Illiquidity, Credit risk and Merton's model
(joint work with J. Dong and L. Korobenko)

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Merton's model of corporate debt

- A corporate bond is a contingent claim on the assets of a firm with pay-off $\min(D, V_T)$. D is the face value of the debt, T is the maturity.
- A geometric Brownian motion $(V_t)_{t\geq 0}$ models the firm's assets:

$$dV_t = \mu V_t dt + \sigma V_t dW_t.$$

  where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $(W_t)_{t\geq 0}$ is a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$.
- The market is endowed with a money market account accumulating interest at a constant rate r.
- The firm's assets are liquidly traded in the market.

The price of the Merton style bond

The liquidity assumption makes the pay-off replicable with the firm's assets and the money market account. Hence, the arbitrage-free price, B_t, of the corporate bond at time t is given by a variation of the Black-Scholes formula:

$$B_t = V_t N(d_1) - D\exp(-r(T-t))N(d_2)$$

where N is the standard normal distribution function and

$$d_1 = \frac{\log(V_t/D) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Curious Observation:

B_t does not depend on μ!

Default probabilities vs the price of the Merton style bond

Remark: $P(V_T < D)$ is interpreted as the default probability of the firm.

Question:

Consider two firms, Firm A and Firm B, which both issue Merton style bonds at time 0 with the same face value and maturity. Suppose that Firm A has a higher probability of default than Firm B. Shouldn't the premium of the Firm A bond be higher than the premium of the Firm B bond?

Default probabilities vs the price of the Merton style bond

Answer:

Not necessarily. Merton model is a counterexample:

- Suppose Firm A and Firm B have the same asset value at time 0, the same volatility, but different drifts with $\mu_B > \mu_A$. Then, the probability of default, $P(V_T < D)$, is higher for Firm A than Firm B.
- However, the prices of the Merton style bonds issued by the two firms are exactly the same.
- This happens because both bonds are replicable, and the prices of the replicating portfolios are exactly the same since the firm values are the same.

Default probabilities vs the price of the Merton style bond

Some insights

- Perfect replication of the pay-off $\min(V_T, D)$ eliminates the effect of the increased default risk due to the drift term in the pricing of the zero coupon bond.
- In the absence of the liquidity assumption, perfect replication is no longer possible, hence the drift term may affect the price.

A model where the firm's value is not a traded asset

Assumptions:

- V_t is not traded but observed.

$$dV_t = \mu_1 V_t dt + \sigma_1 V_t dW_t^1$$

- There is an asset in the market, S_t correlated with V_t. We assume

$$dS_t = \mu_2 S_t dt + \sigma_2 S_t dW_t^2,$$

- (W_t^1, W_t^2) is a two dimensional Brownian motion with correlation ρ.
- All portfolios can be constructed using S_t and the money market account.
- For simplicity assume $r = 0$.
- The market information: $\mathcal{F}_t = \sigma(V_s, S_s, s \leq t)$.

Remark: The above model is incomplete. There is no unique way to set the price of a Merton style bond.

How do we model the price?

- Let b_t be the cost at time t of building an optimal replicating portfolio for $\min(V_T, D)$. That is,

$$(b_t, (\theta^t_s)_{s\in[t,T]}) = \text{argmin}_{P,\theta}\mathbb{E}\left[(P + \int_t^T \theta_s dS_s - \min(V_T, D))^2 | \mathcal{F}_t\right]$$

- Let $c_t = \min_{P,\theta} \mathbb{E}\left[(P + \int_t^T \theta_s dS_s - \min(V_T, D))^2 | \mathcal{F}_t\right]$.
- Let $\kappa > 0$ and $\tilde{c}_t = \frac{c_t}{V_t^2}$. The proposed model for the price of the zero-coupon bond:

$$B_t = b_t \times e^{-\kappa \tilde{c}_t}$$

The interpretation of the pricing formula

- b_t is a benchmark price because it is the price of the closest traded instrument in the market.
- $e^{-\kappa \tilde{c}_t}$ is a discount factor. One can think of it as the compensation due to the extra variability in $\min(V_T, D)$ that can not be hedged by the optimal replicating portfolio.
- $\tilde{c}_t$ is the relative replication error. The importance of the replication error depends on the firm value. Also, this way the price is a monotone function of the firm value which is important to have no-arbitrage.

Finding the optimal replicating portfolio

- The optimization problem $\min_{P,\theta} \mathbb{E}\left[(P + \int_0^T \theta_s dS_s - H)^2\right]$ for a given $H \in \mathcal{L}^2(\mathcal{F}_T)$ is called the mean variance hedging (MVH) problem. Earlier works used martingale decomposition techniques.
- When the underlying processes (here S_t and V_t) are Markov processes, the MVH problem can be formulated as a stochastic control problem.

MVH as a stochastic control problem

Bertsimas, Kogan and Lo (2001) considered $H = F(S_T, V_T)$ and formulated the MVH as the following stochastic optimal control problem:

$$\textit{minimize } \mathbb{E}[(P_T - F(S_T, V_T))^2] \qquad \textit{over all } \theta \in \Theta$$

with the dynamics

$$\begin{aligned} dV_t &= \mu_1 V_t dt + \sigma_1 V_t dW_t^1, \\ dS_t &= \mu_2 S_t dt + \sigma_2 S_t dW_t^2, \\ dP_t &= \theta_t dS_t = \theta_t \mu_2 S_t dt + \theta_t \sigma_2 S_t dW_t^2, \\ P_0 &= p, S_0 = s, V_0 = v. \end{aligned}$$

where Θ is the set of all $\mathbb{R}$-valued predictable S-integrable processes such that $\int \theta dS$ is well-defined.

Remark

In the above formulation the initial cost of the portfolio p is taken as fixed. If we can solve the above problem for any p, then we can optimize over p.

MVH as a stochastic control problem: Dynamic version

$$\text{minimize } \mathbb{E}_{t,p,s,v}[(P_T - F(S_T, V_T))^2] \qquad \text{over all } \theta \in \Theta_t$$

$P_t = p, S_t = s, V_t = v$, and for $s > t$

$$\begin{aligned} dV_s &= \mu_1 V_s ds + \sigma_1 V_s dW_s^1, \\ dS_s &= \mu_2 S_s ds + \sigma_2 S_s dW_s^2, \\ dP_s &= \theta_s dS_s = \theta_s(\mu_2) S_s ds + \theta_s \sigma_2 S_s dW_s^2. \end{aligned}$$

Let $V(t,p,s,v)$ be the optimal value function of this control problem. It is well known that $V(t,p,s,v)$ is characterized as the solution of the Hamilton Jacobi and Bellman (HJB) equation.

Theorem (Bertsimas, Kogan and Lo (2001))

$V(t, p, s, v)$ is quadratic in p, i.e. there are continuous functions $a(t, s, v)$, $b(t, s, v)$ and $c(t, s, v)$ such that

$$V(t, p, s, v) = a(t, s, v) \cdot [p - b(t, s, v)]^2 + c(t, s, v), \qquad 0 \leq t \leq T.$$

$$
\begin{aligned}
\frac{\partial a}{\partial t} &= (\frac{\mu_2}{\sigma_2})^2 a + \mu_2 s \frac{\partial a}{\partial s} + [\frac{2\sigma_1 \rho v \mu_2}{\sigma_2} - \mu_1 v]\frac{\partial a}{\partial v} - \frac{1}{2}v^2 s^2 \frac{\partial^2 a}{\partial s^2} - \frac{1}{2}\sigma_1^2 v^2 \frac{\partial^2 a}{\partial v^2} \\
&\quad - v^2 \sigma_1 s \rho \frac{\partial^2 a}{\partial s \partial v} + \frac{1}{a} v^2 s^2 (\frac{\partial a}{\partial s})^2 + \frac{1}{a}\rho^2 \sigma_1^2 v^2 (\frac{\partial a}{\partial v})^2 + 2v^2 \sigma_1 s \rho \frac{\partial a}{\partial s}\frac{\partial a}{\partial v}, \\
\frac{\partial b}{\partial t} &= [\frac{\sigma_1}{\sigma_2} v \rho \mu_2 - \mu_1 v]\frac{\partial b}{\partial v} - \frac{1}{2}\sigma_1^2 v^2 \frac{\partial^2 b}{\partial v^2} - \frac{1}{2}\sigma_2^2 s^2 \frac{\partial^2 b}{\partial s^2} - \sigma_2 \sigma_1 v s \rho \frac{\partial^2 b}{\partial v \partial s} \\
&\quad + \frac{\sigma_1^2 v^2}{a}(\rho^2 - 1)\frac{\partial a}{\partial v}\frac{\partial b}{\partial v}, \\
\frac{\partial c}{\partial t} &= -\mu_1 v \frac{\partial c}{\partial v} - \mu_2 s \frac{\partial c}{\partial s} - \frac{1}{2}\sigma_1^2 v^2 \frac{\partial^2 c}{\partial v^2} - \sigma_2 \sigma_1 s v \rho \frac{\partial^2 c}{\partial s \partial v} - \frac{1}{2}\sigma_2^2 s^2 \frac{\partial^2 c}{\partial s^2} \\
&\quad + a\sigma_1^2 v^2 (\rho^2 - 1)(\frac{\partial b}{\partial v})^2,
\end{aligned}
$$

with boundary conditions

$$a(T, s, v) = 1, b(T, s, v) = F(s, v), c(T, s, v) = 0.$$

$a(t,s,v) > 0$, hence

- Optimal initial wealth $p^\star(t,s,v)$ that minimizes the quadratic function is $b(t,s,v)$.
- The optimal-replication strategy is the θ corresponding to this initial wealth $p^\star(t,s,v)$.
- The optimal replication error is $\sqrt{c(t,s,v)}$.

The solution of MVH for a Merton style bond

Let $F(S_T, V_T) = \min(V_T, D)$. Because the pay-off is only a function of V_T but not S_T, it turns out that the functions a, b and c are also only functions of t and v but not s. Thanks to this simplification the PDEs for a, b and c become a linear system and can be solved explicitly.

Theorem

(Dong, Korobenko, S.(2016))

1. $a(t, v) = e^{-\left(\frac{\mu_2}{\sigma_2}\right)^2 T - t}$;
2. $b(t, v) = ve^{\left(\mu_1 - \frac{\mu_2 \rho \sigma_1}{\sigma_2}\right)(T-t)} N(d1) + D\,(1 - N(d2))$, *where*

$$\begin{aligned} d1(t, v) &= \frac{\ln\frac{D}{v} - \sigma_1^2(1 - (\frac{1}{2} + \frac{1}{\sigma_1^2}\left[\frac{\sigma_1 \rho \mu_2}{\sigma_2} - \mu_1\right]))(T - t)}{\sigma_1 \sqrt{T - t}}, \\ d2(t, v) &= d1 + \sigma_1 \sqrt{T - t}; \end{aligned}$$

3. $c(t, v) = \sigma^2(1 - \rho^2)\mathbb{E}\left(\int_t^T a_u V_u^2 \left(\frac{\partial b}{\partial v}(u, V_u)\right)^2 du | V_t = v\right)$.

The proposed pricing formula for $\min(V_T, D)$

We propose the following formula to calculate the price of $\min(V_T, D)$:

$$B(t, V_t) = b(t, V_t) \cdot e^{-\kappa \cdot \frac{c(t,V_t)}{V_t^2}},$$

- The price of the contingent claim $B(t, V_t)$ converges to the payoff $\min(V_T, D)$ as t approaches T.
- κ is a preference parameter, a higher value indicates a higher level of risk aversion.
- The mean squared replication error $c(t, V_t)$ has been normalized by V_t^2. This not only gives a better measure of the approximation of the replication (for example $c(t, V_t) = 800$ would be more alarming if the the firm value were 10, as compared to 100), but also makes κ a unitless constant. In general, we need normalization for technical reasons, in particular, to show that the pricing formula is arbitrage free.

A closer look at the function c

We derive a more explicit formula for the function c and analyze its qualitative properties:

Theorem

(Dong, Korobenko, S. 2017)
$\frac{c(t,v)}{v^2}$ *is monotone decreasing in* v. *In particular,*

$$c(t,v) = \sigma_1^2(1-\rho^2)v^2 e^{[(2\mu_1+\sigma_1^2)](T-t)} \tag{1}$$

$$\int_t^T e^{\left(-(\frac{\mu_2}{\sigma_2})^2-\sigma_1^2-2\frac{(\mu_2)}{\sigma_2}\rho\sigma_1\right)(T-u)} E(N(d_1)^2)du \tag{2}$$

where d_1 *is normally distributed with mean*

$$\mu(u,v) = \frac{\ln D - \ln v + (-\mu_1 - \frac{3}{2}\sigma_1^2)(u-t) - \sigma_1^2(1-\alpha)(T-u)}{\sigma_1\sqrt{T-u}}$$

with standard deviation $\sqrt{\frac{u-t}{T-u}}$.

Is the proposed pricing formula arbitrage free?

The Fundamental Theorem of Asset Pricing (Dalbean and Schachermayer):
There is no arbitrage in the sense of *no free lunch with vanishing risk* if and only if there exists an equivalent probability measure Q rendering the price processes sigma martingales.

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Theorem

(Dong, Korobenko, S. 2017) There exist processes $\lambda_1(t)$ and $\lambda_2(t)$ such that

$$B_t = B_0 + \int_0^t N_t d\tilde{W}_t^1$$
$$S_t = S_0 + \int_0^t L_t d\tilde{W}_t^2$$

where $\tilde{W}_t^i = \int_0^t \lambda_i(s)ds + W_t^i$. Moreover, if

$$\mathbb{E}(\exp(\frac{1}{1-\rho^2}\int_0^T \left[\lambda_1^2(s) + \frac{1}{2}\lambda_2^2(s) - 2\rho\lambda_1(s)\lambda_2(s)\right] ds)) < \infty, \qquad (3)$$

then there the exists of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $(\tilde{W}_t^1, \tilde{W}_t^2)_{0\le t\le T}$ is a two dimensional Brownian motion with correlation ρ.

Is the proposed pricing formula arbitrage free?

Theorem

(Dong, Korobenko, S. 2017) Assume $\frac{1}{2} + \frac{\mu_1 - \frac{\mu_2}{\sigma_2}\rho\sigma_1}{\sigma_1^2} > 0$. *We have* $\sup_{t\in[0,T]} |\lambda_i(t)| < K$ *for some deterministic constant* K.

Corollary: (Dong, Korobenko, S. 2017) Assume that $\frac{1}{2} + \frac{\alpha}{\sigma_1^2} > 0$. Let $\tilde{c}(t,v) = c(t,v)/v^2$. For any $\kappa > 0$, $B_t = b(t, V_t)e^{-\kappa\tilde{c}(t,V_t)}$ gives an arbitrage free price for the Merton style bond $\min(V_T, D)$ in the sense of NFLVR.

How do the parameters affect the price?

Overall there are three sets of parameters

1. Parameters of the pay-off: T, D.
2. Parameters of the underlying processes: μ_1, $\theta := \frac{\mu_2}{\sigma_2}$, ρ, σ_1.
3. Risk aversion parameter κ.

We are most interested in μ_1, $\theta := \frac{\mu_2}{\sigma_2}$, ρ, σ_1 and κ and their effects on the yield of the bond.

How do the parameters affect the price?

Some highlights:

- A key quantity is $\alpha = (\mu_1) - \theta\rho\sigma_1$. $b(t, v)$ depends on μ_1, θ and ρ only through the term α.
- $b(t, v)$ increases with α (when everything else is fixed.). Interpretation: The larger the μ_1, the higher the price of the optimal replicating portfolio. The higher the risk premium on the underlying asset S_t the higher the premium of the optimal replicating portfolio.
- If $\rho\theta > 0$ then $b(t, v)$ decreases with σ_1. Numerical calculations show that $\tilde{c}(t, v)$ should be increasing with σ_1 as well. This would mean the yield of the bond increases with σ_1.
- Surprisingly $\tilde{c}(t, v)$ is not monotone in ρ (except in the case $\theta = 0$).
- The yield at date t of a Merton style bond with maturity T is defined as the function

$$y(t, T) = \frac{\log(D) - \log(B_t)}{T - t}$$

Yields increase with κ, most noticably when $\alpha \geq 0$.

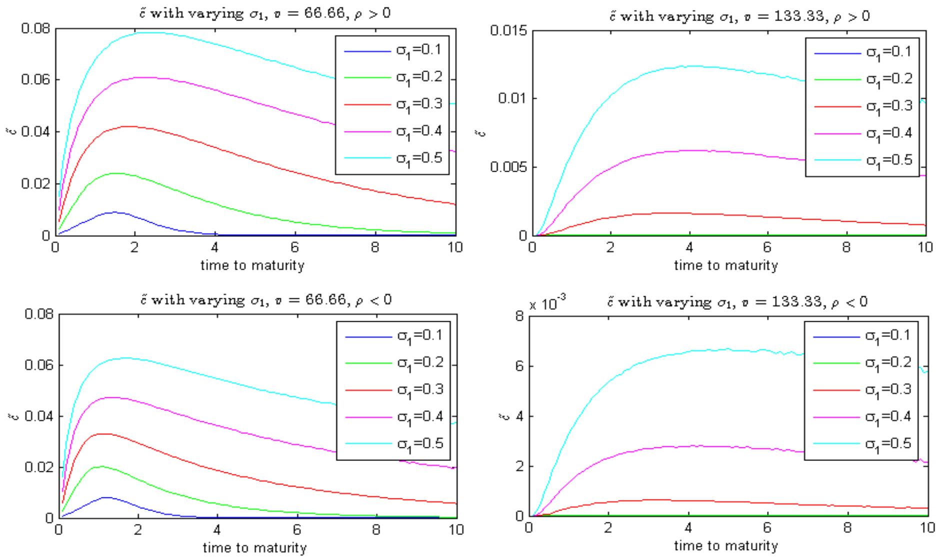

Figure: $\tilde{c}$ with varying σ_1, $\mu_1 = 0$, $\theta = 0.66$

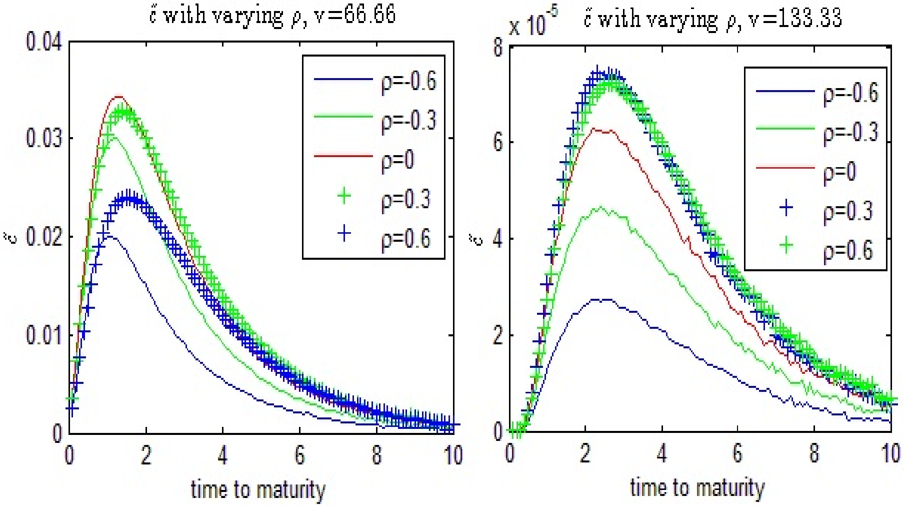

Figure: $\tilde{c}$ with varying ρ, $\mu_1 = 0.2$, $\sigma_1 = 0.2$, $\theta = 1.5$

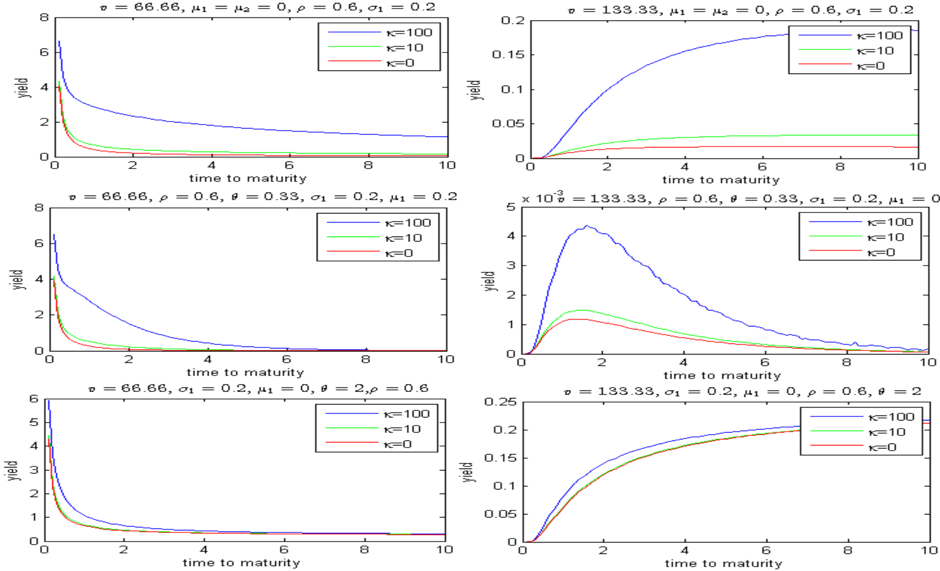

Figure: yield with varying κ for various configurations