Modeling Dynamic User Equilibria as Differential Complementarity Systems (DCS)

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Background & History

How is traffic distributed in a (urban) traffic network and why?

When? Car?
Which route?
Safety, reliability, toll, scenery, ...

Where am I?
Next turn?
Change route?

When? Car?
Which route?
Safety, reliability, toll, scenery, ...

Where am I?
Next turn?
Change route?
Transportation Network Modeling

- Transportation Network Modeling (Traffic Assignment): predict flow distribution in a traffic network, given the total demand (e.g., during the peak period)
- Traffic Equilibrium (Frank Knight, 1924)
- Wardrop First Principle: User Equilibrium (Wardrop, 1952)
  
  The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route
- Wardrop Second Principle: System Optimal (Wardrop, 1952)
  
  At equilibrium, the average journey time is minimum
- Close connection with Nash equilibrium (John Nash, "Beautiful Mind")
Dynamic Traffic Assignment (DTA)

- Intelligent Transportation Systems (ITS) leads to Dynamic Traffic Assignment
- Two major components: Traffic dynamics and traveler behavior
- Three types of players

Literature: Merchant and Nemhauser, 1978; Friesz et al, 1993; Ran and Boyce, 1994; Lo and Szeto, 2002; Mahamassani, 2001; Ben-Akiva et al., 2001; Ban et al., 2008, 2009, 2012a, 2012b; Ma et al., 2014; Ma et al., 2015a, 2015b; Network PDE approach (Bressan and Nguyen, 2015)
Path-based DUE Condition

- **DUE Condition (Ran and Boyce, 1996):**

  If, for each OD pair at each instant of time, the actual travel times experienced by travelers departing at the same time are equal and minimal, the dynamic traffic flow over the network is in a travel-time-based dynamic user equilibrium (DUE).

\[
\begin{aligned}
\eta_p^{rs}(t) &= \pi_p^{rs}(t) \implies f_p^{rs}(t) \geq 0 \\
\eta_p^{rs}(t) &> \pi_p^{rs}(t) \implies f_p^{rs}(t) = 0
\end{aligned}
\]

- $f_p^{rs}(t)$: path flow for path $p$ between OD pair $rs$ at time instant $t$
- $\eta_p^{rs}(t)$: path travel time for path $p$ between OD pair $rs$ at time instant $t$
- $\pi_p^{rs}(t)$: minimum path travel time for all paths between OD pair $rs$ at time instant $t$

\[
0 \leq [\eta_p^{rs}(t) - \pi_p^{rs}(t)] \perp f_p^{rs}(t) \geq 0, \forall r, s, p, t \in [0, T] \quad \text{NCP Formulation (infinite dimensional)}
\]

\[
\sum_{rs} \sum_{p} \int_{0}^{T} [f_p^{rs}(t) - f_p^{rs*}(t)] \eta_p^{rs*}(t) \geq 0 \quad \text{VI Formulation (infinite dimensional)}
\]
Path Travel Time

• **Instantaneous Path Travel Time**
  \[ \psi_p^{rs}(t) = \sum_{a \in p} \tau_a(t) \]

• **Ideal (Actual) Path Travel Time**
  \[ \eta_p^{rh_a}(t) = \eta_p^{rl_a}(t) + \tau_a(t + \eta_p^{rl_a}(t)) \]
Example

\[ \psi^{14}(t) = \tau_a(t) + \tau_b(t) + \tau_c(t) = 9 + 4.5t \]

\[ \eta^{12}(t) = \tau_a(t) = 2 + t \]

\[ \eta^{13}(t) = \tau_a(t) + \tau_b[t + \tau_a(t)] = (2 + t) + \tau_b(2 + 2t) = (2 + t) + [3 + 2(2 + 2t)] = 9 + 5t \]

\[ \eta^{14}(t) = \eta^{13}(t) + \tau_c[t + \eta^{13}(t)] = (9 + 5t) + \tau_c(9 + 6t) = (9 + 5t) + [4 + 1.5(9 + 6t)] = 26.5 + 14t \]
Link-Node Based DUE Condition

- If, from each decision node to every destination node at each instant of time, the actual travel times for all the routes that are being used are equal and minimal, then the dynamic traffic flow over the network is in a travel time based dynamic user equilibrium (DUE) state.

\[
\begin{align*}
\tau_{ij}(t) + \pi^{js}[t + \tau_{ij}(t)] &= \pi^{is}(t) \Rightarrow p_{ij}^s(t) \geq 0 \\
\tau_{ij}(t) + \pi^{js}[t + \tau_{ij}(t)] &> \pi^{is}(t) \Rightarrow p_{ij}^s(t) = 0
\end{align*}
\]

0 ≤ \( p_{ij}^s(t) \) \( \perp \) \{\tau_{ij}(t) + \pi^{js}[t + \tau_{ij}(t)] - \pi^{is}(t)\} ≥ 0 \hspace{1cm} \text{NCP (infinite dimensional)}

Types of DUE

• Simulation-based vs Analytical DUE models
• Instantaneous (reactive) vs. actual (predictive) DUE
• Path-based vs Link-based DUE models
• Continuous-time vs. discrete-time DUE models
Continuous-time vs. Discrete-time

Eg: $\alpha$-PQ model: a continuous-time system

\[
\dot{q}(t) = \begin{cases} 
0 & \text{if } t \in (0, \tau^0) \\
\max (p(t - \tau^0) - \overline{C}, -\alpha q(t)) & t > \tau^0.
\end{cases}
\]

Modeling DUE

User choice model (behavioral) → User equilibrium → VI/NCP (optimization)

Traffic dynamics (physical) → PDEs or ODEs with time delay

A mathematical framework that can properly capture both?
DUE Literature (Analytical Formulations)

• Most on discrete-time DUE problems

• A handful on continuous-time DUE problems: modeling techniques, solution techniques, among others
  – Optimal control method
    • Friesz et al. (1989), Ran and Shimazaki (1989), Ran et al. (1993), Lam et al. (1995)
  – Variational inequality (VI) method

• There has been a lack of a proper mathematical framework to capture both aspects of DUE, choice behavior and system dynamics
  – Discretization scheme of continuous-time models
  – Convergence of continuous trajectories constructed from discrete time solutions
Differential Complementarity System (DCS)

• An ODE parameterized by an algebraic variable that is required to be a solution of a finite dimensional state dependent complementarity problem:

\[
\dot{x}(t) \triangleq \frac{dx(t)}{dt} = f(t, x(t), u(t)) \quad \text{dynamics}
\]

\[
0 \leq u(t) \perp F(t, x(t), u(t)) \geq 0 \quad \text{complementarity constraint}
\]

\[
x(0) = x^0 \quad \text{initial condition},
\]

• DCS is a special case of the Differential Variational Inequality (DVI); see Pang and Stewart (2008) and Friesz (2010) for DVI

• Focus on a special DCS (Pang and Stewart, 2008)

• \( \dot{x}(t) = f(t, x) + B(t, x)u \) ← Dynamics linear on \( u \)

• \( 0 \leq u(t) \perp G(t, x) + F(u(t)) \) ← Complementarity separable on \( u \)
• Let $\Omega = [0, T] \times \mathbb{R}^n$ and two conditions

(A) $f$, $B$, and $G$ are Lipschitz continuous functions on $\Omega$ with Lipschitz constants $L_f$, $L_B$, and $L_G$, respectively;

(B) $B$ is bounded on $\Omega$ with $\sigma_B \equiv \sup_{(t,x) \in \Omega} \| B(t,x) \| < \infty$.

• Convergence + Solution existence

**Theorem 7.1** Let $K \subseteq \mathbb{R}^m$ be a nonempty closed convex set and let $(f, B, G)$ satisfy conditions (A) and (B). Suppose that there exist positive scalars $c_{0,x}$, $c_{1,x}$, $c_{0,u}$, $c_{1,u}$, and $\bar{h}$ such that for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h$,

$$
\| x^{h,i+1} \| \leq c_{0,x} + c_{1,x} \| x^0 \| \quad \text{and} \quad \| u^{h,i+1} \| \leq c_{0,u} + c_{1,u} \| x^0 \|.
$$

(7.5)

There is a sequence $\{h_\nu\} \downarrow 0$ such that the following two limits exist: $\hat{x}^{h_\nu} \rightarrow \hat{x}$ uniformly on $[0, T]$ and $\hat{u}^{h_\nu} \rightarrow \hat{u}$ weakly in $L^2(0, T)$. Furthermore, under either one of the following two conditions:

(a) $F(u) \equiv \Psi(Eu)$, where $E \in \mathbb{R}^{\ell \times m}$ and $\Psi : \mathbb{R}^\ell \rightarrow \mathbb{R}^m$ is Lipschitz continuous, and a constant $c_{2,u} > 0$ exists such that for all $h$ sufficiently small,

$$
\| Eu^{h,i+1} - Eu^{h,i} \| \leq h c_{2,u},
$$

(7.6)

(b) $F(u) \equiv Du$ for some positive semidefinite matrix $D$,

all such limits ($\hat{x}$, $\hat{u}$) are weak solutions of the initial-value DVI (6.2).
User choice model (behavioral) \rightarrow User equilibrium \rightarrow VI* / NCP** \rightarrow DCS*** with Time delay:
A mathematical framework that can properly capture both

Traffic dynamics (physical)

PDEs / ODEs with time delay

*VI: Variational Inequality
**NCP: Nonlinear Complementarity Problem
***DCS: Differential Complementarity System
Instantaneous DUE: Simplification

- Route choice condition of instantaneous DUE: based on prevailing traffic conditions

\[ 0 \leq p_{ij}^s(t) \perp \{ c_{ij}(t) + \eta^{js}(t) - \eta^{is}(t) \} \geq 0 \quad \text{NCP (infinite dimensional)} \]

- Traffic dynamics: Point Queue Model

- A DCS

\[
\dot{q}(t) = u(t) + p(t - t_0) - C
\]

\[ 0 \leq u(t) \perp q(t) \geq 0, \]

\[ v(t) = C - u(t) \]

\[ c_{ij}(t) = \tau_{ij}^0 + (\overline{C}_{ij})^{-1} q_{ij}(t) \]

Observations:
Discontinuities may occur when queue length changes from 0 to nonzero or vice-versa.

DCS Formulation for IDUE

**Route Choice**

\[ 0 \leq p_{ij}^{s}(t) \perp \tau_{ij}^{0} + \sum_{s':(i,j) \in \mathcal{L}_s'} q_{ij}^{s'}(t) \left/ \frac{C_{ij}}{C_{ij}} \right. + \eta_{j}^{s}(t) - \eta_{i}^{s}(t) \geq 0, \quad \forall (i,j) \in \mathcal{L}_s \]

**Flow conservation**

\[ 0 \leq \eta_{i}^{s}(t) \perp \sum_{j:(i,j) \in \mathcal{L}_s} p_{ij}^{s}(t) - \sum_{j:(j,i) \in \mathcal{L}_s} v_{ji}^{s}(t) - d_{i}^{s}(t) \geq 0, \quad \forall i \in \mathcal{N}_s \]

\[ 0 = \eta_{s}^{s}(t) , \]

**Queue dynamics**

\[ v(t) = \bar{C} - u(t) \]

\[ \dot{q}(t) = u(t) + p(t - t_0) - \bar{C} \]

\[ 0 \leq u(t) \perp q(t) \geq 0, \]

\[ c_{ij}(t) = \tau_{ij}^{0} + (\bar{C}_{ij})^{-1} q_{ij}(t) \]

**Destination-based exit capacity**

\[ v_{ij}^{s}(t) = \begin{cases} \frac{p_{ij}^{s}(t')}{p_{ij}(t')} v_{ij}(t) & \text{if } p_{ij}(t') > 0 \\ 0 & \text{if } p_{ij}(t') = 0. \end{cases} \]

\[ t' + \tau_{ij}^{0} + \frac{q_{ij}(t' + \tau_{ij}^{0})}{\bar{C}} = t. \]
Solution Method – Time Decomposition

- We apply **time decomposition** based on free flow link times
  - The DCS model is naturally decomposable to (link, destination pair)
  - Due to instantaneous route choice, the route choice (i.e., to determine the inflow to a link) and traffic dynamics (i.e., to load the inflow to generate queue length and exit flow) can be done separately
  - We divide the entire study period $[0, T]$ to *intervals* based on free flow link travel times, and then divide each interval into *sub-intervals* based on the minimum free flow link travel time; same scheme as in Xu et al. (1999)
  - For the DCS in each sub-interval, the time delayed term (inflow) is already determined in the previous sub-interval and thus known

- Example: three links with $\tau_1^0 = 1; \tau_2^0 = 2.5; \tau_3^0 = 5$; study period $[0, 12.25]$
Solution Method – Time Stepping

• For each sub-interval, IDUE is a DCS without time delay because the delay term (inflow to a link) is already calculated in the previous sub-interval (or known as zero).
• It is solved by the time stepping method for ODE (Shampine and Thompson, 2001)
• The traffic dynamics (network loading) and route choice models are solved separately and iteratively
• We apply the implicit discretization scheme
• Discrete traffic dynamics model: an NCP that has at least one solution
• Discrete route choice model: an NCP that at least one solution
• NCPs are solved using the PATH solver in GAMS
Continuous Time Solution Trajectories

- Construct piece wise linear trajectory based on the discrete solution, for state variables

\[ q_{ij}^s(t) = q_{ij}^{s,t_i} + \frac{t-t_i}{h} (q_{ij}^{s,t_{i+1}} - q_{ij}^{s,t_i}), \text{ for } t \in [t_i, t_{i+1}] \]

\( t_i, t_{i+1} \): discrete time points

\( h = t_{i+1} - t_i \): the length of each discrete time step

- Construct piece wise constant trajectory based on the discrete solution, for algebraic variables

\[ p_{ij}^s(t) = p_{ij}^{s,t_{i+1}}, \text{ for } t \in [t_i, t_{i+1}] \]

\( t_i, t_{i+1} \): discrete time points

\( h = t_{i+1} - t_i \): the length of each discrete time step
Assumption (A)

- For a subinterval $I$, $\hat{p}^s_{ij}(t - \tau^0_{ij})$ is the (known) bounded, nonnegative, integrable inflow rate to link $(i,j)$ for destination $s$.

$\hat{p}^s_{ij}(t - \tau^0_{ij})$ is equal to the weak limit of piecewise constant functor $\hat{p}^{s,h}_{ij}(t)$ as $h \downarrow 0$, i.e., the following conditions hold:

(i) for all $h > 0$ and all $r = 0, 1, \ldots, N_h$,

$$\hat{p}^{s,h}_{ij}(t) = \hat{p}^{s,h,r+1}_{ij} \triangleq \hat{p}^s_{ij}\left(\tau^w_{0,h,r+1} - \tau^0_{ij}\right), \quad \forall t \in (\tau^w_0 + rh, \tau^w_0 + (r + 1)h],$$

(ii) for every continuous function $\phi$ on $I$,

$$\lim_{h \to 0} \int_{\tau^w_0}^{\tau^w_1} \hat{p}^{s,h}_{ij}(t) \phi(t) dt = \int_{\tau^w_0}^{\tau^w_1} \hat{p}^s_{ij}(t) \phi(t) dt.$$
Theorem 1. Suppose that the inflow rate \( \hat{p}_{ij}(t - \tau_{ij}^0) \) satisfy assumption (A) on the interval \( I \triangleq [\tau_{ij}^0, \tau_{ij}^1] \). There exist a sequence of step sizes \( \{h_r\} \downarrow 0 \), an absolutely continuous function \( \hat{q}_{ij}(t) \) on \( I \), and an integral function \( \hat{v}_{ij}(t) \) on \( I \) such that \( \hat{q}_{ij}^{h_r} \to \hat{q} \) uniformly on \( I \) and \( \hat{v}_{ij}^{h_r} \to \hat{v}_{ij} \) on \( L^2(I) \). Moreover, any such limit function \( \hat{q}_{ij}(t) \) is a weak solution of the ODE (3) in the sense that for any two times \( t_2 > t_1 \) in the subinterval \( I \),
\[
\hat{q}_{ij}(t_2) - \hat{q}_{ij}(t_1) = \int_{t_1}^{t_2} \max \left( \hat{p}_{ij}(s - \tau_{ij}^0) - \bar{C}, -\alpha \hat{q}(s) \right) ds = \int_{t_1}^{t_2} \left( \hat{p}_{ij}(s - \tau_{ij}^0) - \hat{v}_{ij}(s) \right) ds;
\]

or equivalently, the ODE (3) and \( \hat{v}_{ij}(t) = \hat{p}_{ij}(t - \tau_{ij}^0) - \frac{d\hat{q}_{ij}(t)}{dt} \) hold for almost all \( t \in I \).

Theorem 3. Suppose that \( \hat{q}_{ij}^{h_r} \to \hat{q}_{ij} \) uniformly on \( I \). Then a sequence of step sizes \( \{h_r\} \downarrow 0 \) exists such that for all \( s \in S \) and all \( (i,j) \in \mathcal{L}_s \), the following weak limits exist in \( L^2(I) \): \( \hat{v}_{ij}^{h_r} \to \hat{v}_{ij}^{s}, \hat{p}_{ij}^{s,h_r} \to \hat{p}_{ij}^{s}, \) and \( \hat{\eta}_{ij}^{s,h_r} \to \hat{\eta}_{ij}^{s} \). Moreover, any such limit \( \left( \hat{v}_{ij}^{s}, \hat{p}_{ij}^{s}, \hat{\eta}_{ij}^{s} \right) \) satisfies for almost all \( t \in I \),
\[
0 \leq \hat{p}_{ij}^{s}(t) \quad \text{and} \quad \tau_{ij}^0 + \frac{\hat{q}_{ij}(t)}{\bar{C}_{ij}} + \hat{\eta}_{ij}^{s}(t) - \hat{\eta}_{ij}^{s}(t) \geq 0, \quad \forall (i,j) \in \mathcal{L}_s
\]
\[
0 \leq \hat{\eta}_{ij}^{s}(t) \quad \text{and} \quad \sum_{j:(i,j) \in \mathcal{L}_s} \hat{p}_{ij}^{s}(t) - \sum_{j:(j,i) \in \mathcal{L}_s} \hat{v}_{ij}^{s}(t) - d_{ij}^{s}(t) = 0, \quad \forall i \in \mathcal{N}_s
\]
\[
0 = \hat{\eta}_{ij}^{s}(t);
\]

moreover, on any subinterval \( I' \subseteq I \) where \( \hat{p}_{ij}^{s,h_r} \to \hat{p}_{ij}^{s} \) uniformly, it holds that
\[
\int_{I'} \hat{p}_{ij}^{s}(t) \left[ \tau_{ij}^0 + \frac{\hat{q}_{ij}(t)}{\bar{C}_{ij}} + \hat{\eta}_{ij}^{s}(t) - \hat{\eta}_{ij}^{s}(t) \right] dt = 0.
\]
Numerical results

- Observation from the instantaneous route choice behavior:
  - Some drivers from 1 to 3 will select link 1, who will come back to the origin at 1 later on.

Example: a small network

Origin: 1; destinations: 2,3,4;
T = 70 min
Link exit capacity: 500 vph for link 4 and 1000 vph for all other links
Free flow travel time: 17 minutes for links 2 and 6, and 10 minutes for all other links
Demand from node 1 to node 2

\[ D(t) = \max(0, 1500 - \frac{24000}{T^2}(t - T/4)^2). \]
A Real-World Case Study

• Real network near the Kichijoji Station, Tokyo, Japan
  – Data Collected in 1997

• IDUE model
  – A simplification of the introduced DUE model

IDUE model results
Evaluation by Micro Simulation Model Calibration Criteria
Real-world data
Results

• Solving IDUE numerically
  – Total time span: 130 minutes
  – Time step length: 5.4 seconds
  – ~100,000 linear complementarity problems (LCP)

• Evaluation by Micro Simulation Model Calibration Criteria (Dowling et al., 2002)
  – Percentages of satisfying links over all observed links
    | $N$ | 1  | 2  | 4  | 5  | 6  | 7  | 13 |
    |-----|----|----|----|----|----|----|----|
    | GEH  | 46%| 44%| 45%| 47%| 44%| 42%| 50%|
    | Volume difference | 39%| 43%| 42%| 42%| 40%| 40%| 44%|
  – Percentages of satisfying links over boundary links
    | $N$ | 1  | 2  | 4  | 5  | 6  | 7  | 13 |
    |-----|----|----|----|----|----|----|----|
    | GEH  | 85%| 88%| 89%| 89%| 86%| 86%| 86%|
    | Volume difference | 90%| 88%| 86%| 85%| 82%| 81%| 83%|

<85%  >85%
Path Travel Times

• Ten randomly selected paths
Observations

- IDUE is an oversimplified version of UDE
  - Point queue model cannot capture spillback
  - Route choice is based on instantaneous travel times
  - IDUE does not model traffic signals
- Despite all these, IDUE can predict:
  - The volumes of 50% of the links accurately
  - The volumes of 90% of the boundary links accurately
  - The travel times of selected paths to a reasonable extent

Predictive DUE

• Uses actual travel times (more complex)
• More realistic traffic dynamics model, especially to capture possible queue spillback
• Queue spillback leads to flow interactions of adjacent links, i.e., the time decomposition (separation) idea for point queue model would not work here

• Traffic dynamics model: capture traffic realism and also be efficient for network (large-scale) applications
• They should focus on capturing inter-link interactions, i.e., link transmission model (LTM) type is preferred
Traffic Dynamics

• Describe how traffic evolves over time and space (surface traffic in particular)
• LWR model (1950s)
• Cell Transmission Model (CTM; Daganzo, 1994)
• Link Transmission Model (LTM; Yperman, 2007)
• Double Queue Model (DQM; Osorio, 2011; Ma et al., 2014)
Double Queue Model

- The two queues are not independent
- Queue capacity: Upper-bound of upstream queue
- Non-negativity: Lower-bound of downstream queue
- The two queues work as “gates” to regulate flow in/out of the link, respecting traffic flow dynamics

Double Queue Model

\[ 0 \leq q_{ij}^d(t) \leq p_{ij}(t) \leq q_{ij}^u(t) \leq Q_{ij} \]

- Spillback happens when \( q_{ij}^u(t) = Q_{ij} \) occurs
- *Spillback* means congestion propagates to the entrance of a link, i.e., the inflow will be determined by the downstream condition (at an early time). This may or may not restrict the inflow rate to the link.
- Spillback does not mean traffic jam (or stoppage)
- Spillback does not necessarily lead to restriction to the inflow, if the inflow is small enough
Link 1: 2.9 mi
Link 2: 0.7 mi

Number of variables

<table>
<thead>
<tr>
<th>Link 1</th>
<th>Link 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 per link</td>
<td>4 per link</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

2 per cell

30 cells per mi

156
42
Nodal model with fixed discharging priorities

• In practice, most intersections need to specify the discharging priorities for the incoming roads.
Nodal model with fixed discharging priorities

- Discharging priorities at a general intermediate node
Spillbacks in capacitated queue dynamics

- Exit flow cannot reach the one that defined in PQ model when downstream link is congested.

\[ \delta_{n^{\text{in}}_i} (t) \] : exit flow ‘withheld’ due to the congestion at the downstream link(s)

\[ \eta_{ij} (t) \] : inflow that needs to be ‘withheld’ at a link because the upstream queue reaches its queue storage capacity
DCS-based Nodal Model

• “If-then-else” rules can be reformulated as complementarity conditions; see Ban et al. (2012)
• Min/max operators can be reformulated as complementarity conditions, e.g.:
  \[ x = \min(a, b) \iff 0 \leq a - x \perp b - x \geq 0 \]
• Using complementarity to explicitly express the relationship of state variables (\(q\)’s) and auxiliary variables (\(\delta\) and \(\eta\))


Ma, R., Ban, X., Pang, J.S., 2015. Continuous-time dynamic user equilibria for single-destination traffic networks with queue spillbacks. Submitted to *Transportation Science (2nd revision)*.
DCS-based Nodal Model

\[
\delta_{n_i^m, i}(t) \triangleq \min \left\{ \frac{C_{n_i^m, i} - \mu_{n_i^m, i}(t)}{\sum_{j: (i, j) \in \mathcal{L}'} \eta_{ij}(t)} - \sum_{1 \leq \hat{m} \leq m-1} \delta_{n_i^\hat{m}, i}(t) \right\}, \quad m = 1, \ldots, M_i
\]

\[
\delta_{n_i^m, i}(t) = \max \left( 0, \delta_{n_i^m, i}(t) \right), \quad m = 1, \ldots, M_i.
\]

\[
0 \leq v_{n_i^m, i}(t) \perp \sum_{j: (i, j) \in \mathcal{L}'} \eta_{ij}(t) - \sum_{1 \leq \hat{m} \leq m-1} \delta_{n_i^\hat{m}, i}(t) - \delta_{n_i^m, i}(t) \geq 0, \quad m = 1, \ldots, M_i.
\]

\[
0 \leq \delta_{n_i^m, i}(t) \perp \delta_{n_i^m, i}(t) - \delta_{n_i^m, i}(t) \geq 0 \quad m = 1, \ldots, M_i.
\]

where \( \delta_{n_i^m, i}(t) = C_{n_i^m, i} - \mu_{n_i^m, i}(t) - v_{n_i^m, i}(t) \).

\[
0 \leq q_{ij}^d(t) \perp \mu_{ij}(t) \geq 0
\]

\[
0 \leq \eta_{ij}(t) \perp Q_{ij} - q_{ij}^u(t) \geq 0.
\]
The DUE Model

- The double queue dynamics (link-level, constant delays)
- The nodal model (network level)

- Flow conservation
- Route choice (time-varying, state-dependent delays)
- Departure time choice

- A DCS with time-varying, state-dependent delays
- Approximation via constant-delays
Numerical example

- A simple network with one OD pair, node 1 to 5
- Dummy origin: node 6
- Dummy destination: node 7

<table>
<thead>
<tr>
<th>link</th>
<th>$\tau_{ij}^0$</th>
<th>$\tau_{ij}^\omega$</th>
<th>$\bar{C}_{ij}$</th>
<th>$\bar{Q}_{ij}$</th>
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<tr>
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Discharging priorities with spillbacks

Exit flow and queues of link 1-3

Exit flow and queues of link 2-3
Open Questions

• Travel time discontinuities

• Stability and convergence of the model

• Integration with real time data / decisions
  – Beyond model parameter estimation and periodic updating
  – Serves as network structure for statistical estimation/prediction
  – Fundamental changes to the modeling framework
Thank You

• Questions?

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