G$ has an arithmetic subgroup $\Gamma$, such that the containment is true for every representation $\rho$. We will not assume the audience is acquainted with Galois cohomology or the theorem of Tits.

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**Definition**

$\Gamma$ universally arithmetic $\iff \forall \rho: \tilde{G} \to \text{GL}_n(\mathbb{R}), \exists M \in \text{GL}_n(\mathbb{R}), M \rho(\Gamma) M^{-1} \subseteq \text{GL}_n(\mathbb{Z})$.

**Eg.** $\Gamma$ irreducible in $G_1 \times G_2$

$\Rightarrow \pi_1(\Gamma)$ is dense in $G_1$, so $\not\subseteq (G_1)_\mathbb{Z}$

$\Rightarrow \Gamma$ not universally arithmetic.

$\Gamma$ universally arithmetic in $G_1 \times G_2$

$\Rightarrow \Gamma \simeq \Gamma_1 \times \Gamma_2$ and $\Gamma_i$ univ arith in $G_i$.

**Warning.** Converse is not true.

**Prop.** List (up to commens) all univ arith subgrps of simple $G$. (could also do semisimple)

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**Proposition (Morris [2004])**

Every $G$ has a universally arithmetic subgroup $\Gamma$.

Reps of $G \leftrightarrow$ Reps of $g$ (characteristic 0)

So can restate (stronger) in terms of Lie algebras.

**Definition**

Semisimple Lie algebra $g_\mathbb{Q}$ over $\mathbb{Q}$ is $\mathbb{R}$-universal:

$\forall$ homo $\rho: g_\mathbb{Q} \to \text{gl}_n(\mathbb{R})$ (Q-linear),

$\exists M \in \text{GL}_n(\mathbb{R}), M \rho(g) M^{-1} \subseteq \text{gl}_n(\mathbb{Q})$.

**Proposition (Raghunathan [1982], Morris [2004])**

Every semisimple $g_\mathbb{R}$ has an $\mathbb{R}$-universal $\mathbb{Q}$-form:

$\forall g_\mathbb{R}$, $\exists$ $\mathbb{R}$-universal $g_\mathbb{Q}$, $g_\mathbb{R} \cong g_\mathbb{Q} \otimes \mathbb{R}$.

**Eg.** $g_\mathbb{Q}$ split $\Rightarrow$ irred rep is highest-weight module $V\lambda$

$\Rightarrow V\lambda \otimes \mathbb{C}$ irred $\Rightarrow g_\mathbb{Q}$ weakly $\mathbb{R}$-universal.

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**Proposition (Raghunathan [1982], Morris [2004])**

Every semisimple $g_\mathbb{R}$ has an $\mathbb{R}$-universal $\mathbb{Q}$-form.

- My original proof was tedious (and explicit).
- Better pf (conceptual) by Prasad-Rapinchuk [2006].
- Today: another proof (direct, natural).

**Exercise**

$g_\mathbb{Q}$ $\mathbb{R}$-univ $\Rightarrow$ irred reps over $\mathbb{Q}$ remain irred over $\mathbb{R}$

$\Leftrightarrow$ “weakly $\mathbb{R}$-universal”

Converse often holds (split over quad ext, same $\ast$-action)

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**Proposition (Morris [2004])**

$\forall \rho: G \to \text{GL}_n(\mathbb{R}), \exists M \in \text{GL}_n(\mathbb{R}), M \rho(\Gamma) M^{-1} \subseteq \text{GL}_n(\mathbb{Z})$.

$G = \text{linear semisimple Lie group}$ (no compact factors)

$\pm G(\mathbb{R})$ semisimple over $\mathbb{R}$ (no anisotropic factors)

$\Gamma = \text{arithmetic subgroup}$

**Definition**

$\forall \rho: \tilde{G} \to \text{GL}_n(\mathbb{R}), M \rho(\Gamma) M^{-1} \subseteq \text{GL}_n(\mathbb{Z})$.

**Eg.** $\text{SL}_n(\mathbb{Z})$ is universally arithmetic in $\text{SL}_n(\mathbb{R})$.

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**Proposition (Raghunathan [1982], Morris [2004])**

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**Eg.** $g_\mathbb{Q}$ split $\Rightarrow$ irred rep is highest-weight module $V\lambda$

$\Rightarrow V\lambda \otimes \mathbb{C}$ irred $\Rightarrow g_\mathbb{Q}$ weakly $\mathbb{R}$-universal.

In general, $\rho \otimes \mathbb{C} = \sum_i V\lambda_i, \{\lambda_i\} = \text{Gal}(\overline{F}/F)$-orbit.

Today, assume $\text{Aut } g = G$ (or $g$ is “inner form”).

So all $\lambda_i$’s equal: $\rho \otimes \mathbb{C} = mV\lambda$. Notation: $\rho = \rho_\lambda$.

**Schur’s Lemma**

$\rho: g_F \to \text{gl}_n(F)$

$C_L(\rho) = C_{\text{Mat}_{\text{deg}(L)}(\rho(g_F))}$

$\rho$ irreducible over $L$ $\iff$ $C_L(\rho)$ is division algebra.

**Cor.** $g_\mathbb{Q}$ weakly $\mathbb{R}$-universal $\iff$

$\forall$ Q-irred $\rho, C_Q(\rho) \otimes \mathbb{R} = C_{\mathbb{R}}(\rho)$ is div alg.

Tits: use Galois cohomology to calculate $C_F(\rho_\lambda)$.
**How to use Galois cohomology**

\[ g_C = sl_2(C) = \text{complex semisimple Lie algebra} \]
\[ g = sl_2(R) = \text{split} \ R \text{-form} \]
\[ \sigma: sl_2(C) \rightarrow sl_2(C) \text{ complex conjugation} \]

\[ g_R = \text{any} \ R \text{-form} = \text{R-span of} \ C \text{-basis, consts in} \ R. \]
So \[ g_C = g + Ig_R. \] Cplx conj w.r.t. \[ g_R \] is \( \sigma_R \): \[ g_C \rightarrow g_C. \]

\[ \sigma \text{ and } \sigma_R \text{ are conjugate-linear Lie alg auts of} \ g_C, \]
so \[ t_R := \sigma_R \sigma^{-1} \in \text{Aut}(g_C) = G_C. \] **Note:** \( t_R \sigma t_R = 1. \)

\[ \{ \text{forms of} \ g \} \rightarrow G_C \quad t_R \in H^1(\text{Gal}(C/R); G_C) \]

\[ \exists \alpha_R: g_R \cong g_R' \iff \exists \alpha \in \text{Aut}(g_C), \sigma_R' = \alpha \sigma_R \alpha^{-1} \]
\[ \iff \exists x \in G_C, t_R' = \sigma_R' \sigma^{-1} = x t_R \sigma x^{-1} \sigma^{-1} = x t_R x^{-1} \]
\( t_R' \) is “cohomologous” to \( t_R \)

**Cor.** \( g_Q \text{-univ} \iff \forall \lambda, \lambda(z_R) = 1 \Rightarrow \lambda(z_Q) = 1. \)

**Proposition (Raghunathan [1982], Morris [2004])**

Every semisimple \( g_R \) has an \( R \)-universal \( Q \)-form.

**Outline of proof** (for inner forms).

\[ t_R \in G_C = \text{adjoint grp}, \quad z_R := t_R \bar{t}_R \in Z(\tilde{G}_C) =: Z_C. \]
\[ \bar{Z}_R = Z_R = z \in Z_R = Z_Q. \] (inner form)

So \( \tilde{G}_C := \tilde{G}_C/z \) is a \( Q \)-group.

\[ t_R \bar{t}_R = z = 1 \text{ in } \tilde{G}_C \quad \Rightarrow \quad t_R \text{ is coho to } t_Q \in \hat{G}_Q[1]. \]
\[ H^1(Q[i]/Q; \hat{G}_Q[1]) \rightarrow H^1(C/R; \hat{G}_C) \quad \sim \text{Kneser} \]

Let \( g_Q \) be \( Q \)-form corresponding to \( t_Q \)
\[ = \text{fixed points of } t_Q \sigma \text{ in } g_Q[1]; \]
\( \lambda(z_R) = 1 \Rightarrow \lambda(z_R) = 1 \Rightarrow \lambda(z_Q) = 1. \] □

**Tits calculated \( C_R(\rho) \) from \( \lambda \) [1971]**

\[ z_R := t_R \bar{t}_R \in Z := Z(\tilde{G}_C) \text{ in } \tilde{G}_C. \]
In fact, \( z_R = z_R \), so \( z_R \in Z_R. \)
So \( \lambda(z_R) \in \{ \pm 1 \}. \)
\[ C_R(\rho) \equiv \frac{-1, \lambda(z_R)}{R} \equiv \begin{cases} \mathbb{R} & \text{if } \lambda(z_R) = 1; \\ \mathbb{H} & \text{if } \lambda(z_R) = -1. \end{cases} \]

\( Q \)-form \( g_Q \sim t_Q \in G_L \) if \( g_Q \) splits over \( L = Q[i]. \)
\[ C_Q(\rho) \equiv \frac{-1, \lambda(z_Q)}{Q} \equiv \begin{cases} \mathbb{Q} & \text{if } \lambda(z_Q) = 1; \\ \mathbb{H}_Q & \text{if } \lambda(z_Q) = -1. \end{cases} \]

**Cor.** \( g_Q \) weakly \( R \)-universal \quad \text{(split over quadratic extension)}
\[ \iff \forall \ Q \text{-irred } \rho, \quad C_Q(\rho) \otimes R = C_R(\rho) \text{ is div alg} \]
\[ \iff \forall \lambda, \quad \lambda(z_Q) \neq 1 \Rightarrow \lambda(z_R) \neq 1. \]

A list of references is in the bibliography of:

Dave Witte Morris:
A cohomological proof that real representations of semisimple Lie algebras have \( Q \)-forms (preprint).
arxiv:1410.2339