Equilibrium with Transaction Costs

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Based on
"Existence of a Radner equilibrium in a model with transaction costs,"
https://arxiv.org/abs/1702.01706

Equilibrium Basics

We are interested in a Radner equilibrium with a single consumption good where the traded security incurs a proportional transaction cost.

Inputs:

- Preferences
- Initial allocations
- Income and dividends
- Transaction cost parameter, $\lambda \in [0,1)$

Outputs:

- Prices (interest rate)
- Optimal investment and consumption strategies

Questions: Does such an equilibrium exist? How do equilibrium prices depend on λ?

Existing Literature

Exogenous Assets: Vayanos (RFS '98), Lo-Mamaysky-Wang (JPE '04), Davila (w.p. '16)

No Existence: Buss-Dumas (w.p. '13), Buss-Uppal-Vilkov (w.p. '14)

Continuum of Agents: Vayanos (RFS '98), Vayanos-Vila (ET '99), Huang (JET '03)

Approximation: Muhle-Karbe-Herdegen (w.p. '17)

Goal: Establish the existence of an (exact) equilibrium with a finite number of agents, in which all traded asset prices are derived endogenously.

Model Inputs

Consider a model with a single consumption good (the numeraire) and times $t \geq 0$.

Two Agents: $c \mapsto -e^{-\alpha_i c}$ for $\alpha_i > 0$, $i = 1, 2$

Time Preferences: $t \mapsto -e^{-\beta_i t}$ for $\beta_i > 0$, $i = 1, 2$

Income: Agents $i = 1, 2$ have income processes

$$dY_{it} = \mu_i dt + \sigma_i dZ_{it}, \quad Y_{i0} \in \mathbb{R},$$

where the Z_i are possibly correlated Brownian motions.

Traded Annuity: A share in the annuity, A, guarantees its holder an income stream at a rate of 1 per unit time at all times in the future. The annuity is in one-net supply.

Transaction Costs: $\lambda \in [0,1)$

Shadow Prices

Introduced for transaction costs by Jouini-Kallal (1995) and Cvitanić-Karatzas (1996).

	Transaction Cost Market	(Frictionless) Shadow Market
Price:	A_t	$\tilde{A}_t$
Buy:	$A_t(1+\lambda)$	$\tilde{A}_t = (1+\lambda)A_t$
Sell:	$A_t(1-\lambda)$	$\tilde{A}_t = (1-\lambda)A_t$
No Trade:	$\tilde{A}_t \in [(1-\lambda)A_t, (1+\lambda)A_t]$	
Optimal:	$\hat{\theta}_t$	$\tilde{\theta}_t = \hat{\theta}_t$

Remark: Since buying/selling/not trading is *investor-specific*, so is $\tilde{A}_i$, $i = 1, 2$.

Single-Agent Utility Maximization

We study the shadow market problems directly and focus on shadow annuity values $\tilde{A}_i$ such that r_i is constant and strictly positive. In this case,

$$\tilde{A}_{it} = \tilde{A}_i = \frac{1}{r_i} \text{ (constant).}$$

Wealth evolution: For initial wealth $X^c_{i0} = \theta_{i0}\tilde{A}_i = \theta_{i0}/r_i$,

$$dX^c_{it} = \left(X^c_{it} r_i + Y_{it} - c_t\right) dt.$$

Admissibility: We say that a consumption stream c is admissible and write $c \in \mathcal{A}_i$ if

$$\mathbb{E}\left[e^{-\beta_i t - \alpha_i r_i X^c_{it} - \alpha_i Y_{it}}\right] \longrightarrow 0 \quad \text{as } t \to \infty.$$

Optimization:

$$\sup_{c \in \mathcal{A}_i} \mathbb{E}\left[-\int_0^\infty e^{-\beta_i t - \alpha_i c_t} dt\right] \tag{1}$$

Equilibrium

For transaction cost parameter $\lambda \in [0,1)$, the processes $(\tilde{A}_i, \hat{c}_i, \hat{\theta}_i)_{i=1,2}$ constitute an *equilibrium with transaction costs* if

1. Smooth trading: $\hat{\theta}_i$ is differentiable in t.
2. Ratio relationships hold:

$$\frac{\tilde{A}_{1t}}{\tilde{A}_{2t}} \begin{cases} \in \left[\frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda}\right] & \text{always} \\ = \frac{1+\lambda}{1-\lambda} & \text{if agent 1 buys: } \hat{\theta}'_{1t} > 0 \\ = \frac{1-\lambda}{1+\lambda} & \text{if agent 1 sells: } \hat{\theta}'_{1t} < 0 \end{cases}$$

3. Market clearing holds: $\hat{\theta}_{1t} + \hat{\theta}_{2t} = 1$ and

$$\sum_i Y_{it} + 1 = \sum_i \hat{c}_{it} + 2\lambda \left|\hat{\theta}'_{1t}\right| A_t,$$

where $A_t := \frac{\tilde{A}_{it}}{1+\lambda}$ if agent i buys: $\hat{\theta}'_{it} > 0$.

4. Individual optimality: $\hat{c}_i$ and $\hat{\theta}_i$ are optimal for (1) with $\tilde{A}_i$ as the shadow price.

Existence of an Equilibrium (I)

Theorem (W. 2017) Suppose $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i \sigma_i^2}{2} > 0$, $i = 1, 2$. Then there exists an equilibrium with optimal wealth and consumption streams

$$\hat{X}_{it} = \frac{\theta_{i0}}{r_i} + \frac{t}{\alpha_i r_i}\left(1 - \tilde{\beta}_i\right),$$

$$\hat{c}_{it} = r_i \hat{X}_{it} + Y_{it} + \frac{1}{\alpha_i r_i}\left(\tilde{\beta}_i - 1\right).$$

Case 1. A no trade equilibrium occurs if

$$\frac{\tilde{\beta}_2}{\tilde{\beta}_1} \in \left[\frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda}\right].$$

In this case, $r_i = \tilde{\beta}_i$, and there is a range of interest rates r that are consistent with equilibrium.

Existence of an Equilibrium (II)

Case 2. An equilibrium in which agent 1 buys and agent 2 sells at all times occurs if

$$\frac{\tilde{\beta}_2}{\tilde{\beta}_1} > \frac{1+\lambda}{1-\lambda}.$$

The equilibrium interest rate is given uniquely by

$$r = \frac{\frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2}}{\frac{1}{\alpha_1(1+\lambda)} + \frac{1}{\alpha_2(1-\lambda)}},$$

and $r = (1+\lambda)r_1 = (1-\lambda)r_2$.

Note: An analogous result holds for $\frac{\tilde{\beta}_2}{\tilde{\beta}_1} < \frac{1-\lambda}{1+\lambda}$.

Interest Rates for Small λ

Question: How does the equilibrium interest rate depend on λ as $\lambda \to 0$?

Answer: It depends on the agents' risk aversion parameters.

For $\tilde{\beta}_2 > \tilde{\beta}_1 > 0$ and $\lambda \in [0, \frac{\tilde{\beta}_2 - \tilde{\beta}_1}{\tilde{\beta}_1 + \tilde{\beta}_2})$, there exists an equilibrium such that trade occurs and $r = r(\lambda)$ is given by

$$r(\lambda) = \frac{\frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2}}{\frac{1}{\alpha_1(1+\lambda)} + \frac{1}{\alpha_2(1-\lambda)}}.$$

THANK YOU!