Fundamental Limits of Robust Interference Management
– An Information Theoretic Perspective

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Fundamental Limits

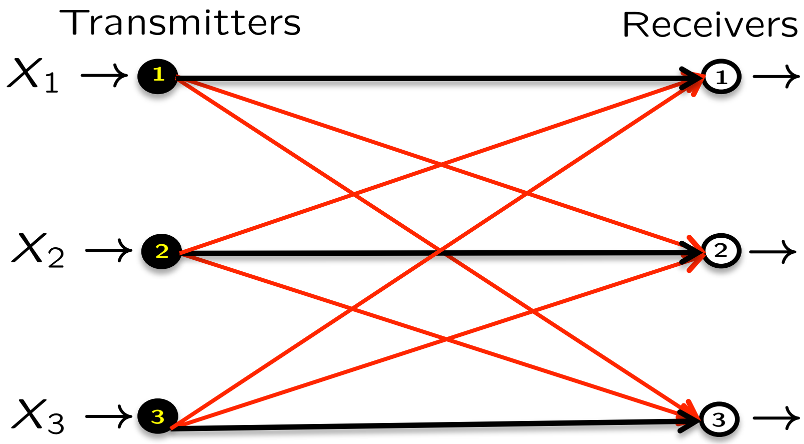
"Noise Barrier" $\rightarrow$ Shannon

"Bandwidth Barrier" $\rightarrow$ MIMO

"Interference Barrier" $\rightarrow$?

TDMA/FDMA/CDMA, Cellular-reuse patterns, Zero forcing, Dirty Paper Coding, Successive Interference Cancellation, Opportunistic Beamforming, Interference Alignment, Interference Neutralization, Joint Space-Time Precoding, Rate Splitting, Interference Forwarding, Elevated Multiplexing, Improper Signaling

...



$$\rightarrow Y_1 = \sqrt{P}G_{11}X_1 + \sqrt{P}G_{12}X_2 + \sqrt{P}G_{13}X_3 + Z_1$$

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$E|X_i|^2 \leq 1$ $\quad Z_i \sim \mathcal{N}(0,1)$ $\quad G_{ij} = \hat{G}_{ij} + \epsilon \tilde{G}_{ij}$ $\quad$ CSIT: $\hat{G}$ $\quad$ Perfect CSIR

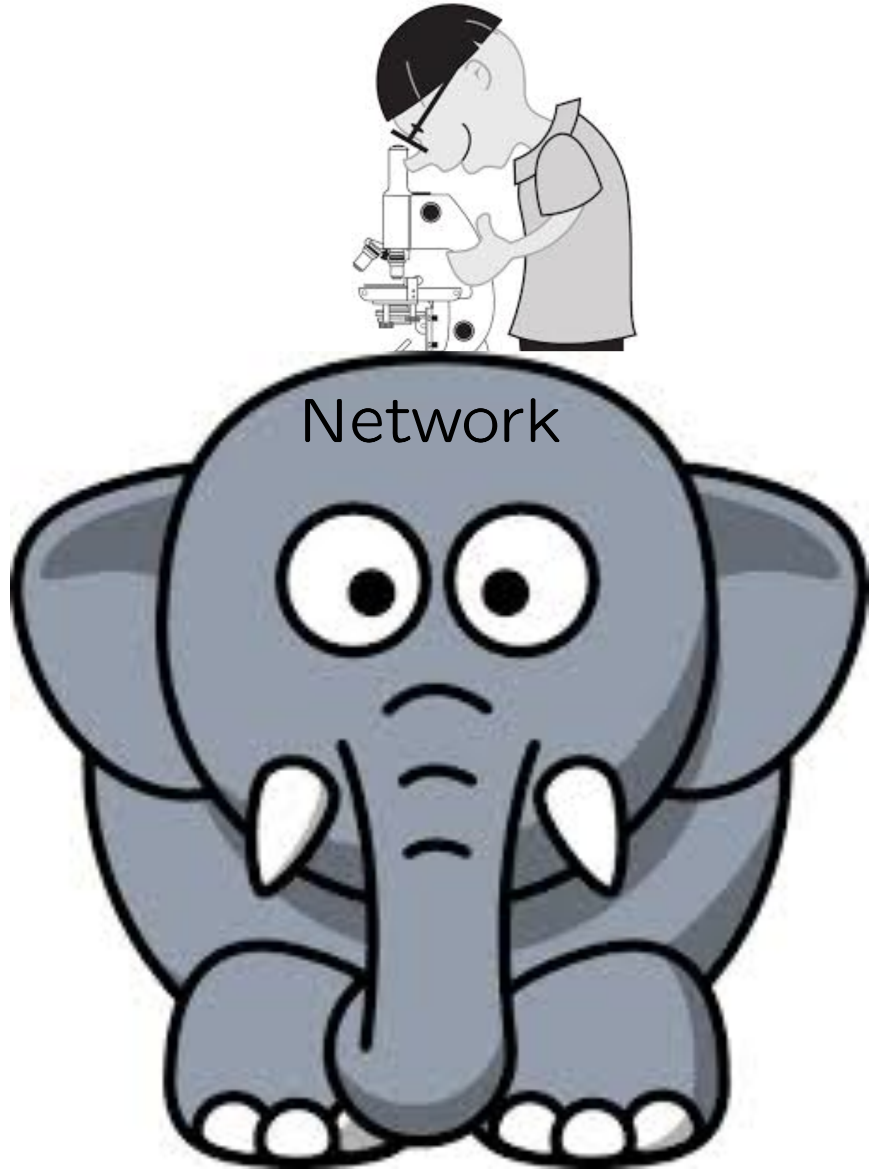
One hop

MISO Broadcast Channel: All messages shared by all Tx.

X Channel: Each Tx has an independent message for each Rx.

Interference Channel: Each Tx has an independent message for its corresponding Rx.

Exact Capacity



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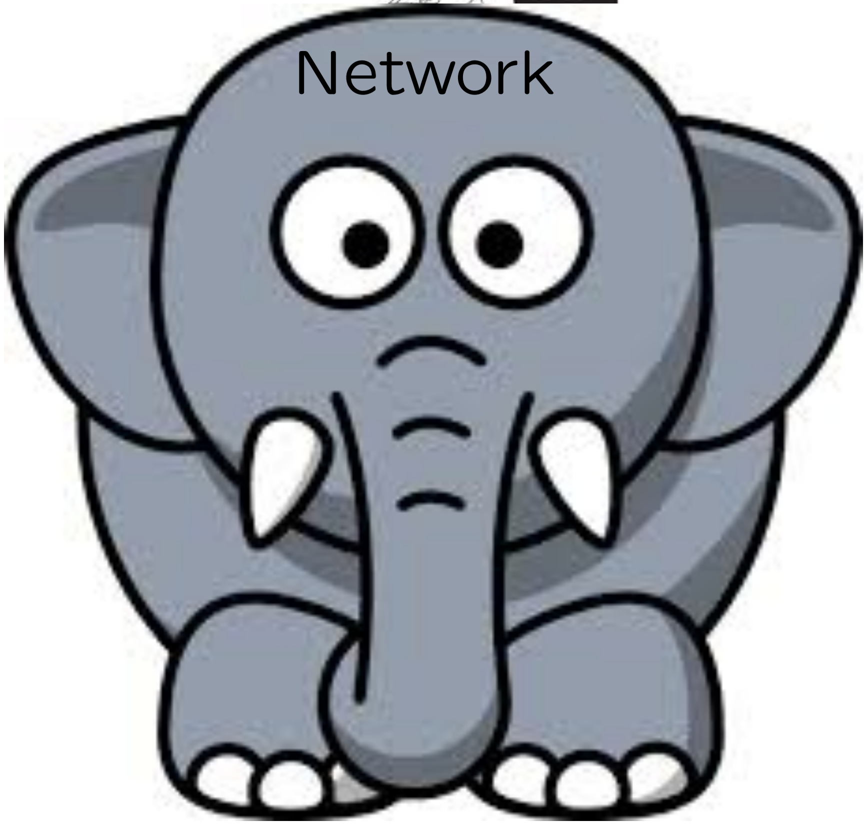
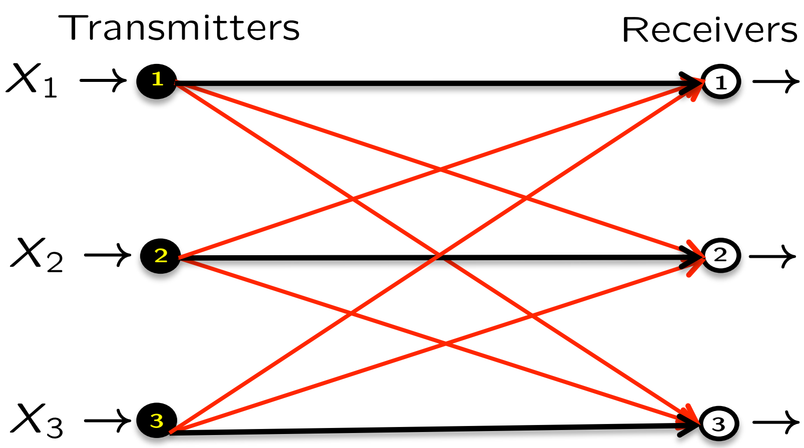


Approximate Capacity

Degrees of Freedom (DoF)

Limitations of DoF



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DoF $= \lim_{P \to \infty} \frac{\text{Network Capacity}}{\text{Link Capacity}} \approx$ No. of interference free links that can be created in the network

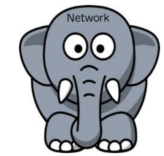
Unable to capture diversity of channel strength levels.

Strong channel $\approx$ Weak channel

Unable to capture diversity of channel knowledge levels.

Finite Precision CSIT $\approx$ No CSIT

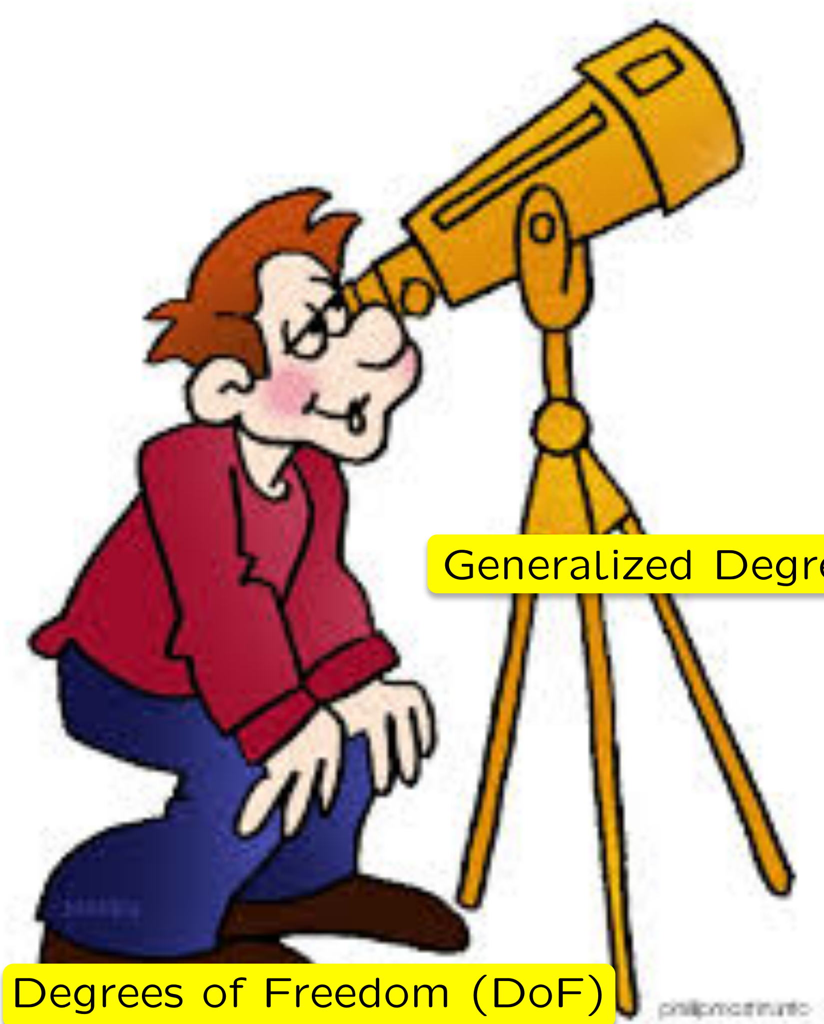
Perfect CSIT: DoF depend on Channels Rational/Irrational



Capacity

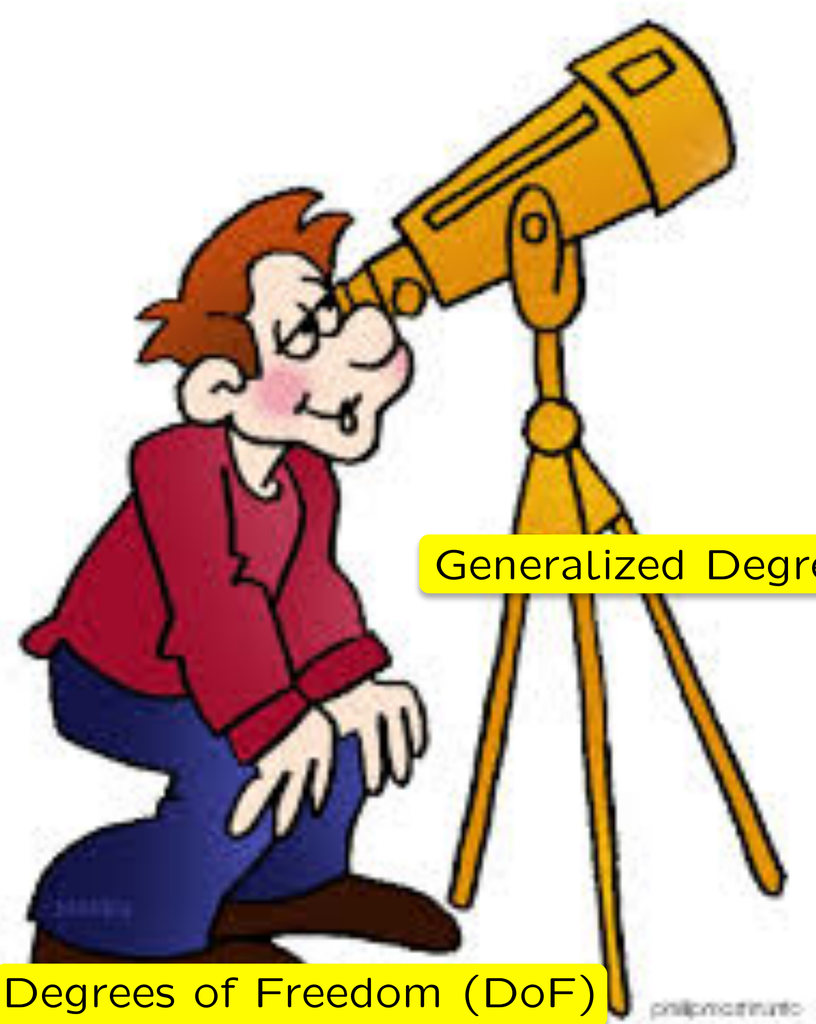
Bounded Gap Approximations

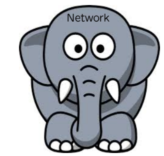
Generalized Degrees of Freedom (GDoF)



Degrees of Freedom (DoF)

Progressive Refinements Approach





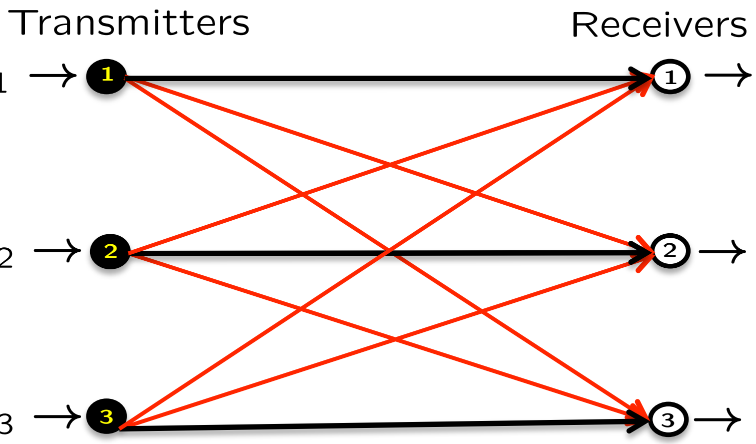
Generalized Degrees of Freedom (GDoF)

Degrees of Freedom (DoF)

Generalized Degrees of Freedom

[Etkin, Tse, Wang, IT Trans. 2008]

(Basis for Deterministic Models of [Avestimehr, Diggavi, Tse, IT Trans. 2011])



$X_1 \rightarrow$ 1 $\rightarrow Y_1 = \sqrt{P^{\alpha_{11}}} G_{21} X_1 + \sqrt{P^{\alpha_{12}}} X_2 + \sqrt{P^{\alpha_{13}}} X_3 + Z_1$

$X_2 \rightarrow$ 2 $\rightarrow Y_2 = \sqrt{P^{\alpha_{21}}} G_{21} X_1 + \sqrt{P^{\alpha_{22}}} X_2 + \sqrt{P^{\alpha_{23}}} X_3 + Z_2$

$X_3 \rightarrow$ 3 $\rightarrow Y_3 = \sqrt{P^{\alpha_{31}}} G_{31} X_1 + \sqrt{P^{\alpha_{32}}} X_2 + \sqrt{P^{\alpha_{33}}} X_3 + Z_3$

$E|X_i|^2 \leq 1$

$Z_i \sim \mathcal{N}(0,1)$

$$G_{ij} = \hat{G}_{ij} + \sqrt{P^{-\beta_{ij}}} \tilde{G}_{ij}$$

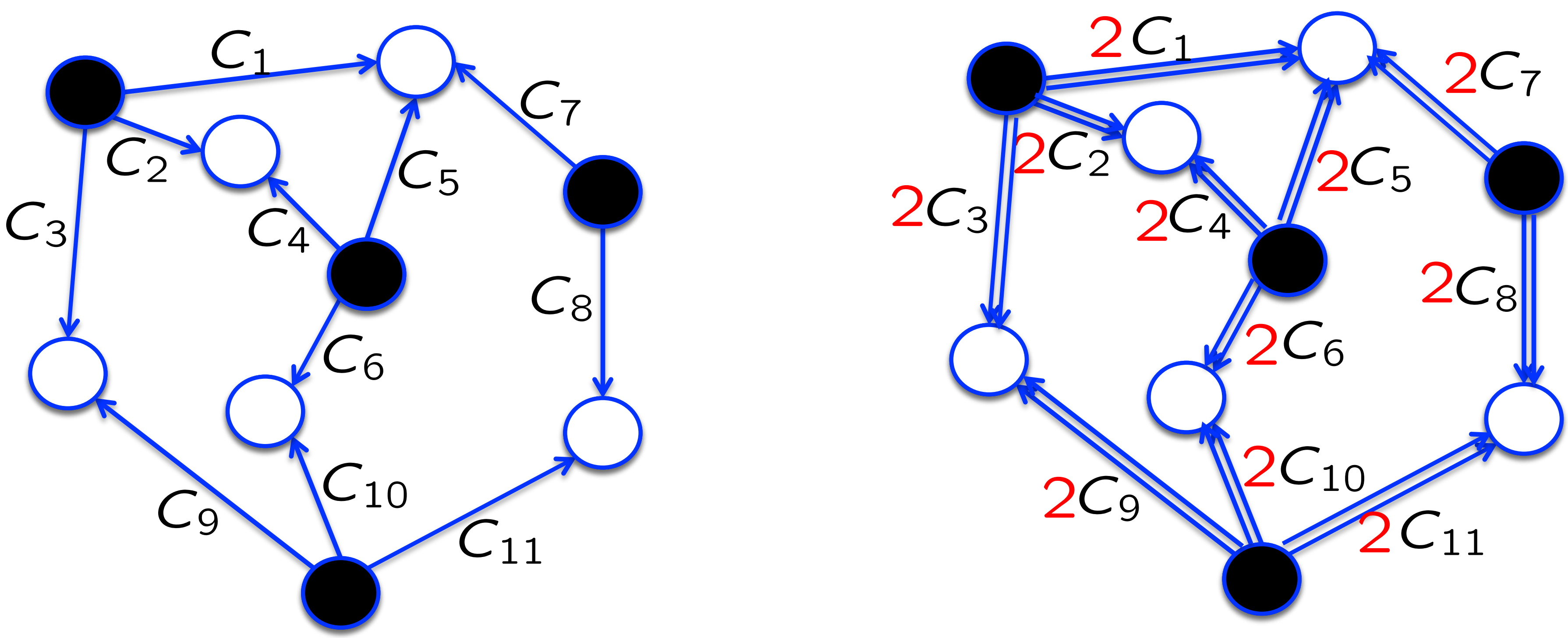
CSIT: $\hat{G}$

Perfect CSIR

$$\text{GDoF} = \lim_{P \to \infty} \frac{\text{Network Capacity}}{\text{Link Capacity}}$$

FAQ: Linear scaling of powers for DoF was understandable.
But what does the exponential scaling of powers mean?

(Networks of Capacitated Links)



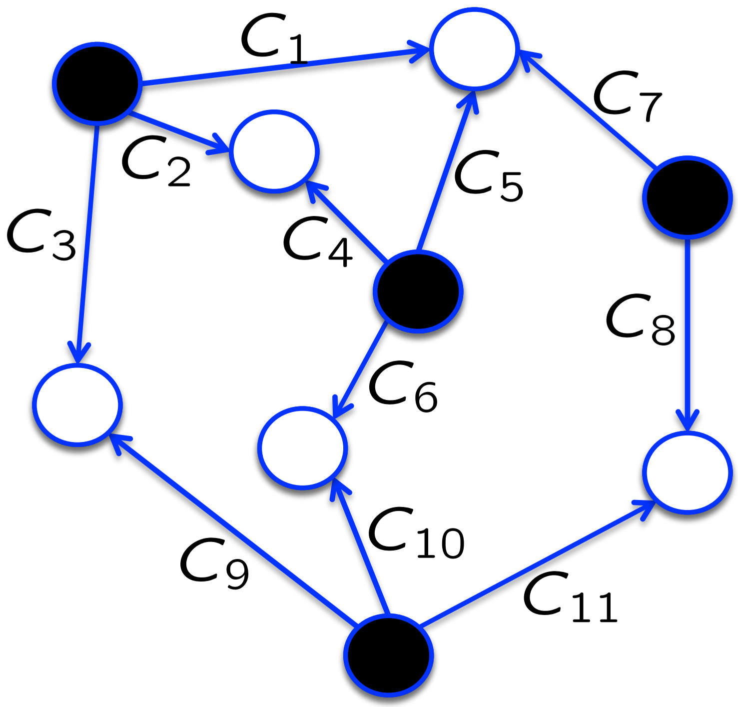
Network Capacity: $\mathcal{C}$

Network Capacity: $\mathcal{C}'$

$$\mathcal{C}' = 2\mathcal{C}$$

If we wanted $\mathcal{C}$, but it was somehow easier to find $\mathcal{C}'$,
then we could find $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}'/2$.

(Networks of Capacitated Links)



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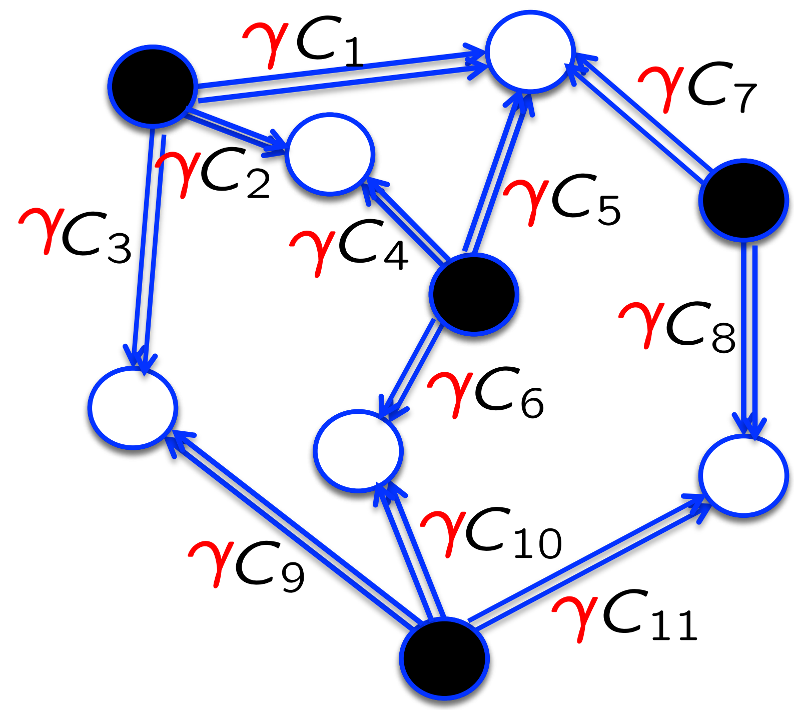
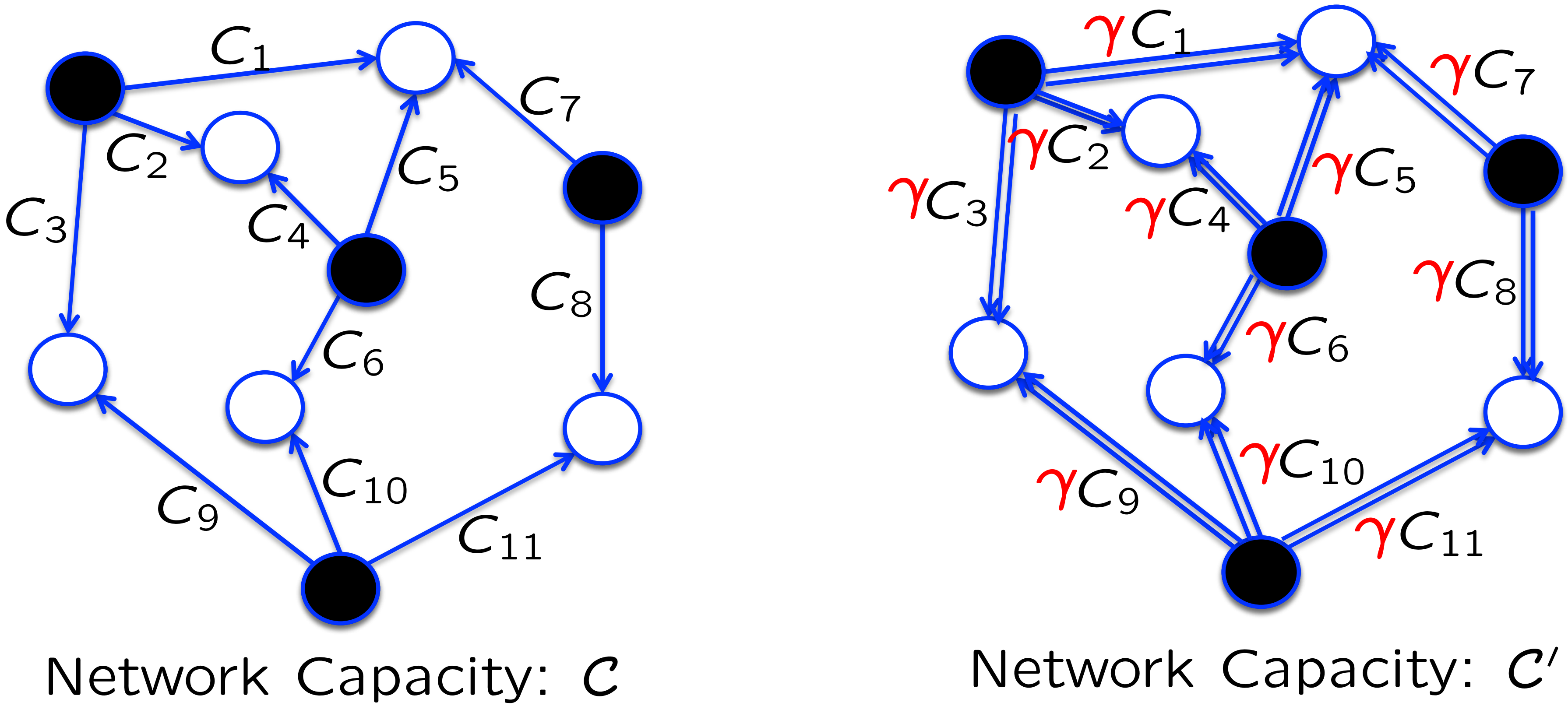


Network Capacity: $\mathcal{C}'$

If we wanted $\mathcal{C}$, but it was somehow easier to find $\mathcal{C}'$,
then we could find $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}'/\gamma$

Intuition extended to Wireless Networks

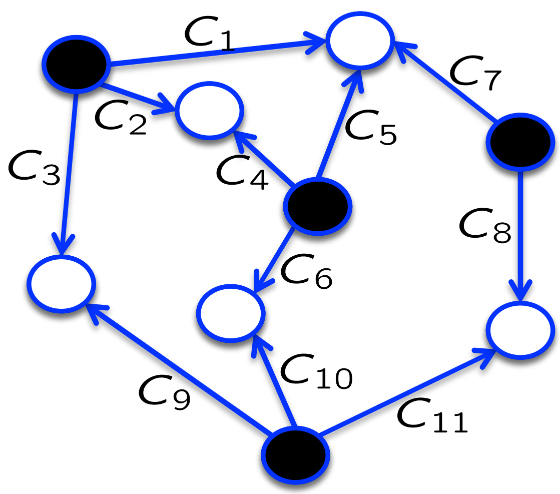


Intuition: $\mathcal{C}' \approx \gamma \mathcal{C}$

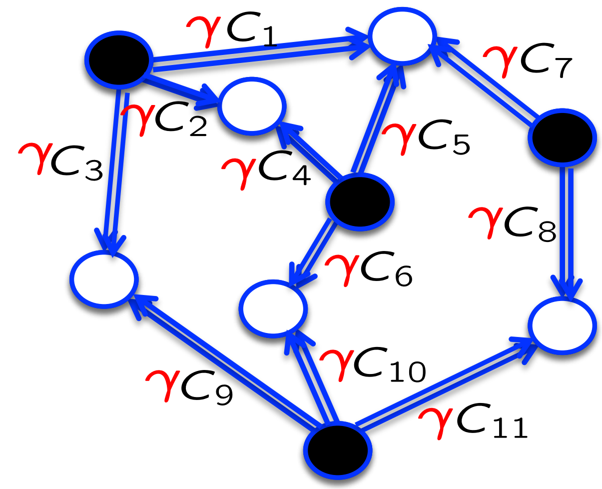
If we wanted $\mathcal{C}$, but it was somehow easier to find $\mathcal{C}'$, then we could find $\mathcal{C}$ **approximately** as $\mathcal{C} \approx \mathcal{C}'/\gamma$

(**Spatial Invariance Conjecture**, Jafar, ITA 2014) Suppose we double the number of antennas at every node (generic channels). Then the network DoF must double as well.

Wireless Network



Network Capacity: $\mathcal{C}$



Network Capacity: $\mathcal{C}'$

$$Y_i = \sqrt{\mathrm{SNR}_i} X_i + N_i$$

Link Capacity $C_i \approx \log(\mathrm{SNR}_i)$

Measure SNR in dB scale
define $\alpha_i = \log_{10}(\mathrm{SNR}_i)$

Define $10^\gamma = P$

$$Y = \sqrt{P^{\alpha_i}} X + N$$

(GDoF Setting)

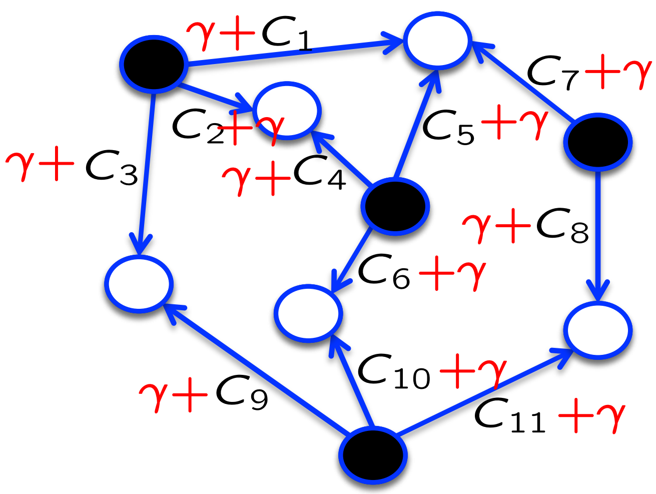
$$\gamma C_i \approx \log(\mathrm{SNR}_i^\gamma)$$

$$\gamma C_i \approx \log(10^{\gamma \alpha_i})$$

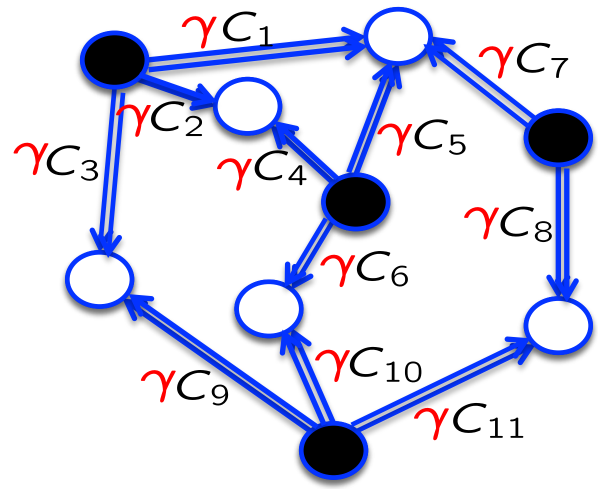
$$\gamma C_i \approx \log(P^{\alpha_i})$$

$$\mathcal{C} \approx \lim_{\gamma \to \infty} \frac{\mathcal{C}'}{\gamma} = \lim_{P \to \infty} \frac{\mathcal{C}'}{\log(P)}$$

DoF versus GDoF



DoF Perspective

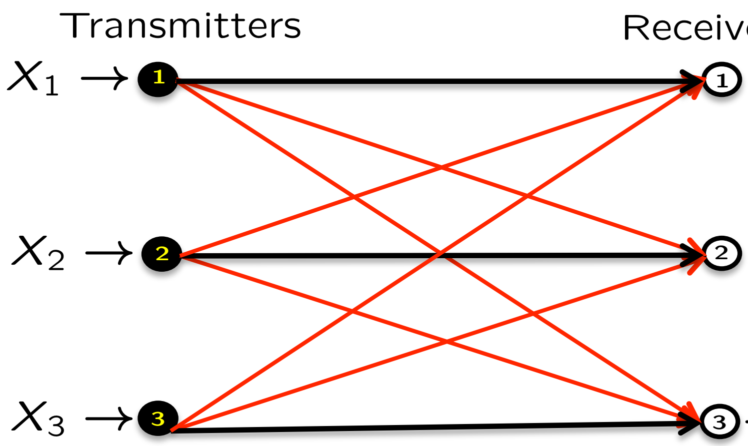


GDoF Perspective

As $\gamma \to \infty$ all channels equally strong relative strengths preserved

DoF metric scales SNR linearly ($C_i \to C_i + \gamma$)
GDoF scales SNR exponentially ($C_i \to \gamma C_i$).

Generalized Degrees of Freedom



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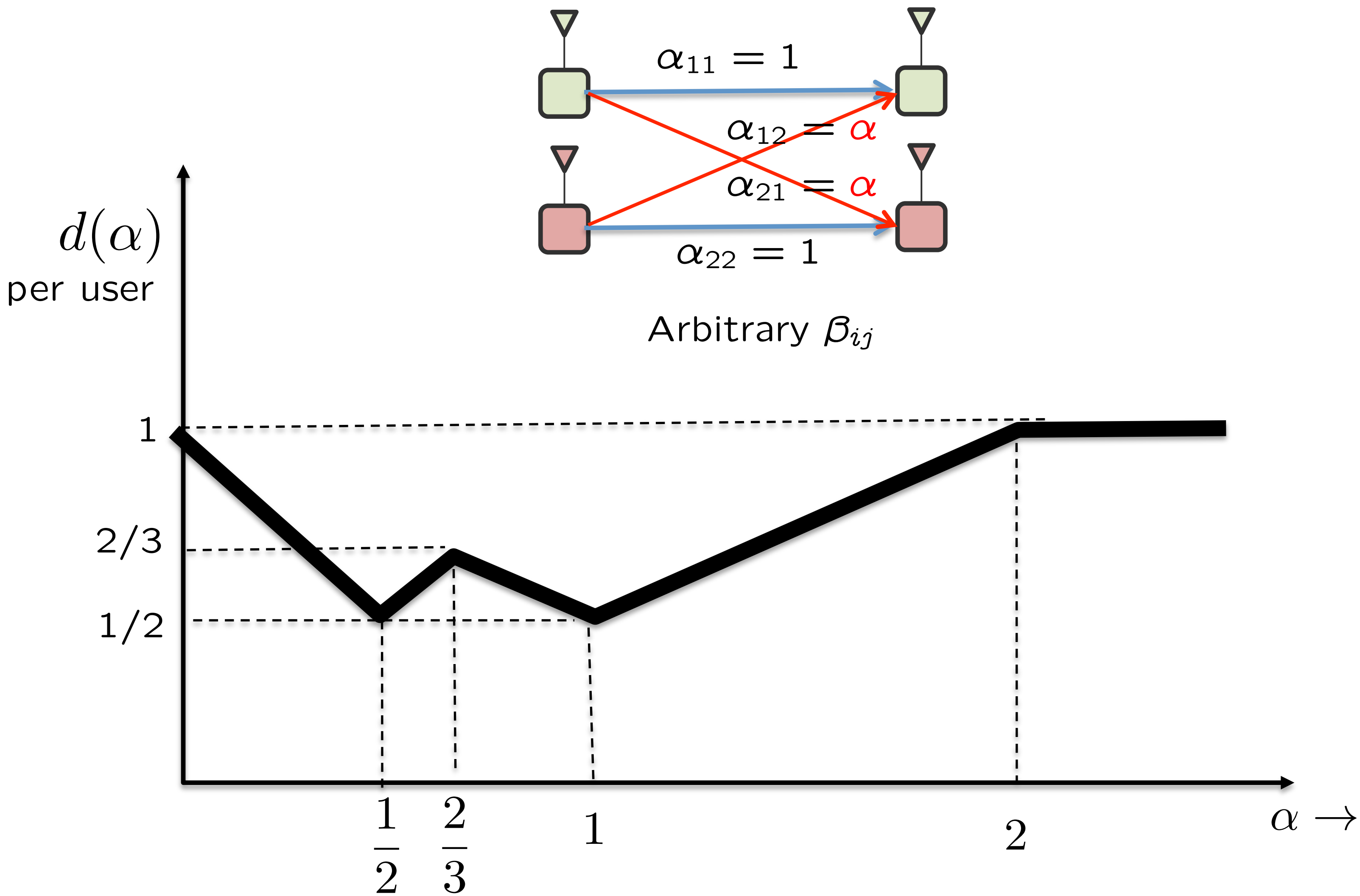
The Next Frontier

GDoF Characterizations under arbitrary levels of channel knowledge, arbitrary levels of channel strengths

Some Examples of Robust GDoF Results

- 2 user interference channel with arbitrary α_{ij}, β_{ij}
- 2 user MISO BC with arbitrary α_{ij}, β_{ij}
- K user IC: Optimality of treating interference as noise
- Topological Interference Management $\equiv$ Index Coding
- Optimality of TDMA/coloring for Topological Interference Management and Index Coding

R. Etkin, D. Tse, H. Wang, **Gaussian Interference Channel Capacity to Within One Bit**, 54(12), IT Trans. 2008.



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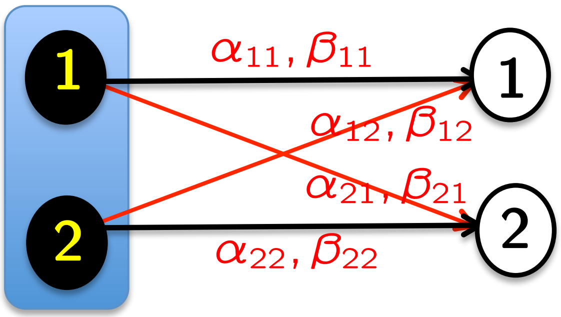
2 User MISO BC

Arbitrary channel strengths and uncertainty levels

[ArXiv 1602.02203, Davoodi, Yuan, Jafar]



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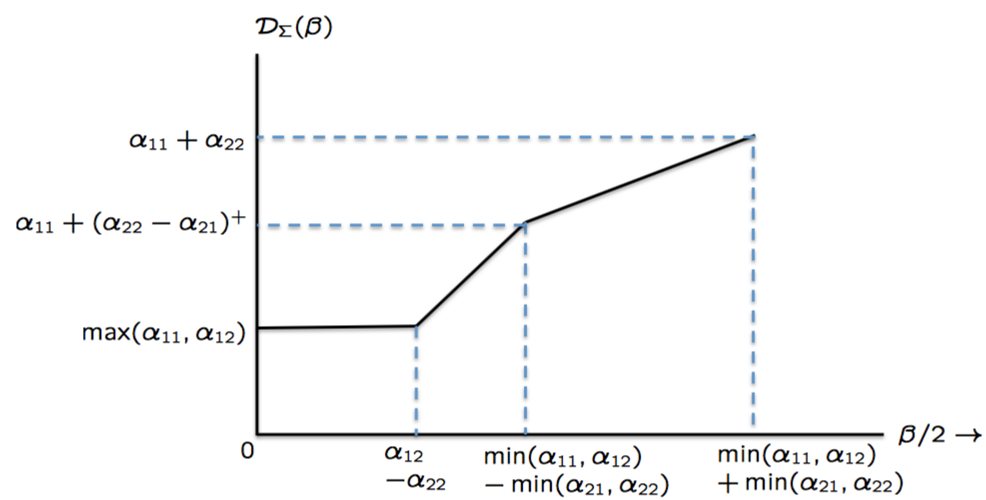
G_{kl} &= \hat{G}_{kl} + \sqrt{P^{-\beta_{kl}}} \tilde{G}_{kl}
\end{aligned}
$$

$$D_\Sigma = \min(D_1, D_2)$$

$$
\begin{aligned}
D_1 &= \max(\alpha_{11}, \alpha_{12}) + \max(\alpha_{21} - \alpha_{11} + \min(\beta_{11}, \beta_{12}), \alpha_{22} - \alpha_{12} + \min(\beta_{11}, \beta_{12}), 0) \\
D_2 &= \max(\alpha_{21}, \alpha_{22}) + \max(\alpha_{11} - \alpha_{21} + \min(\beta_{21}, \beta_{22}), \alpha_{12} - \alpha_{22} + \min(\beta_{21}, \beta_{22}), 0)
\end{aligned}
$$

- Does not depend on strongest CSIT for each receiver.
- Optimal to serve only User 1 iff each Tx antenna prefers User 1 over User 2 by at least β_1. $\alpha_{11} - \alpha_{21} \geq \beta_1$ and $\alpha_{12} - \alpha_{22} \geq \beta_1$
- Sum GDoF vs CSIT budget ($\beta = \sum \beta_{ij}$) when each Tx antennas prefer User 1, and $\alpha_{11} + \alpha_{22} \geq \alpha_{12} + \alpha_{21}$

Aligned Image Sets



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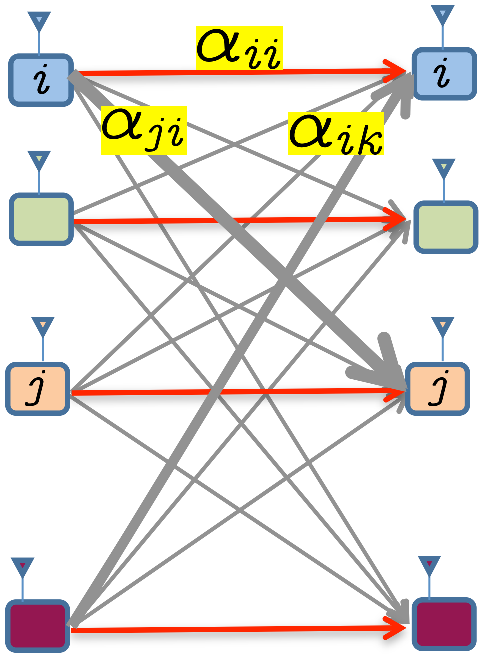
TIN Optimality Condition

Strongest Interference **from** User i

Strongest Interference **to** User i

$$\alpha_{ii} \geq \max_{j:j\neq i}\{\alpha_{ji}\} + \max_{k:k\neq i}\{\alpha_{ik}\}$$



K-user Gaussian IC

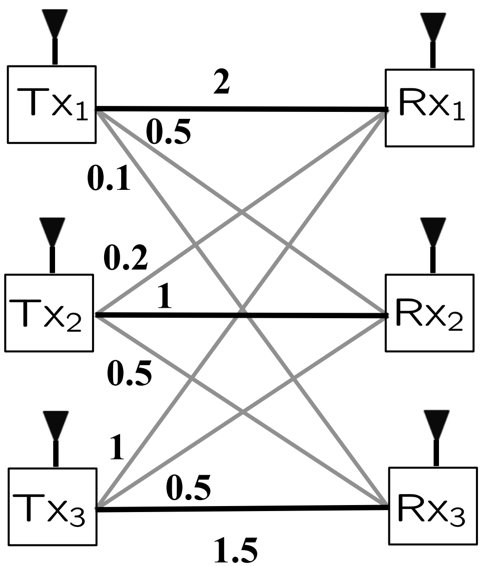
GDoF Region of TIN-optimal IC

Direct Characterization of GDoF Region:

Depends only on **max** strengths,

even though power optimization is fully involved.

(FM elimination of power control variables)



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\end{aligned}
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$$\Pi_K = \{(1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$$

all possible cycles in the channel

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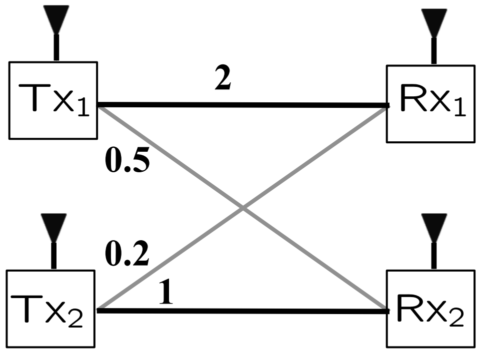

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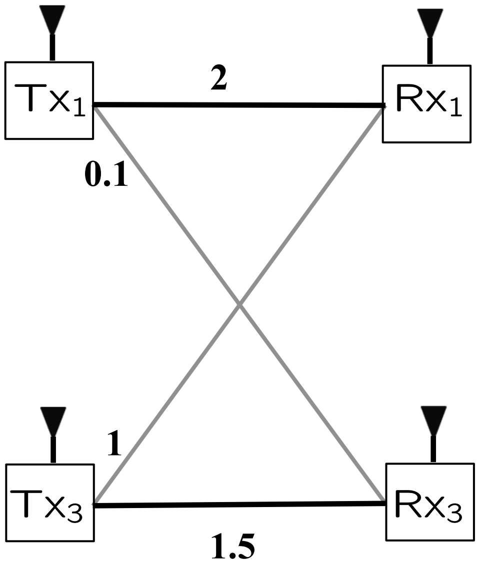

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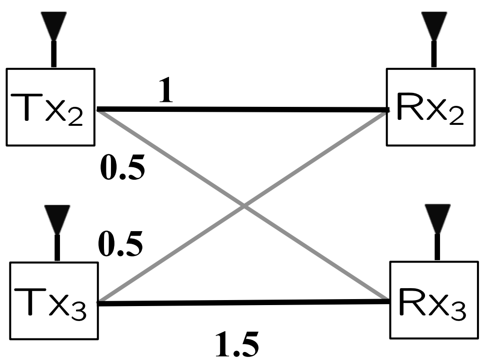

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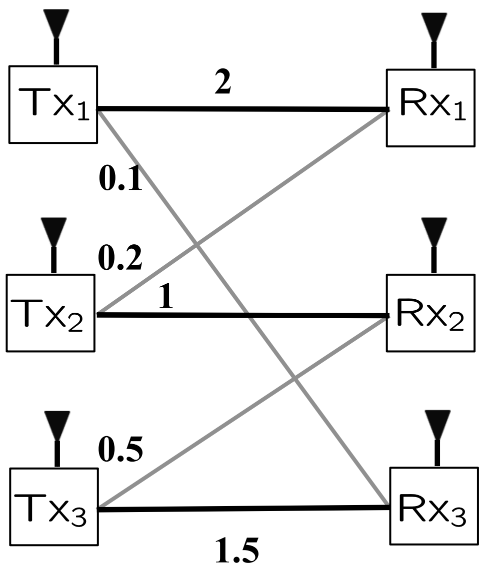

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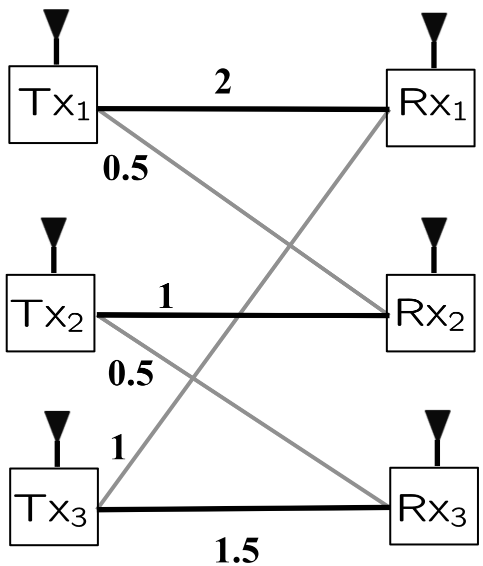

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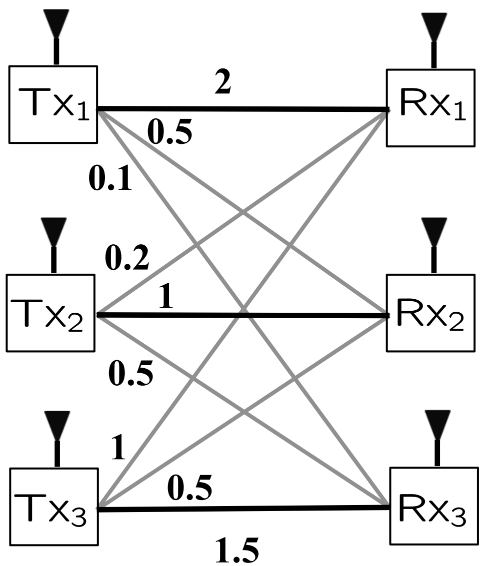

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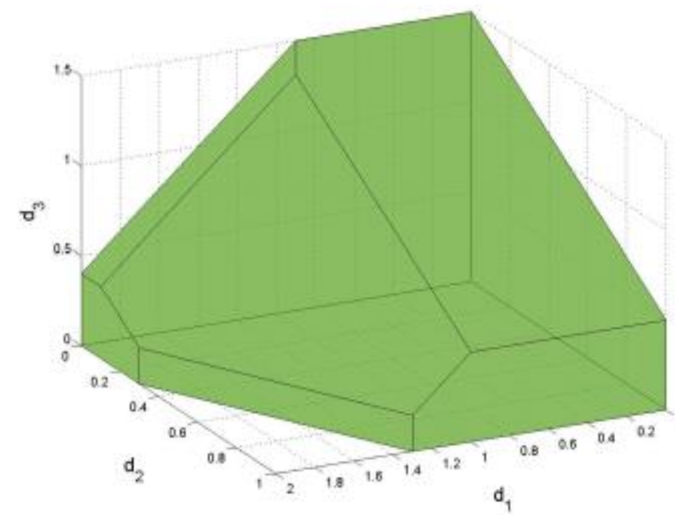

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Naderializadeh, Avestimehr, **ITLinQ: A New Approach for Spectrum Sharing in D2D Communication Systems**, JSAC 2014.

Yi, Caire, **Optimality of treating interference as noise: A combinatorial perspective**, TIT 62(8), 2016.

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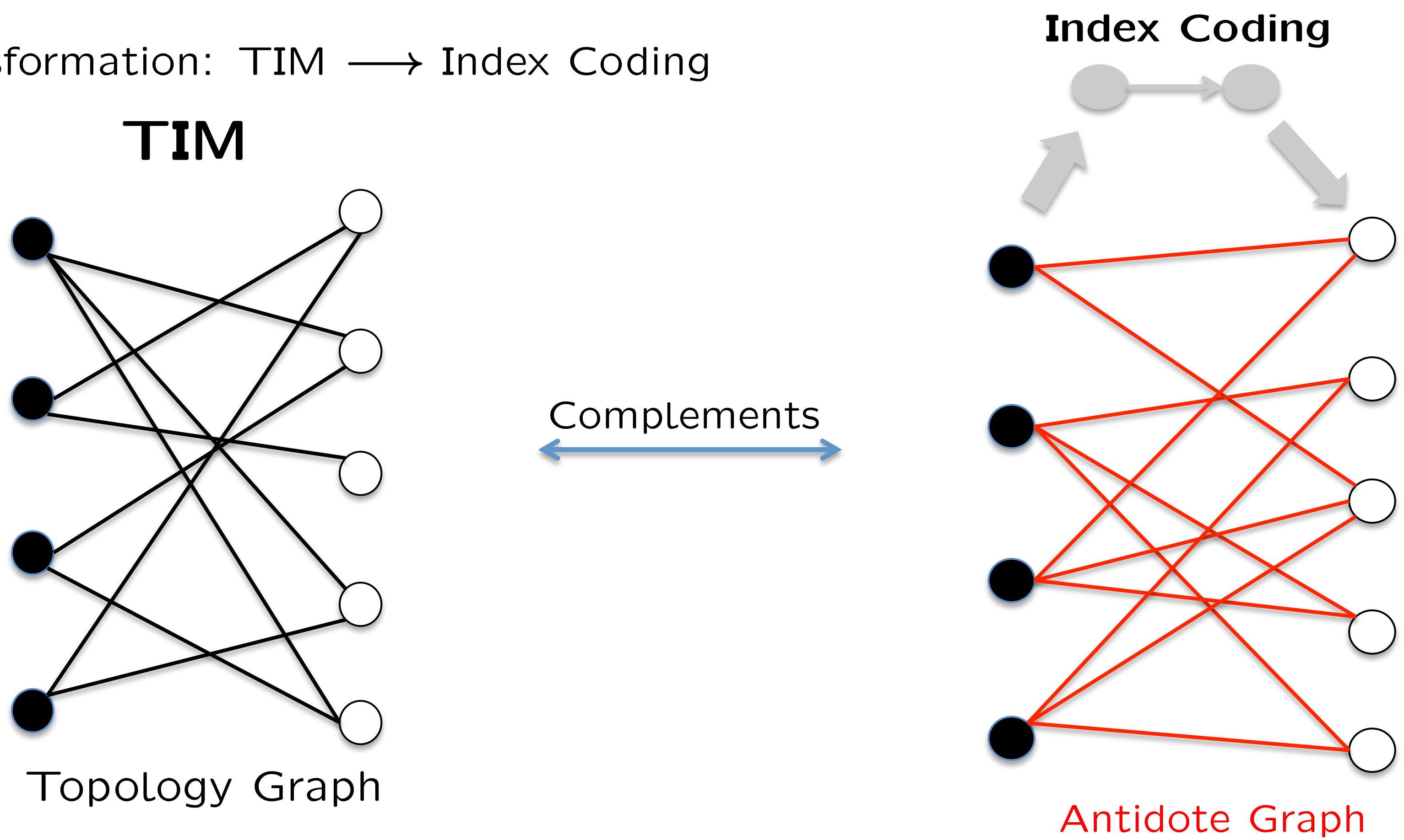
Topological Interference Management

[Jafar, IT Trans. 2014]

$\alpha_{ij} \in \{0, 1\}, \beta_{ij} = 0$

Transformation: TIM $\longrightarrow$ Index Coding



Given

The topology graph is the complement of the antidote graph

Arbitrary Multiple Unicast Message Set (each message has unique source, unique destination)

1. Index Coding Capacity Region includes TIM DoF region.

2. Equivalent for linear schemes.

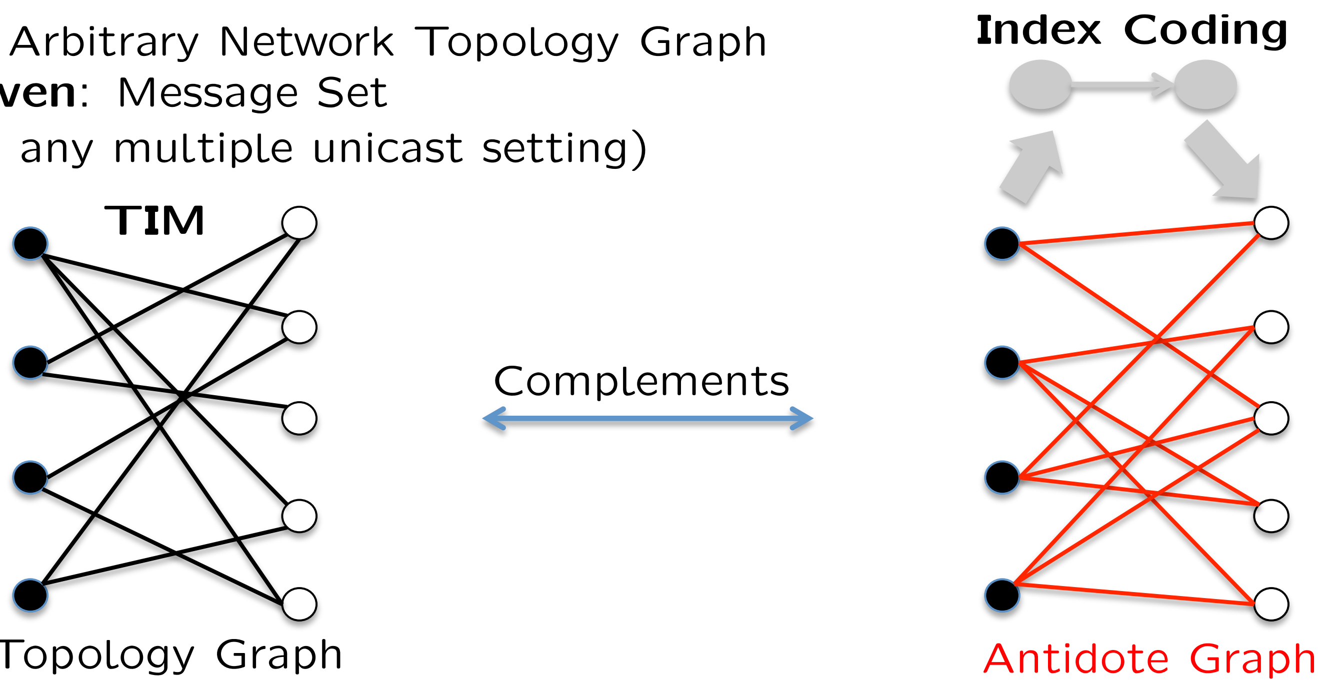
Some Examples of Robust GDoF Results

- 2 user interference channel with arbitrary α_{ij}, β_{ij}
- 2 user MISO BC with arbitrary α_{ij}, β_{ij}
- K user IC: Optimality of treating interference as noise
- Topological Interference Management $\equiv$ Index Coding
- Optimality of TDMA/coloring for
  Topological Interference Management and Index Coding

Optimality of TDMA (Fractional Coloring) for TIM (Index Coding)

[Yi, Sun, Jafar, Gesbert, ArXiv:1501.07870]

Given: Arbitrary Network Topology Graph
Not Given: Message Set
(can be any multiple unicast setting)



Is TDMA optimal?

Is Fractional Coloring Optimal?

Optimal for every possible multiple **unicast** message set.
Optimal for entire DoF/capacity **region**.

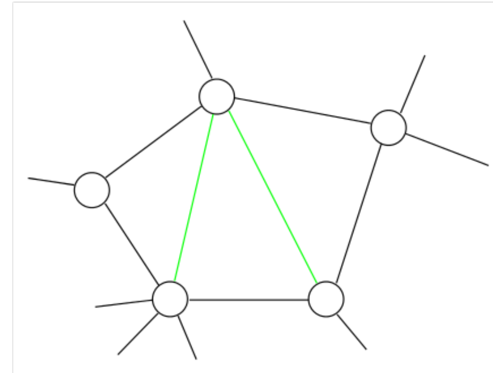
Answer for both TIM and Index Coding:

Yes, if and only if network topology graph is **chordal bipartite**.

https://en.wikipedia.org/wiki/Chordal_graph

Chordal Graphs

Every cycle that can have a chord, must have a chord.

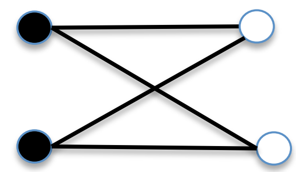


A graph is **chordal** if
it has no chordless cycles of length ≥ 4.

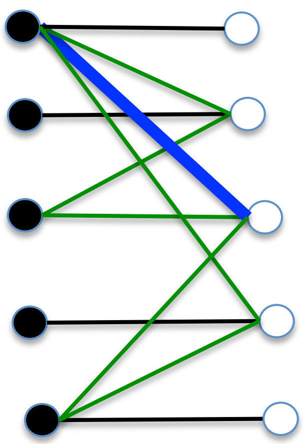
Network Topology graphs are Bipartite Graphs.

What about bi-partite graphs?

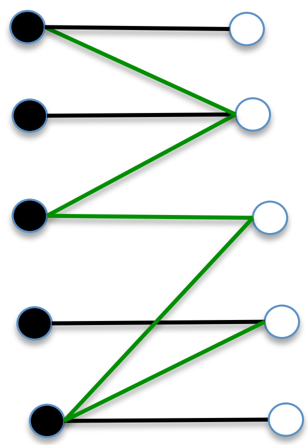
Cycles cannot have odd length.
Cycles of length 4 cannot have chords.



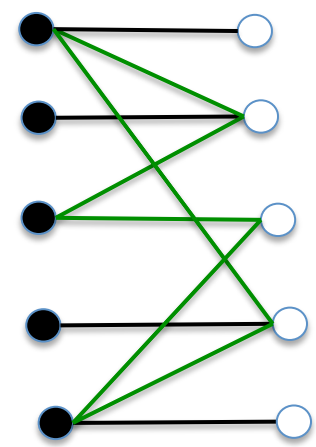
A bipartite graph is **chordal bipartite** if
it has no chordless cycles of length ≥ 6.



Chordal bipartite.



Chordal bipartite.

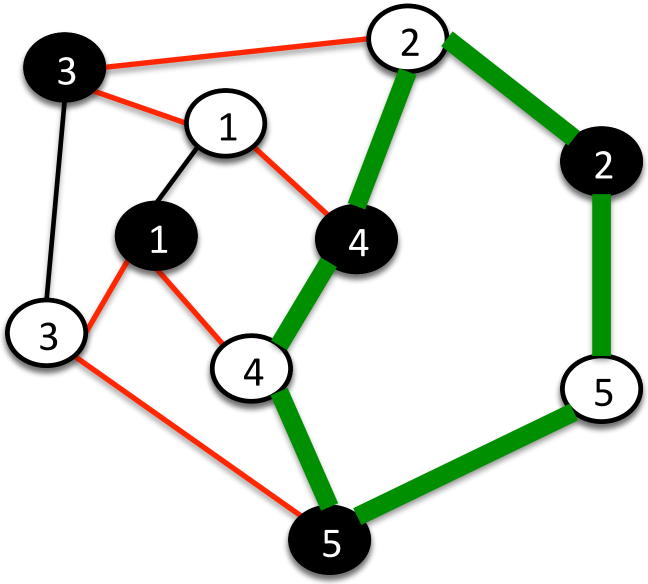


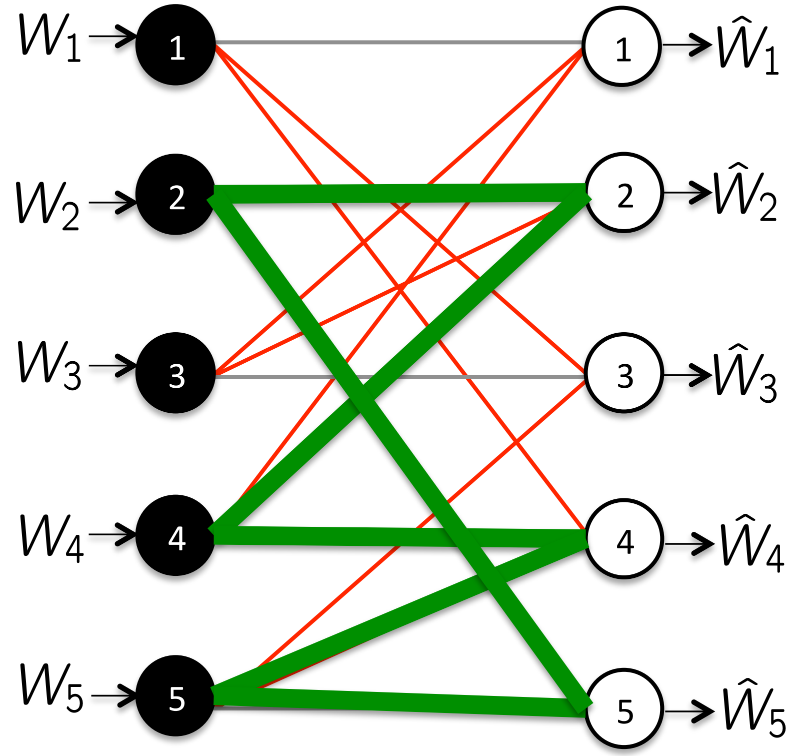
Not chordal bipartite.

TDMA/fractional-coloring **optimal**
for **all** message sets.

TDMA/fractional-coloring **sub-optimal**
for **some** message sets.

Sanity Check



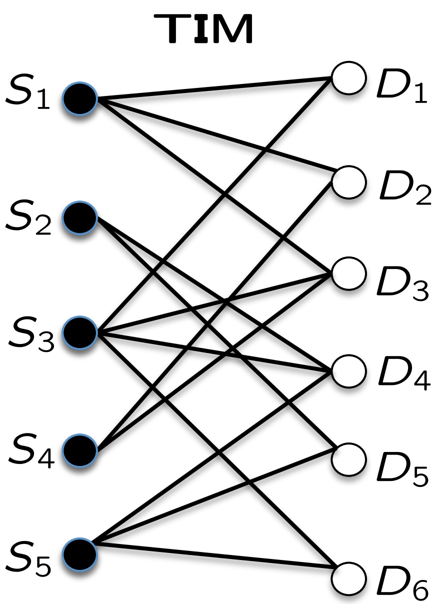


Recall that TDMA is suboptimal.

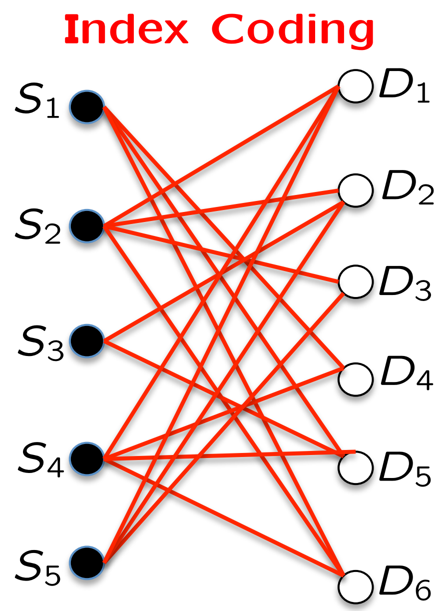
Cannot be Chordal Bipartite.

Must have a chordless cycle of length ≥ 6

Applying the Result



Topology Graph



Antidote Graph

Topology Graph is Chordal Bipartite ✔

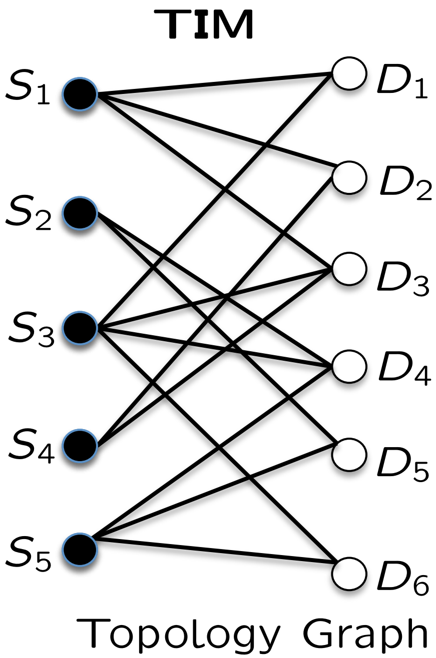
$\Rightarrow$ For any unicast message set

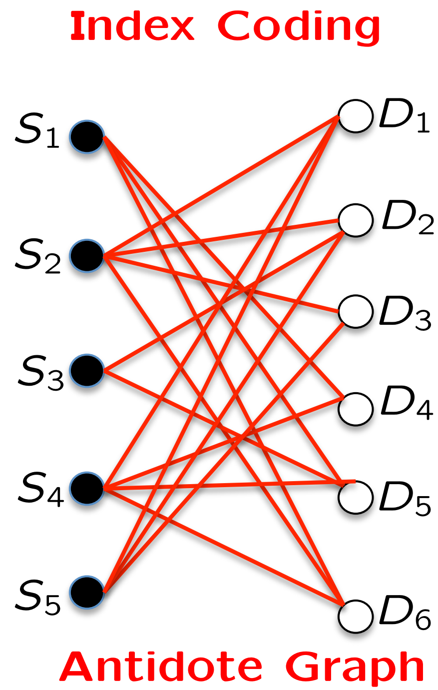
- TDMA achieves TIM DoF region.
- Fractional Coloring Achieves Index Coding Capacity Region.

Consider the messages: $W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{63}$.

What is the DoF/Capacity Region?

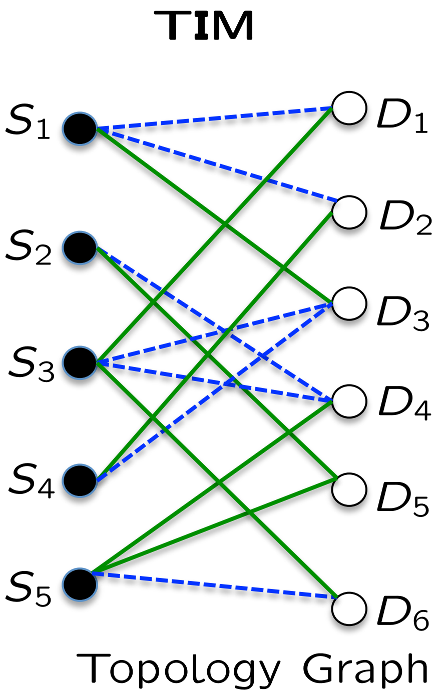
DoF/Capacity Region

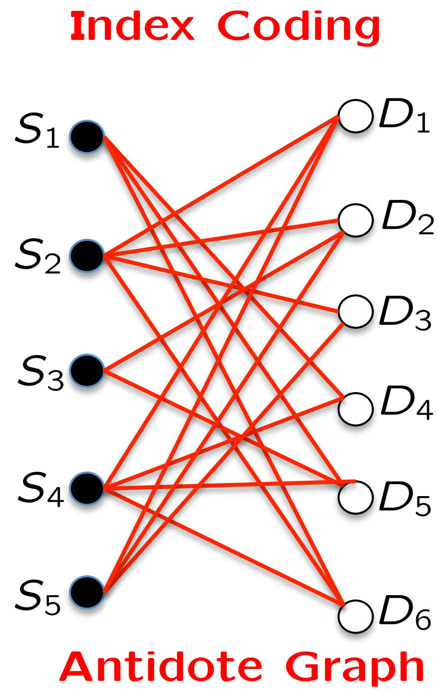




Messages: $W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{63}$

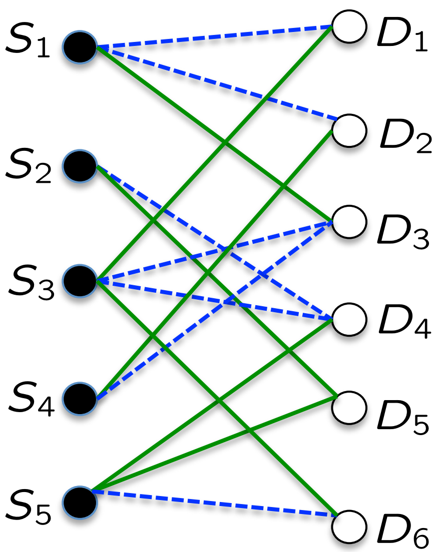
DoF/Capacity Region





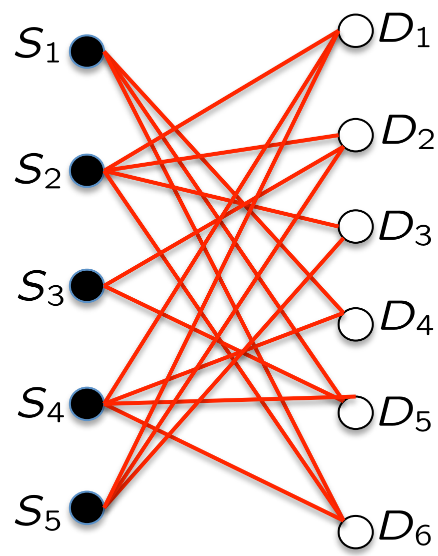
Messages: $W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{63}$

TIM



Topology Graph

Index Coding

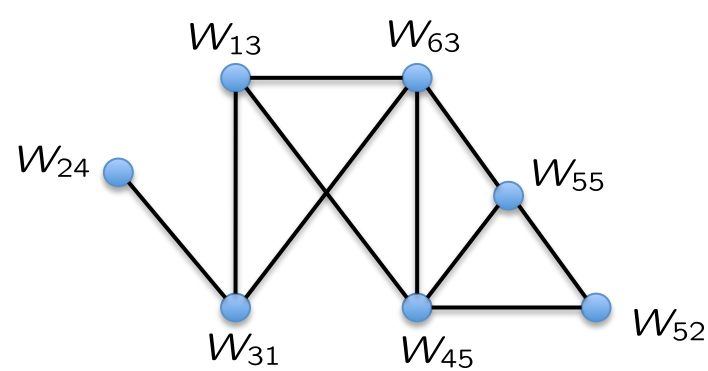


Antidote Graph

Messages: $W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{63}$

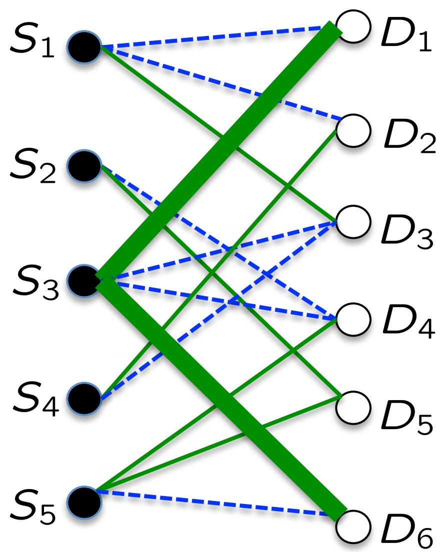
Message Conflict Graph

Messages conflict if
they come from the same source,
are intended for the same destination,
or if the source of one interferes
with the destination of the other



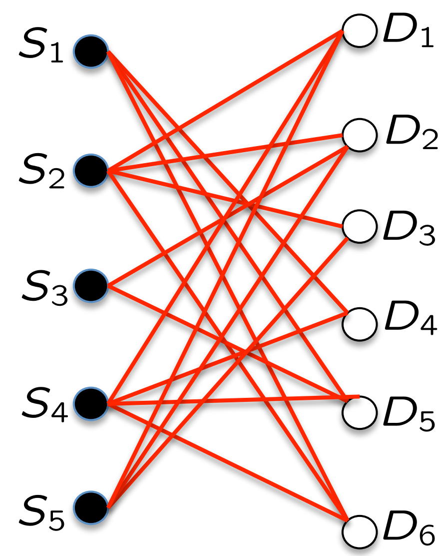
Message Conflict Graph

TIM



Topology Graph

Index Coding



Antidote Graph

Messages: $W_{13}, W_{24}, W_{31}, W_{45}, W_{52}, W_{55}, W_{63}$

Message Conflict Graph

Messages conflict if
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or if the source of one interferes
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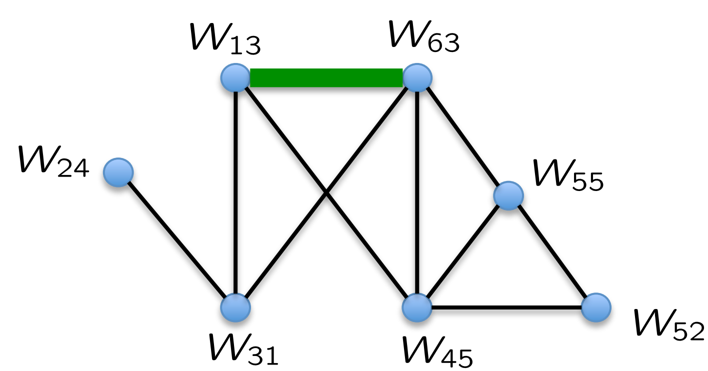
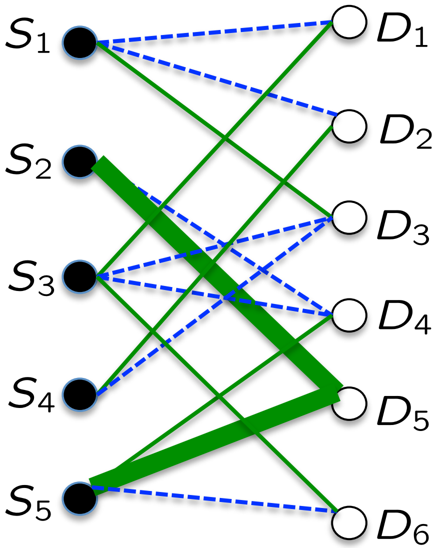


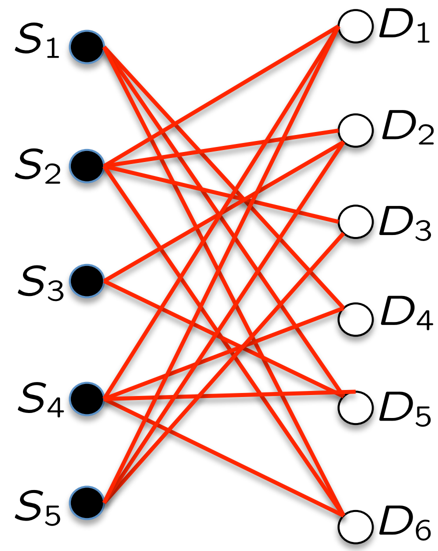
Message Conflict Graph

TIM



Topology Graph

Index Coding



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Messages conflict if
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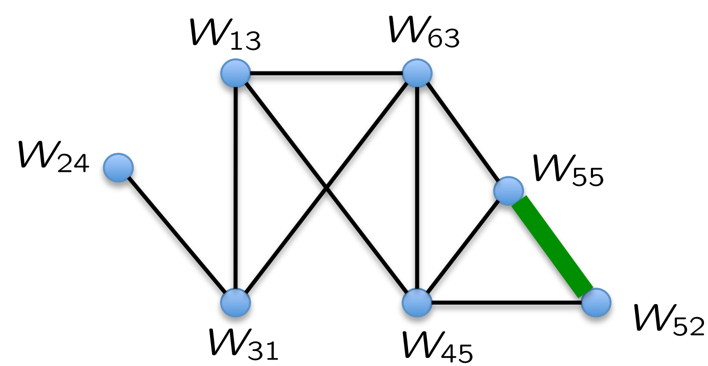
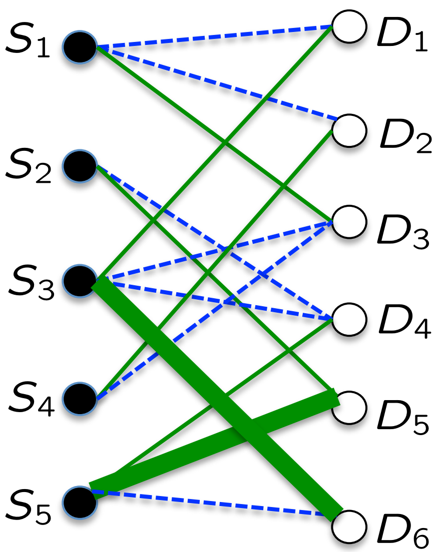


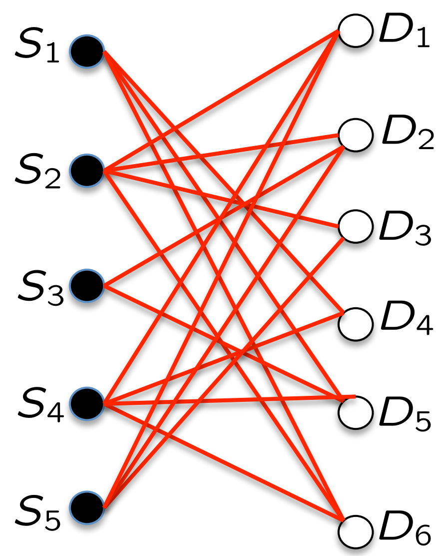
Message Conflict Graph

TIM



Topology Graph

Index Coding



Antidote Graph

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Message Conflict Graph

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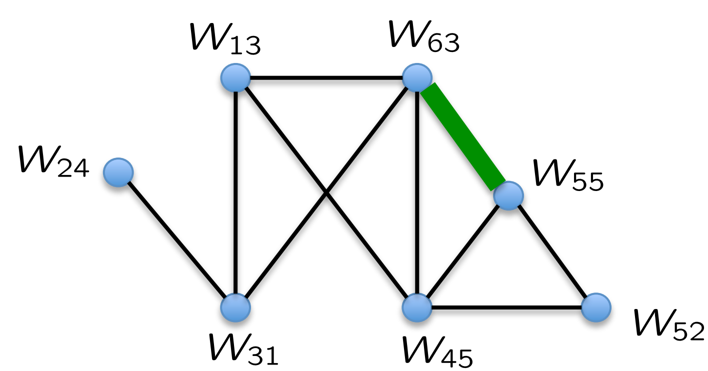
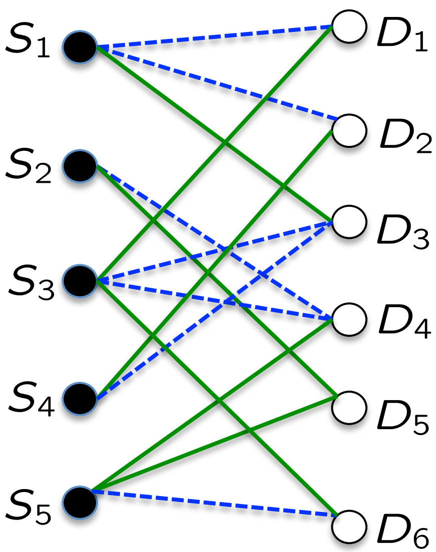


Message Conflict Graph

DoF/Capacity Region

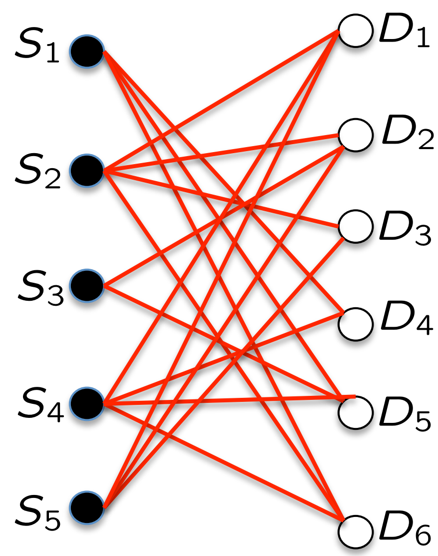
(Clique Inequalities)

TIM



Topology Graph

Index Coding



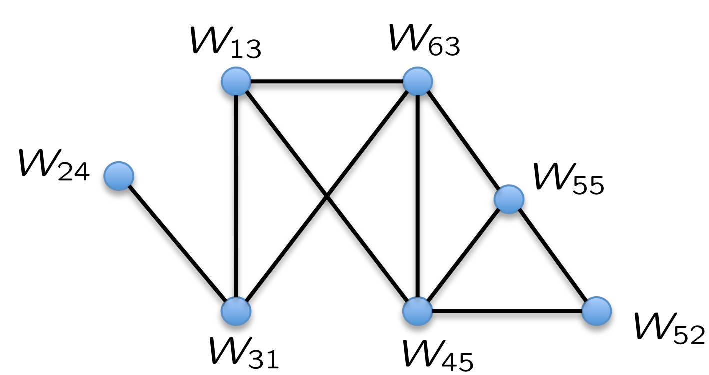
Antidote Graph

TIM DoF Region

$$\begin{aligned}
d_{24} + d_{31} &\leq 1 \\
d_{13} + d_{31} + d_{63} &\leq 1 \\
d_{13} + d_{45} + d_{63} &\leq 1 \\
d_{45} + d_{55} + d_{63} &\leq 1 \\
d_{45} + d_{55} + d_{52} &\leq 1
\end{aligned}$$

Index Coding Capacity Region

$$\begin{aligned}
R_{24} + R_{31} &\leq 1 \\
R_{13} + R_{31} + R_{63} &\leq 1 \\
R_{13} + d_{45} + R_{63} &\leq 1 \\
R_{45} + R_{55} + R_{63} &\leq 1 \\
R_{45} + R_{55} + R_{52} &\leq 1
\end{aligned}$$

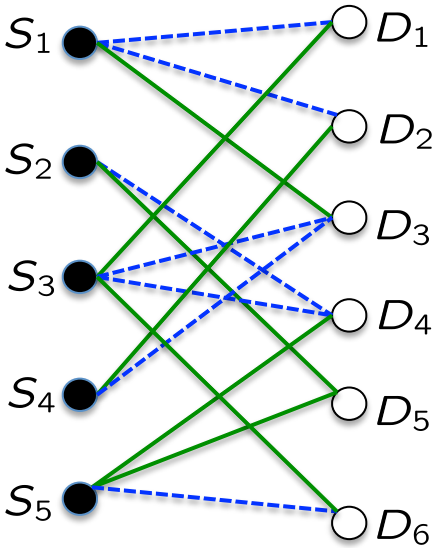


Message Conflict Graph

DoF (Capacity) Region (Clique Inequalities)
For each clique of message conflict graph, sum of DoF (rates) ≤ 1.

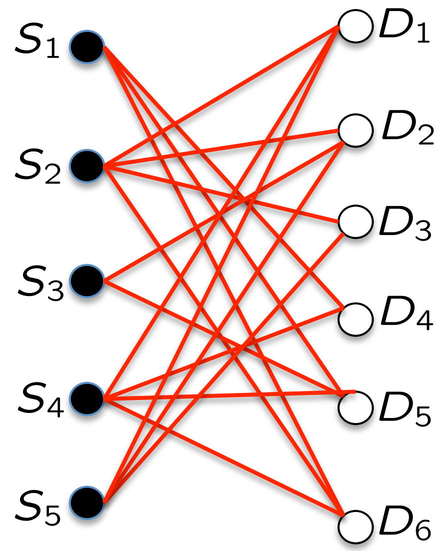
DoF/Capacity Region

TIM



Topology Graph

Index Coding



Antidote Graph

TIM

If the Topology Graph
is chordal-bipartite
then for any multiple unicast message set
the DoF region
is achieved by TDMA
and is described by the clique inequalities
of the message conflict graph.

Index Coding

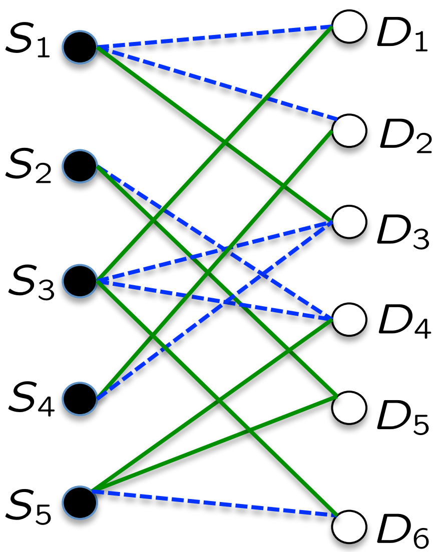
If the complement of the antidote graph
is chordal-bipartite
then for any multiple unicast message set
the Capacity region
is achieved by Fractional Coloring
and is described by the clique inequalities
of the message conflict graph.

Next: Proof.

Demand Graph

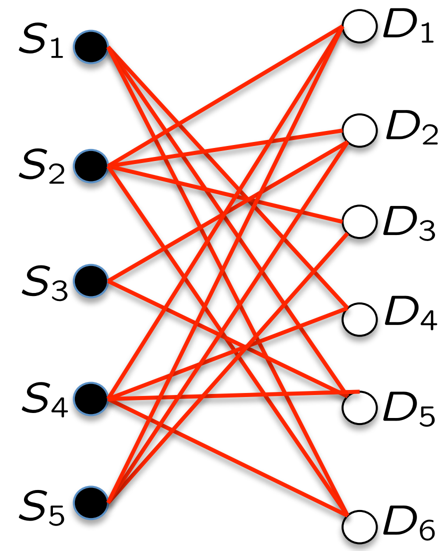
[Neely, Tehrani, Zhang, IT Trans. 2013]

TIM



Topology Graph

Index Coding



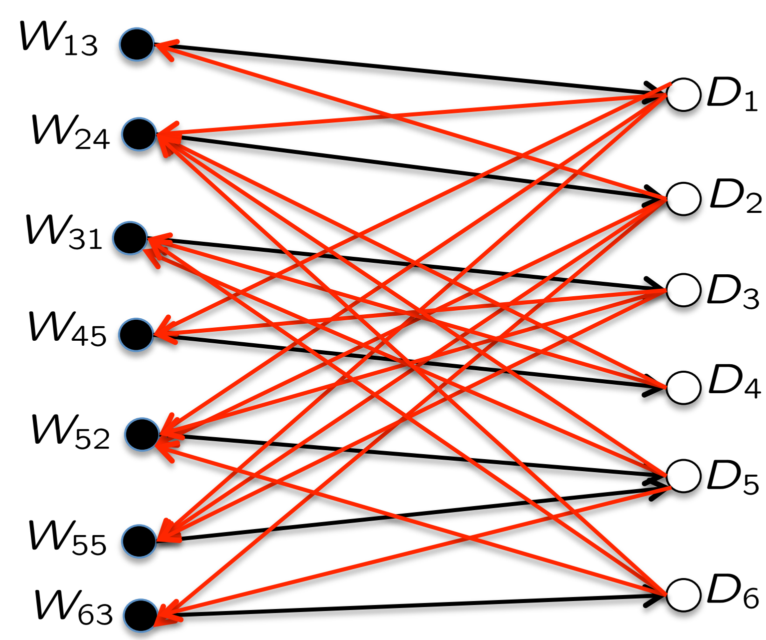
Antidote Graph

Demand Graph

W_{ij} on the left
D_i on the right
Edge from W_{ij} to its D_i
Edge from D_i to W_{jk}
if D_i is not connected to S_k.

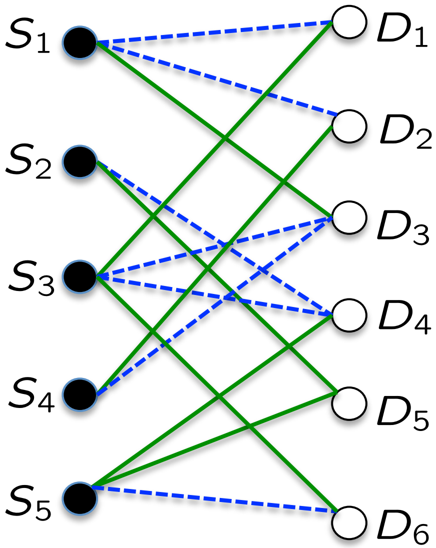
Acyclic Demand Graph Bound

If Demand Graph is acyclic for
a subset of messages,
then the sum-rate/DoF
for that set of messages is ≤ 1.



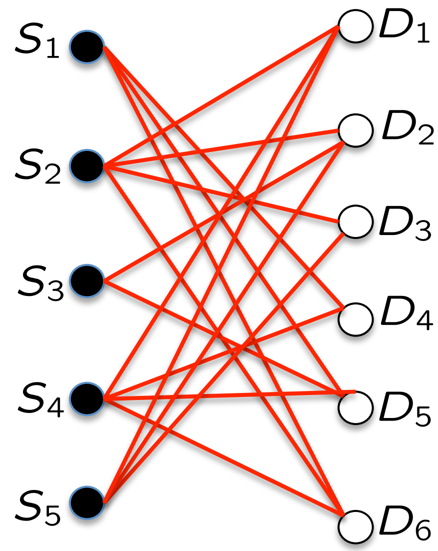
Proof of Converse (Outer Bound)

TIM



Topology Graph

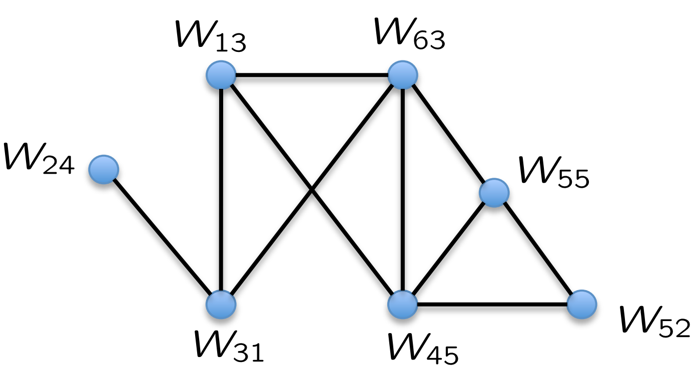
Index Coding



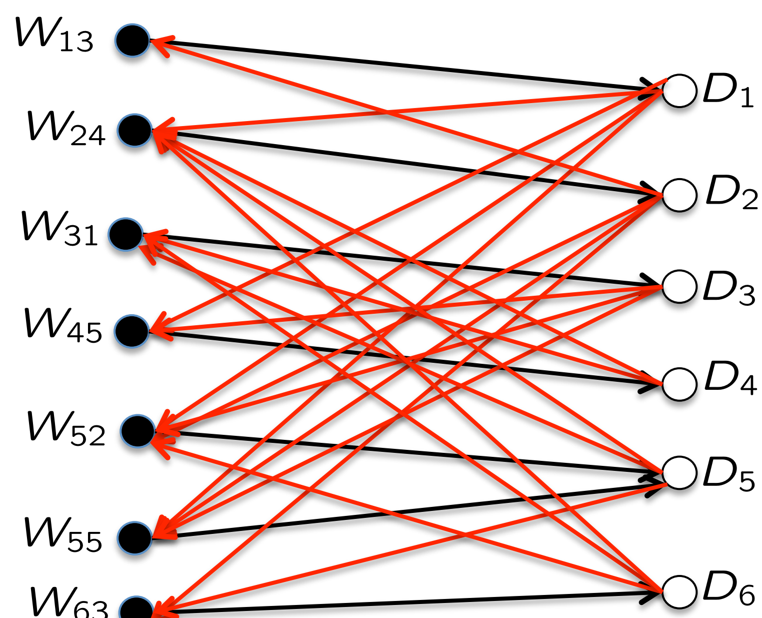
Antidote Graph

Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.



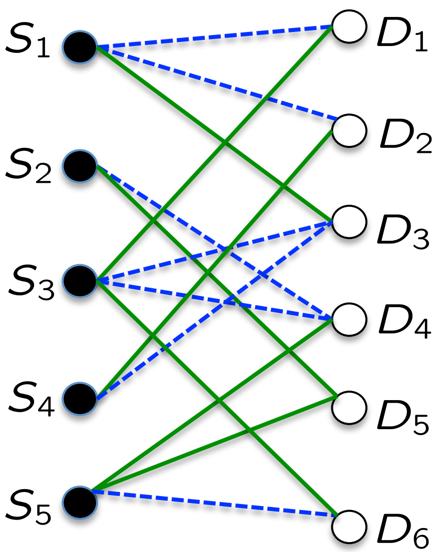
Message Conflict Graph



Demand Graph

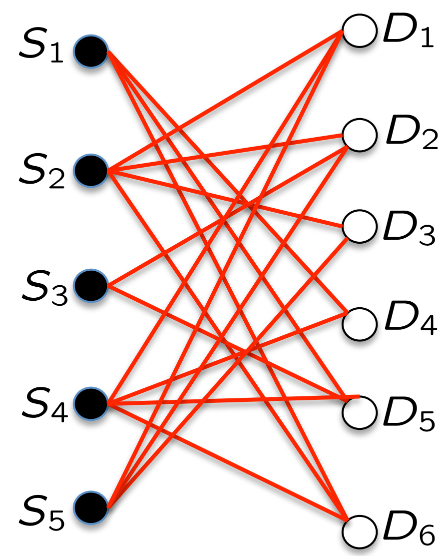
Proof of Converse (Outer Bound)

TIM



Topology Graph

Index Coding



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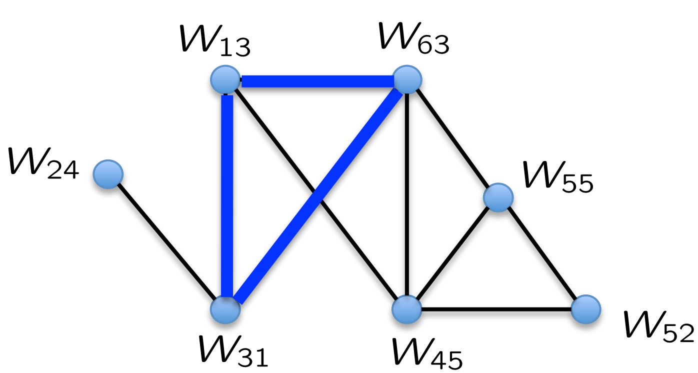
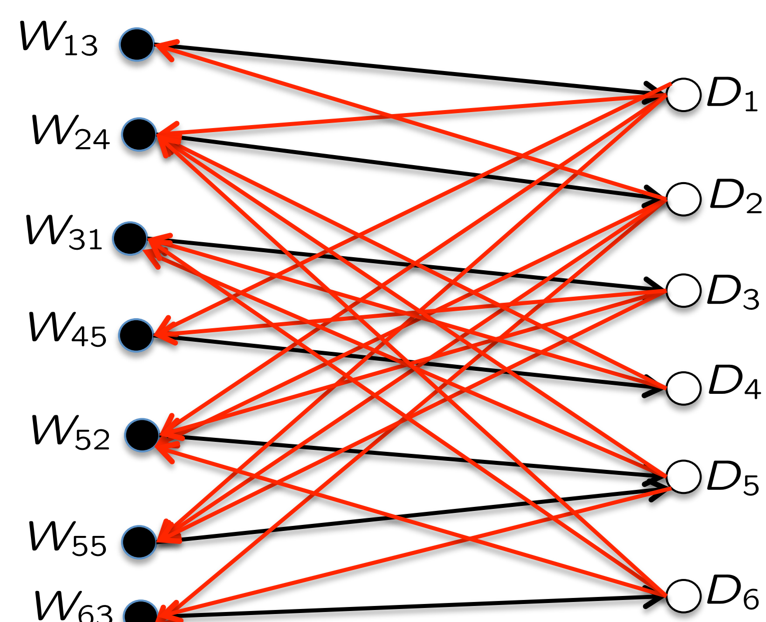
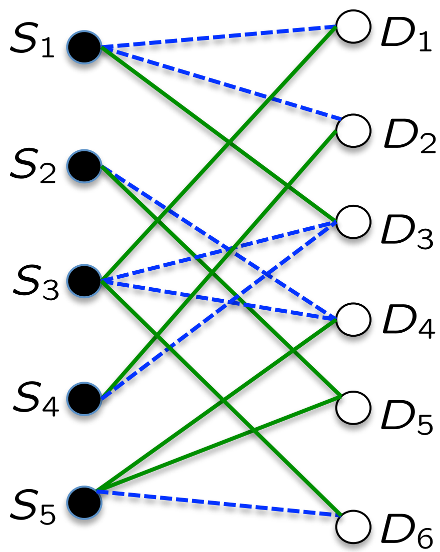


Message Conflict Graph



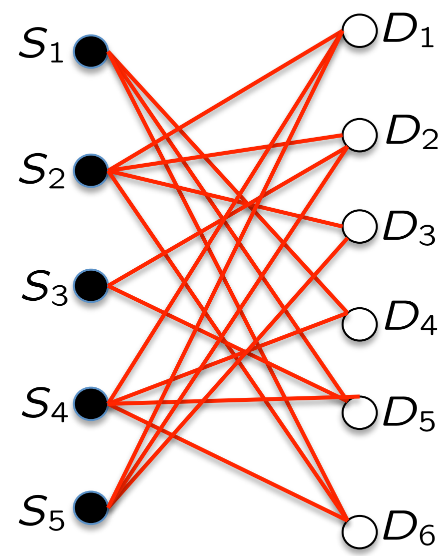
Demand Graph

TIM



Topology Graph

Index Coding



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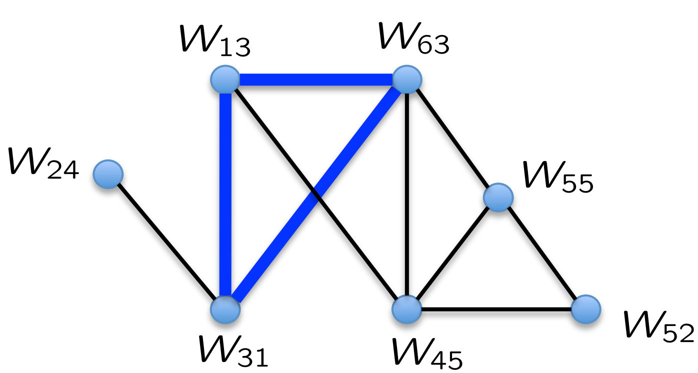
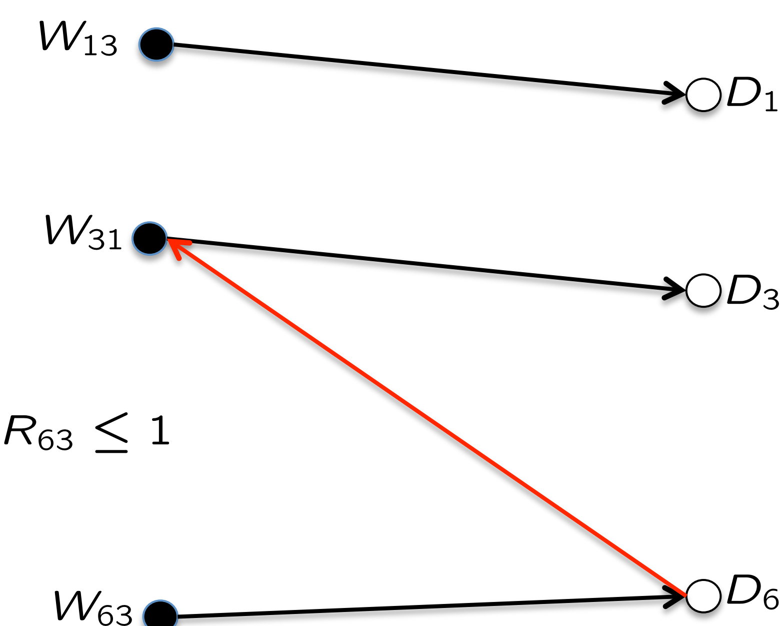


Message Conflict Graph

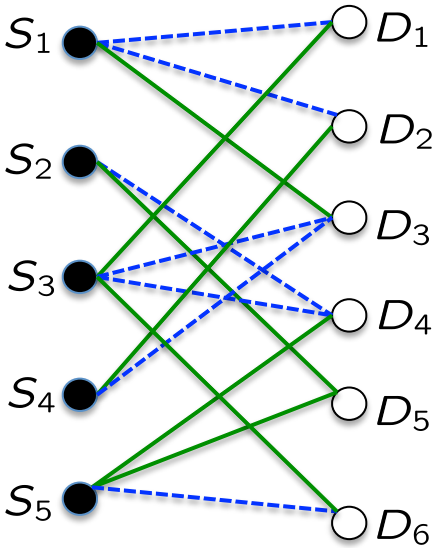
$$R_{13} + R_{31} + R_{63} \leq 1$$



Demand Graph

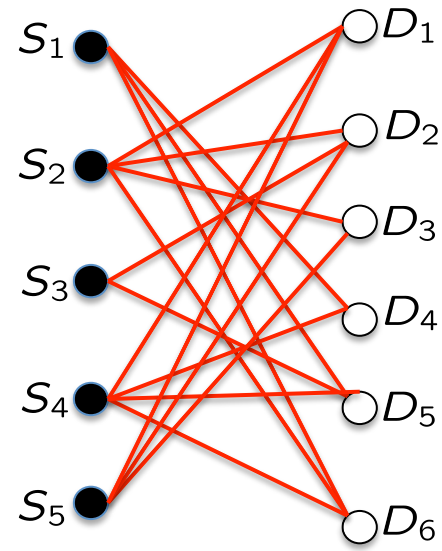
Proof of Converse (Outer Bound)

TIM



Topology Graph

Index Coding



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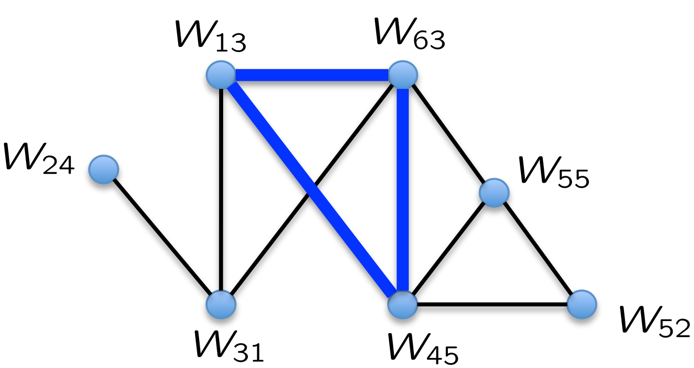
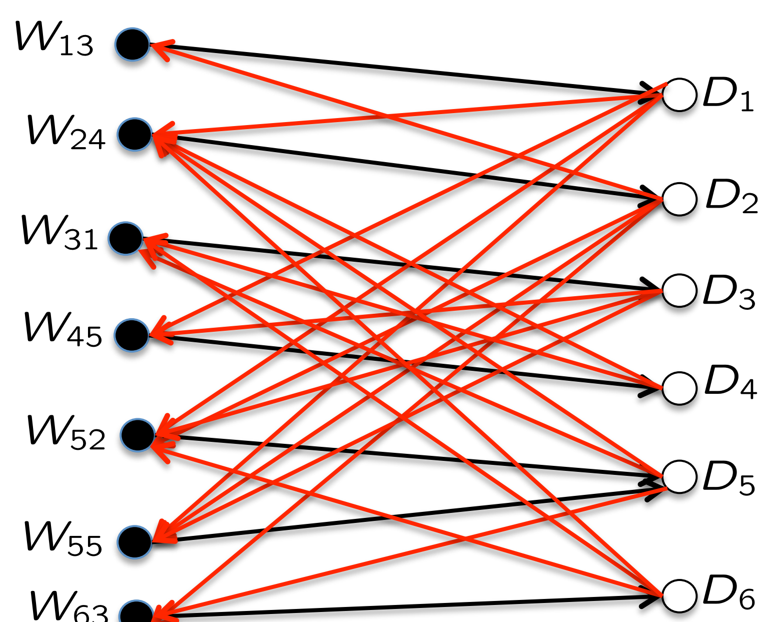
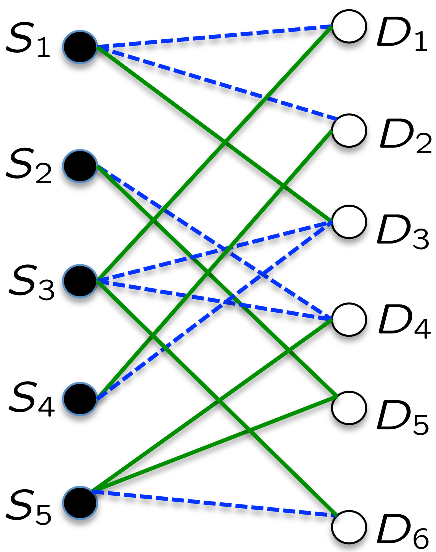


Message Conflict Graph



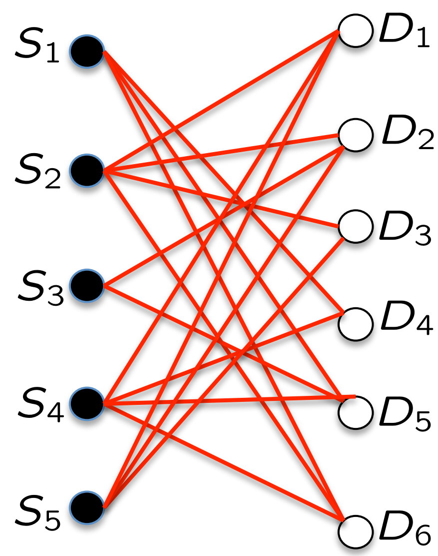
Demand Graph

TIM



Topology Graph

Index Coding



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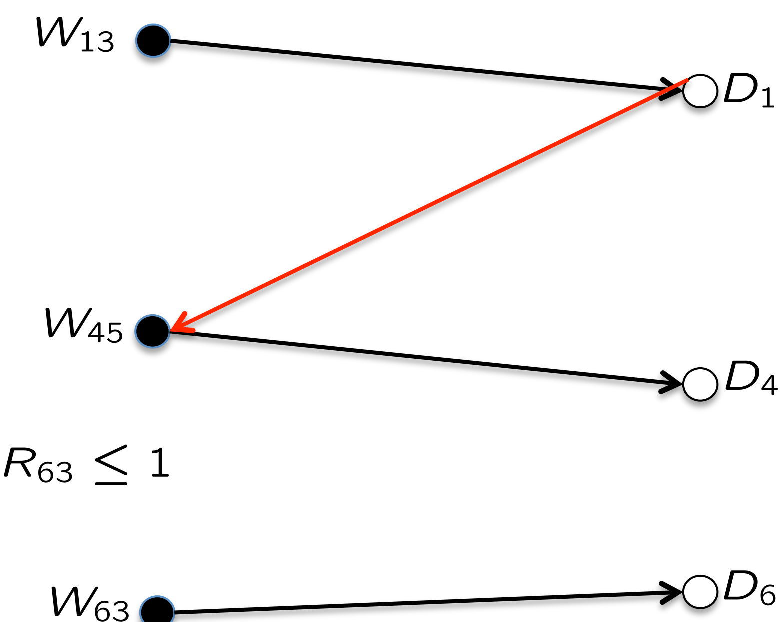
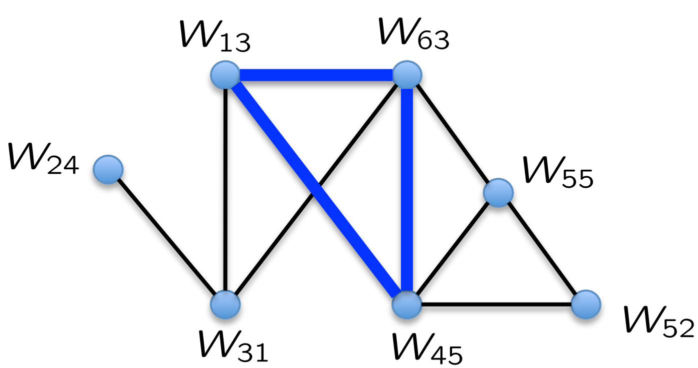


Demand Graph

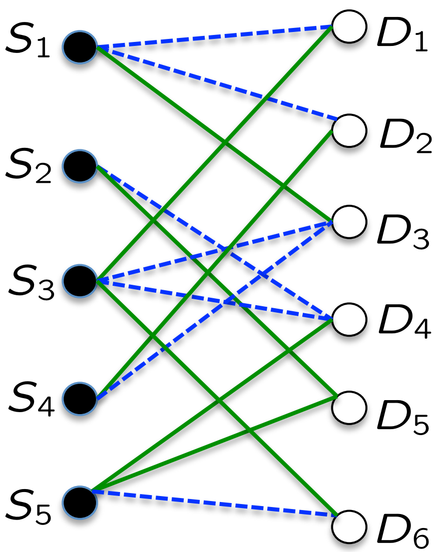


Message Conflict Graph

$$R_{13} + R_{45} + R_{63} \leq 1$$

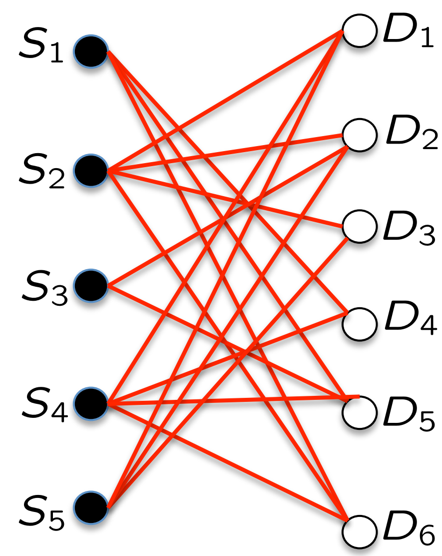
Proof of Converse (Outer Bound)

TIM



Topology Graph

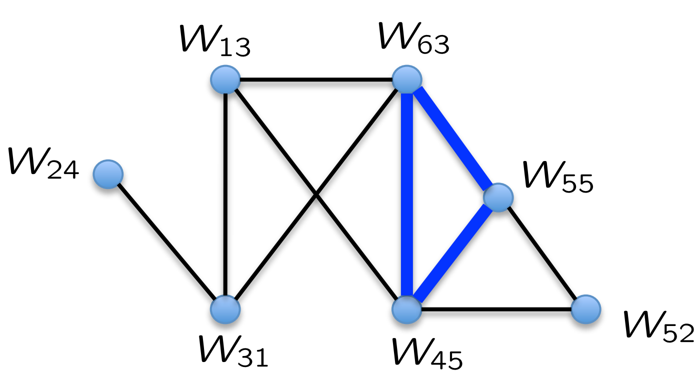
Index Coding



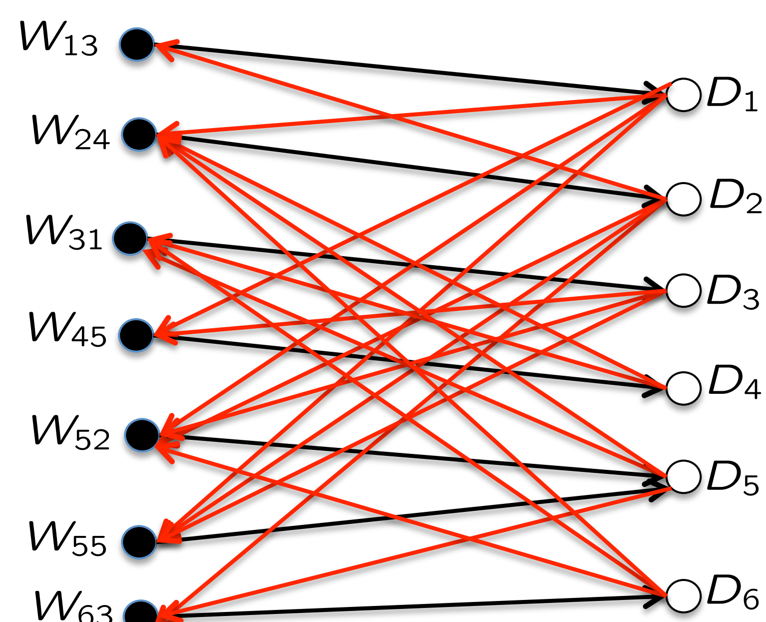
Antidote Graph

Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

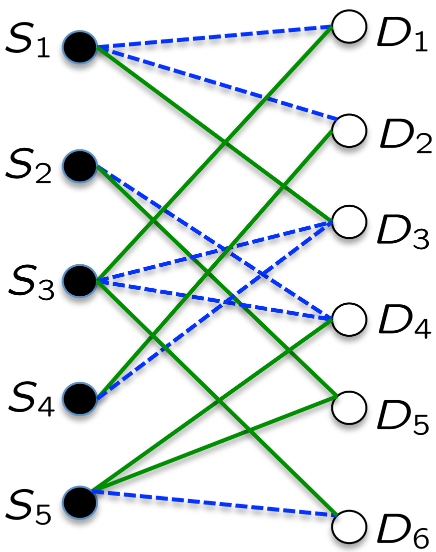


Message Conflict Graph



Demand Graph

TIM



Topology Graph

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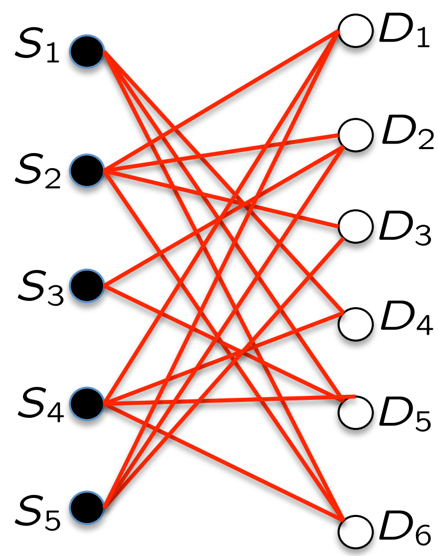

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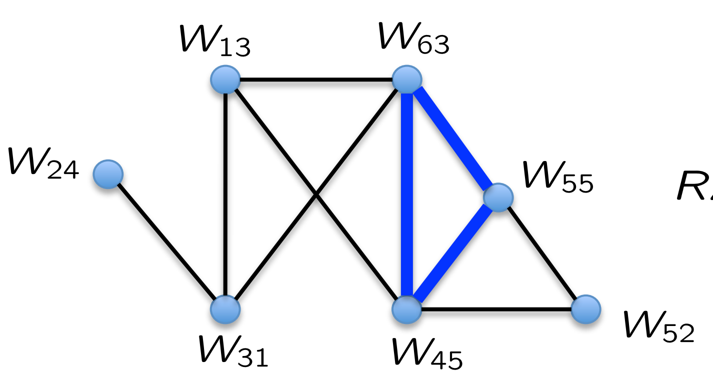
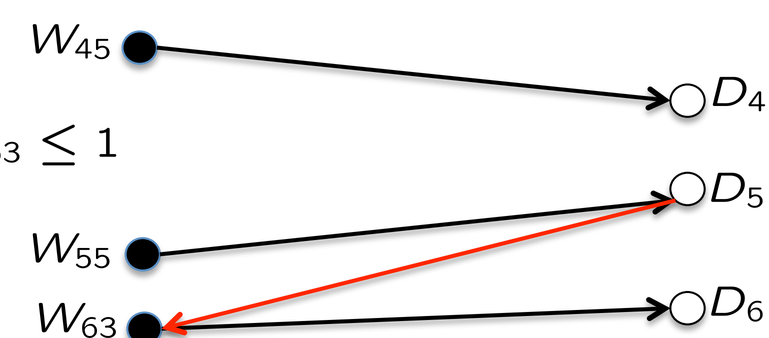


Message Conflict Graph

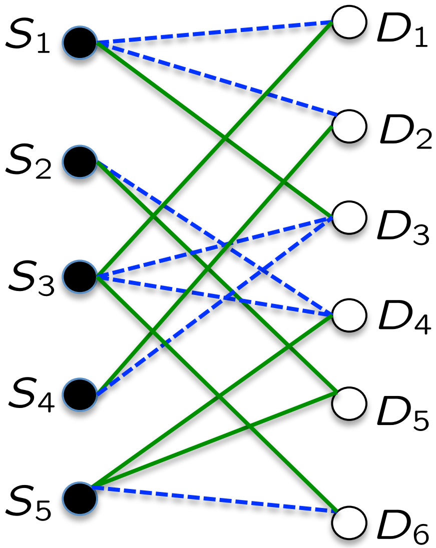
$$R_{45} + R_{55} + R_{63} \leq 1$$



Demand Graph

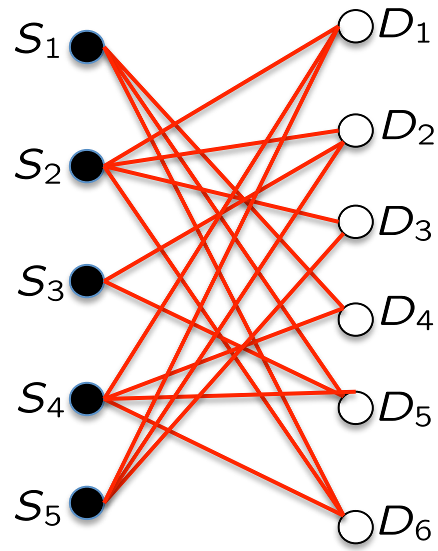
Proof of Converse (Outer Bound)

TIM



Topology Graph

Index Coding



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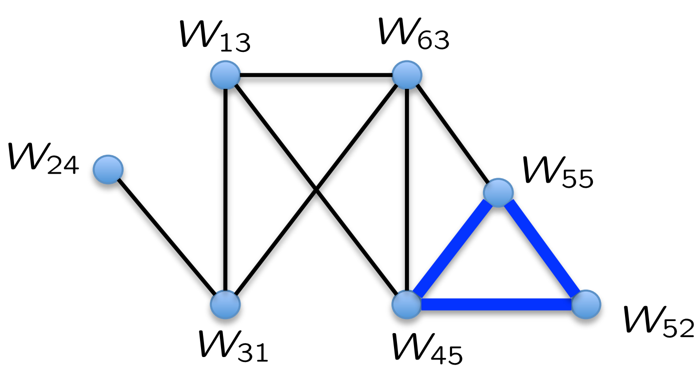
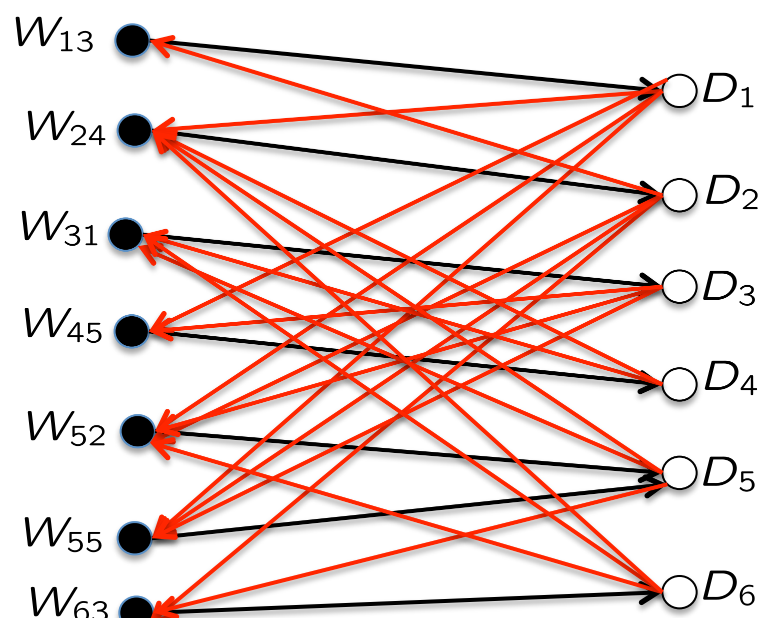
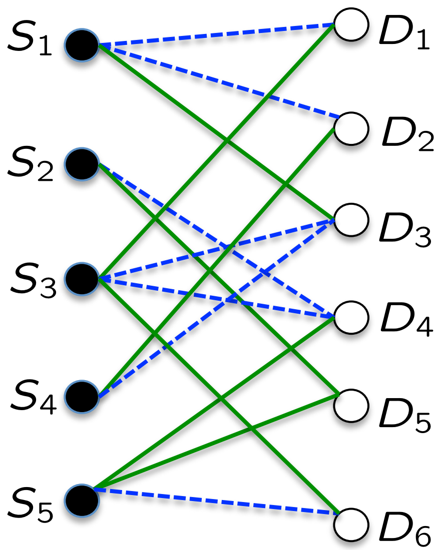


Message Conflict Graph



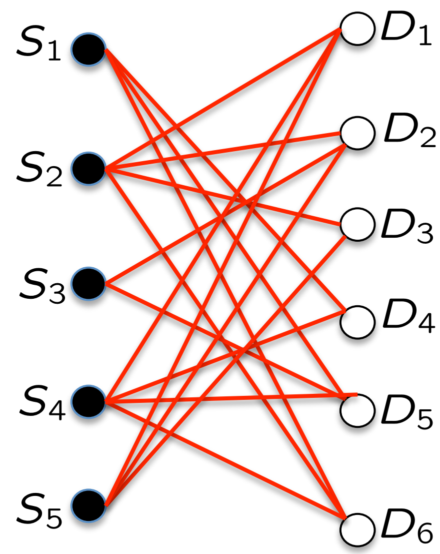
Demand Graph

TIM



Topology Graph

Index Coding

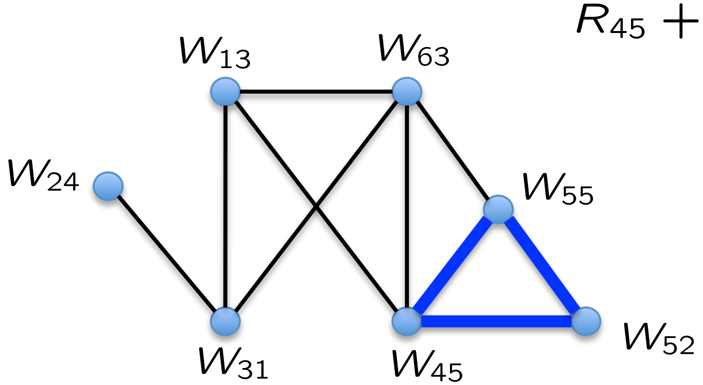


Antidote Graph

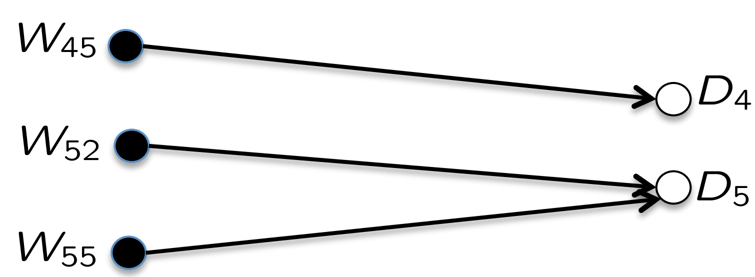
Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

$$R_{45} + R_{52} + R_{55} \leq 1$$

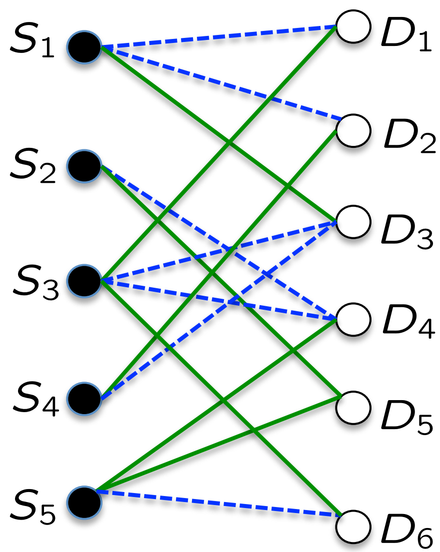


Message Conflict Graph



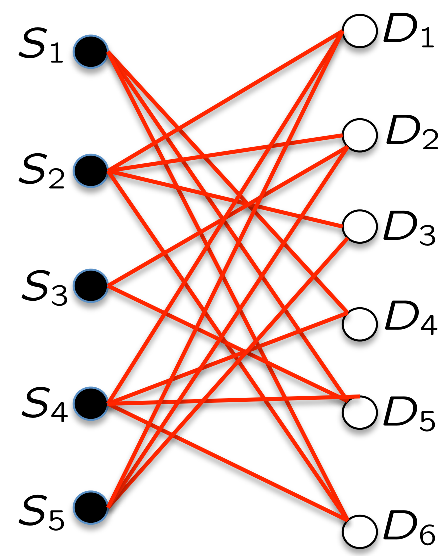
Demand Graph

TIM



Topology Graph

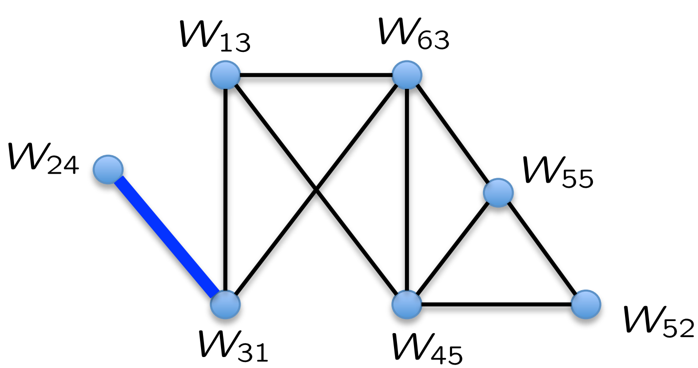
Index Coding



Antidote Graph

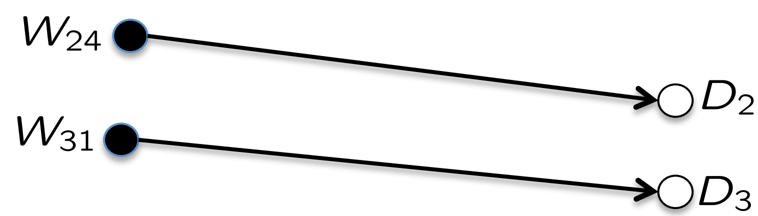
Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.



Message Conflict Graph

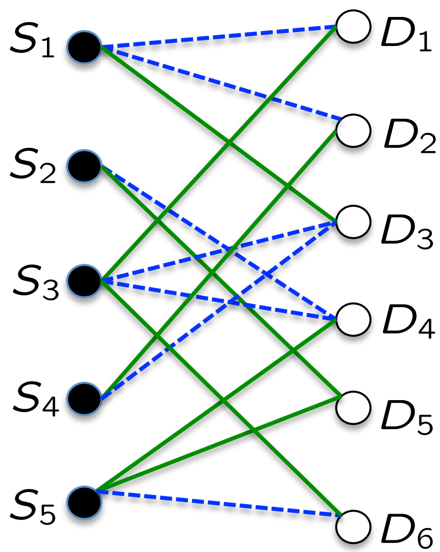
$$R_{24} + R_{31} \leq 1$$



Demand Graph

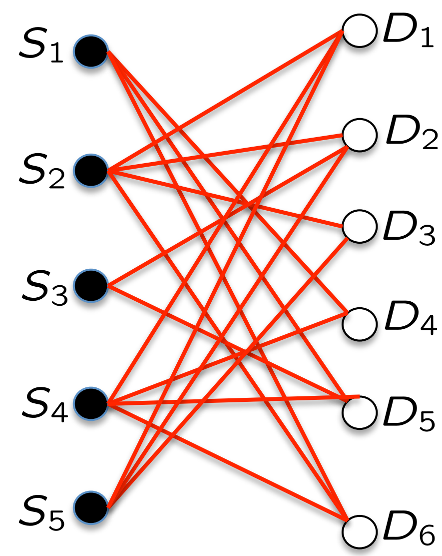
Proof of Converse (Outer Bound)

TIM



Topology Graph

Index Coding



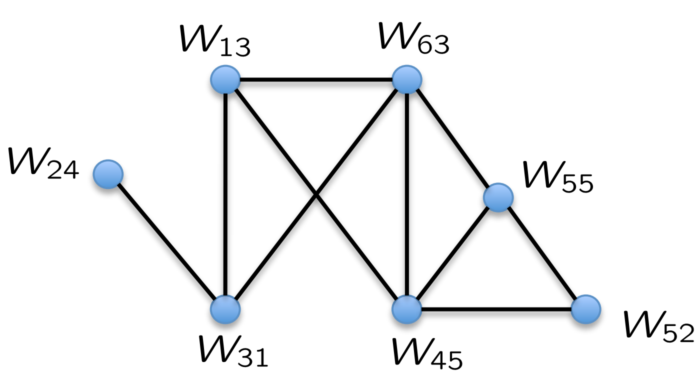
Antidote Graph

Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

$$
\begin{aligned}
R_{24} + R_{31} &\leq 1 \\
R_{13} + R_{31} + R_{63} &\leq 1 \\
R_{13} + d_{45} + R_{63} &\leq 1 \\
R_{45} + R_{55} + R_{63} &\leq 1 \\
R_{45} + R_{55} + R_{52} &\leq 1
\end{aligned}
$$

Clique Inequalities provide Capacity Region Outer Bounds



Message Conflict Graph

True for this example.
General Proof?

Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite
then every clique in Message Conflict Graph
induces an Acyclic Demand Graph.

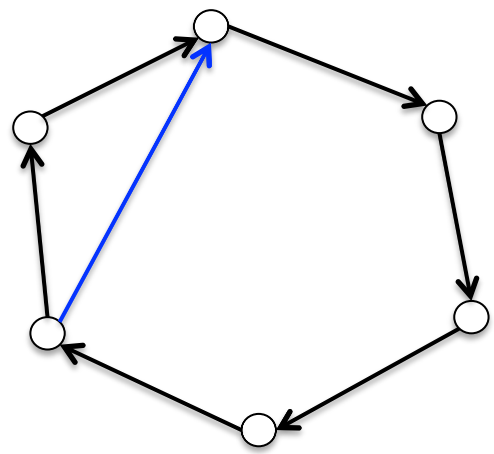
Proof by Contradiction:

Suppose a clique in message conflict graph
induces a demand graph that has directed cycles.

Choose the shortest such cycle.

It must be a chordless cycle.

(Otherwise the chord creates a shorter directed cycle)



Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

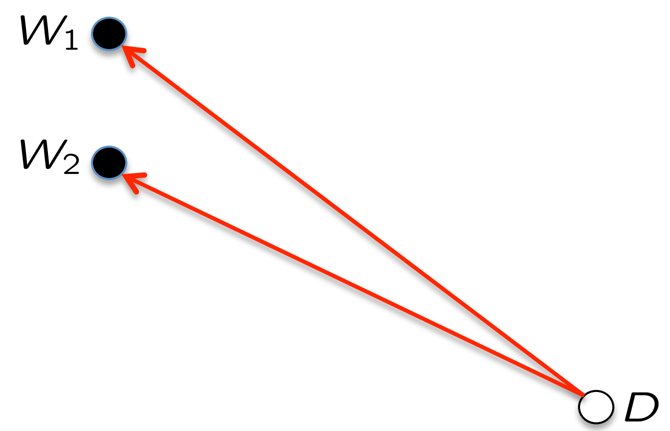
Proof by Contradiction:

Suppose a clique in message conflict graph induces a demand graph that has directed cycles.

Choose the shortest such cycle.

It must be a chordless cycle.

A chordless cycle in a demand graph, cannot involve multiple messages from the same source.



Suppose W_1, W_2 come from the same source.

W_1 must have an incoming edge from some destination D in the induced cycle.

W_2 must also have an incoming edge from **same** destination D in the induced cycle.

D has multiple outgoing edges, so the cycle cannot be chordless.

Claim

If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

Proof by Contradiction:

Suppose a clique in message conflict graph induces a demand graph that has directed cycles.

Choose the shortest such cycle.

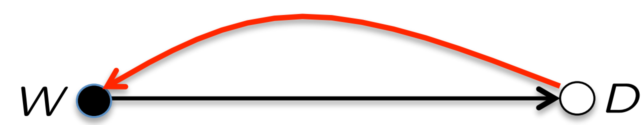
It must be a chordless cycle.

A chordless cycle in a demand graph, cannot involve multiple messages from the same source.

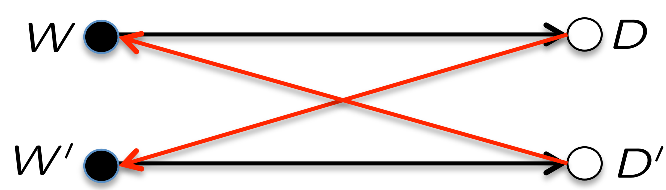
Let the length of this chordless cycle be n.

n must be even (because the demand graph is bi-partite).

$n \neq 2$ because the destination must hear its desired source.



$n \neq 4$ because then the messages do not conflict.
(the messages must conflict because they form a clique in the message conflict graph)



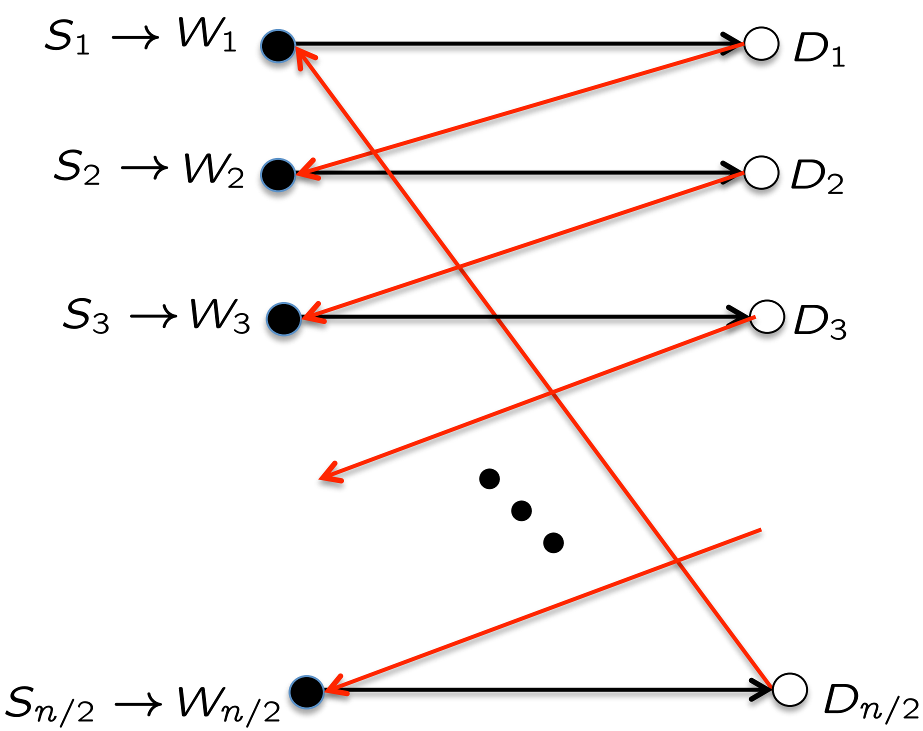
So $n \in \{6, 8, 10, \cdots\}$

Claim

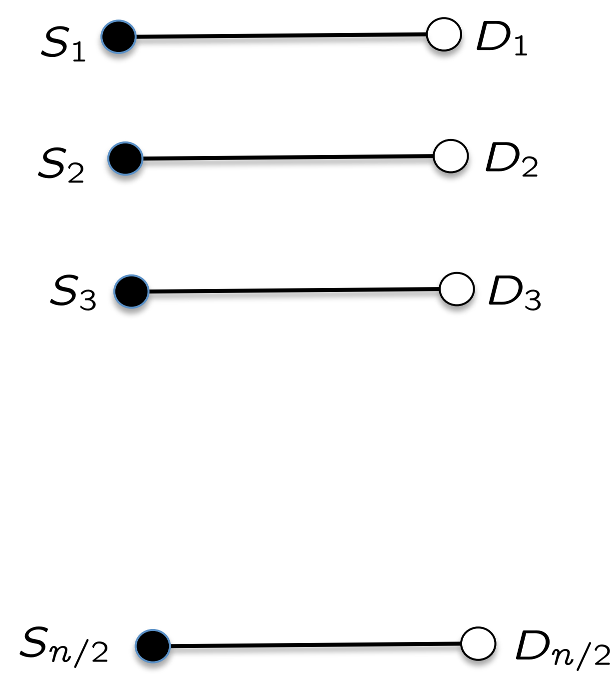
If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

Proof by Contradiction: (continued)

Chordless cycle in induced demand graph of length $n \in \{6, 8, 10, \cdots\}$



Chordless Cycle in Demand Graph



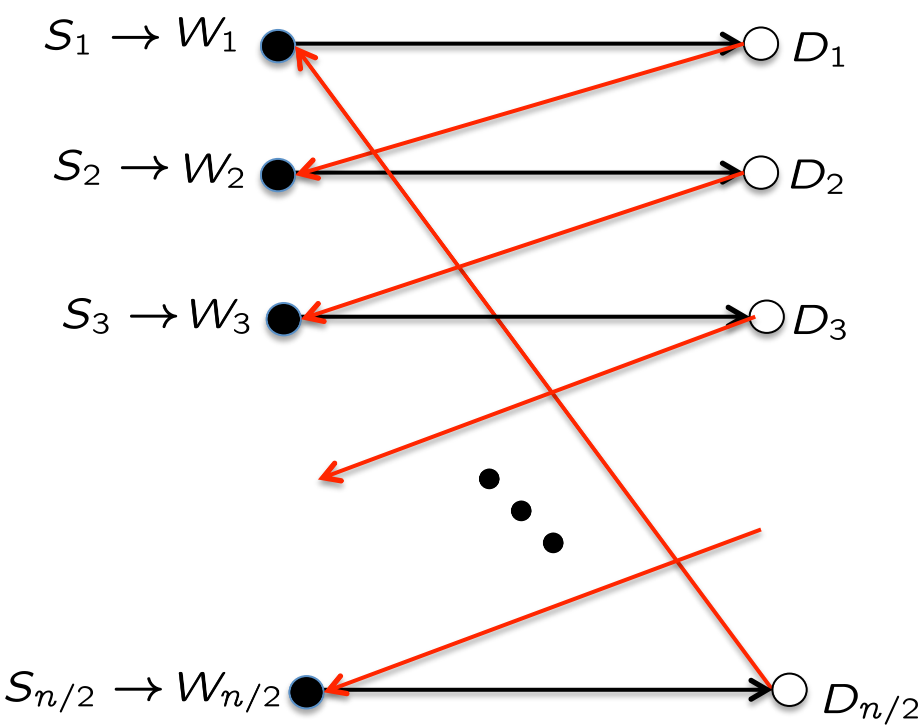
Implies Chordless Cycle of Length 6 in Topology Graph $\mathcal{G}$

Claim

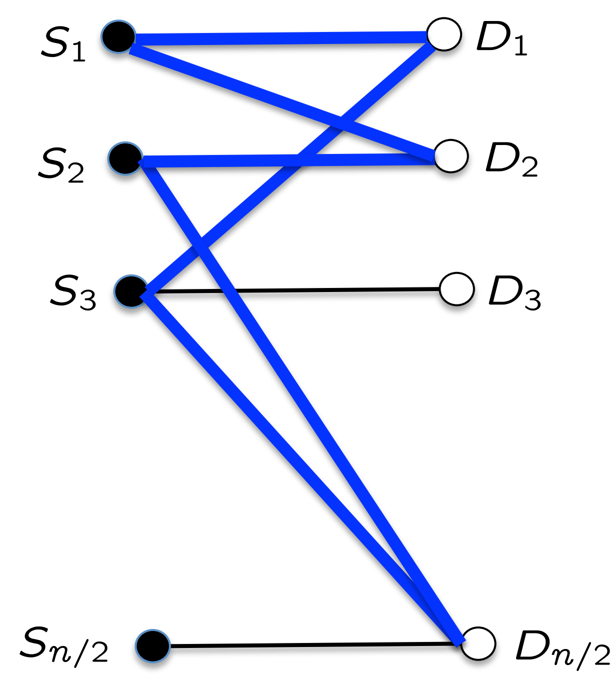
If Topology Graph $\mathcal{G}$ is chordal bipartite then every clique in Message Conflict Graph induces an Acyclic Demand Graph.

Contradiction!

Proof by Contradiction: (continued)



Chordless Cycle in Demand Graph

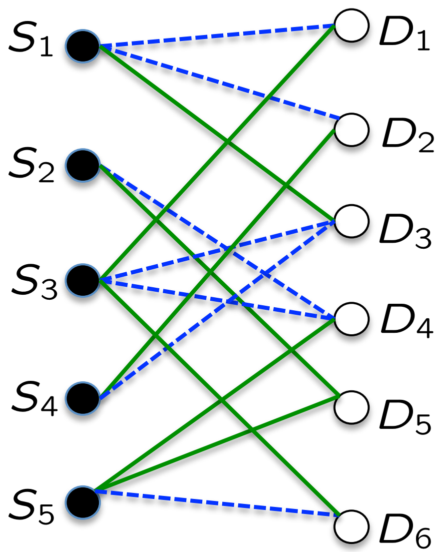


Implies Chordless Cycle of Length 6 in Topology Graph $\mathcal{G}$

But $\mathcal{G}$ is chordal bipartite.

Achievability

Whatever we need, in graph theory there is a theorem for that.



Topology Graph

Converse

✔ If network topology graph $\mathcal{G}$ is chordal bipartite, then the clique inequalities of the message conflict graph are outer bounds on the capacity region.

Achievability

Why is this outer bound achievable with TDMA?

Message Conflict Graph is $\mathcal{G}_e^2$, i.e., the square of the line graph of $\mathcal{G}$

If $\mathcal{G}$ is chordal bipartite, then it is weakly chordal.

If $\mathcal{G}$ is weakly chordal, then $\mathcal{G}_e^2$ is also weakly chordal.

Weakly chordal graphs are perfect graphs.

The region described by clique inequalities of perfect graphs has integral vertices

Every vertex of outer bound region is achievable by TDMA.

TDMA achieves the entire DoF region.

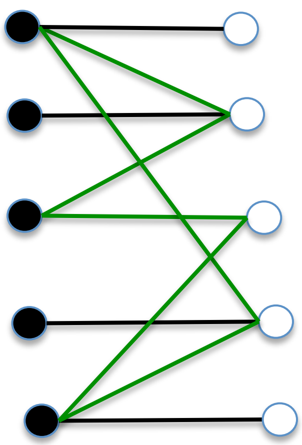
Fractional coloring achieves the entire capacity region.

What if Topology Graph is Not Chordal Bipartite?

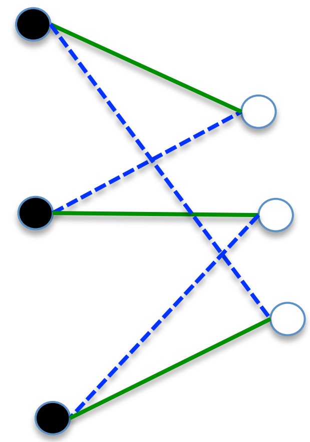
Contains chordless cycle of length ≥ 6

Then there exists a multiple unicast message set for which TDMA is not optimal.

Case 1: $n/2$ is odd.



(Not Chordal Bipartite) Topology Graph



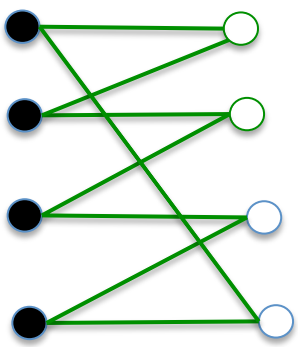
TDMA (fractional coloring) can only achieve total DoF (rate) ≤ 1.

Multicast achieves DoF (rate) 3/2.

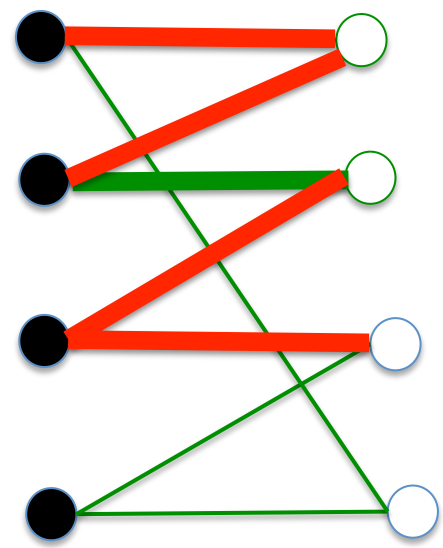
What if Topology Graph is Not Chordal Bipartite?

Then there exists a multiple unicast message set for which TDMA is not optimal.

Case 2: $n/2$ is even.



Topology Graph
(Not Chordal Bipartite)



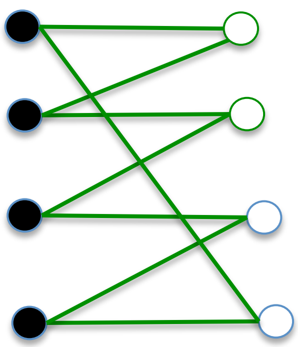
TDMA (fractional coloring) can only achieve total DoF (rate) ≤ 2.

Interference Alignment achieves DoF (rate) $= 8/3$ [Jafar, IT Trans 2014]

What if Topology Graph is Not Chordal Bipartite?

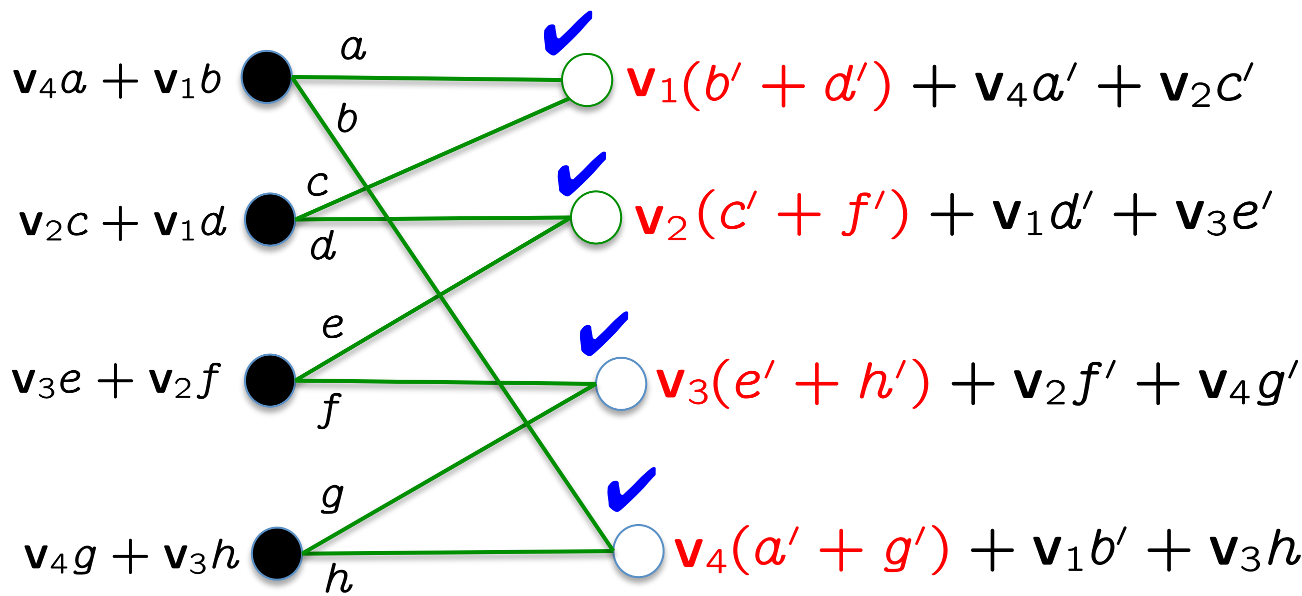
Then there exists a multiple unicast message set for which TDMA is not optimal.

Case 2: $n/2$ is even.



Topology Graph

(Not Chordal Bipartite)



$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are 3×1 vectors.
Any $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ are linearly independent.

Align interference to D_i along $\mathbf{v}_i$.

8 symbols sent over 3 channel uses.

Interference Alignment achieves 8/3 DoF.

(Equiv. IA achieves rate 8/3 for Index Coding)

TDMA (fractional coloring) is not optimal.

(Incidentally, 8/3 is optimal.)

Conclusion

GDoF

Fundamental limits
Robust insights
Vast scope
Challenging
Not always beyond reach