Modeling Dynamic User Equilibria as Differential Complementarity Systems (DCS)

Xuegang (Jeff) Ban, Rensselaer Polytechnic Institute (RPI)

Workshop II: Traffic Estimation

Institute of Pure and Applied Mathematics (IPAM), UCLA

October 13, 2015

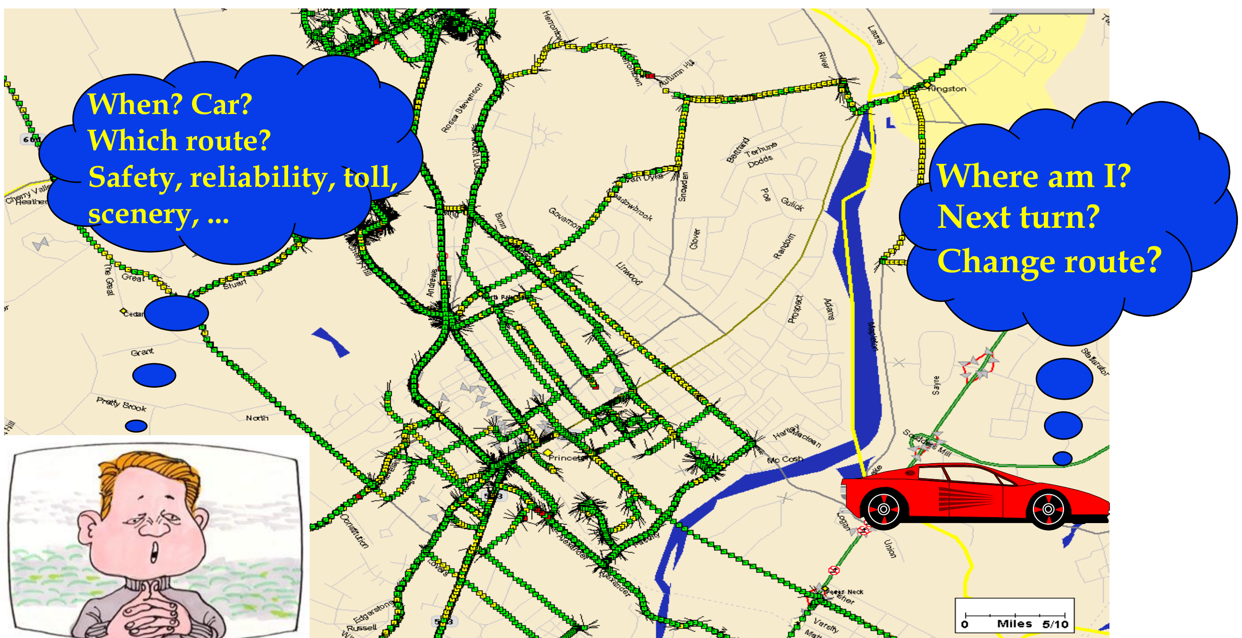
Collaborators

- **Dr. Jong-Shi Pang**, University of Southern California
- ***Dr. Rui Ma***, University of California, Davis

- **Dr. Henry Liu**, University of Michigan
- ***Dr. Wilfredo Yushimito***, Adolfo Ibáñez University, Chile
- **Dr. Wen Yuan Szeto**, Hong Kong University, China
- **Dr. Bin Ran**, University of Wisconsin-Madison
- **Dr. Michael C. Ferris,** University of Wisconsin-Madison

Background & History

How is traffic distributed in a (urban) traffic network and why?



Transportation Network Modeling

- Transportation Network Modeling (Traffic Assignment): predict flow distribution in a traffic network, given the total demand (e.g., during the peak period)
- Traffic Equilibrium (Frank Knight, 1924)
- Wardrop First Principle: User Equilibrium (Wardrop, 1952)

  The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route
- Wardrop Second Principle: System Optimal (Wardop, 1952)

  At equilibrium, the average journey time is minimum
- Close connection with Nash equilibrium (John Nash, <Beautiful Mind>)

Dynamic Traffic Assignment (DTA)

- Intelligent Transportation Systems (ITS) leads to Dynamic Traffic Assignment
- Two major components: Traffic dynamics and traveler behavior
- Three types of players

Literature: Merchant and Nemhauser, 1978; Friesz et al, 1993; Ran and Boyce, 1994; Lo and Szeto, 2002; Mahamassani, 2001; Ben-Akiva et al., 2001; *Ban et al., 2008, 2009, 2012a, 2012b; Ma et al., 2014; Ma et al., 2015a, 2015b;* **Network PDE** approach (Bressan and Nguyen, 2015)

Dynamic User Equilibrium (DUE)

Auto

Cournot-Nash Equilibrium

Freight

Dynamic System Optimum (DSO)

Transit

Path-based DUE Condition

- DUE Condition (Ran and Boyce, 1996):

  If, for each OD pair at each instant of time, the actual travel times experienced by travelers departing at the same time are equal and minimal, the dynamic traffic flow over the network is in a travel-time-based dynamic user equilibrium (DUE).

$$\begin{cases} \eta_p^{rs}(t) = \pi^{rs}(t) \Rightarrow f_p^{rs}(t) \geq 0 \\ \eta_p^{rs}(t) > \pi^{rs}(t) \Rightarrow f_p^{rs}(t) = 0 \end{cases}$$

$f_p^{rs}(t)$: path flow for path p between OD pair rs at time instant t

$\eta_p^{rs}(t)$: path travel time for path p between OD pair rs at time instant t

$\pi^{rs}(t)$: minimum path travel time for all paths between OD pair rs at time instant t

$\Downarrow$

$$0 \leq [\eta_p^{rs}(t) - \pi^{rs}(t)] \perp f_p^{rs}(t) \geq 0, \forall r, s, p, t \in [0, T]$$ NCP Formulation (infinite dimensional)

$\Downarrow$

$$\sum_{rs}\sum_{p}\int_0^T [f_p^{rs}(t) - f_p^{rs^*}(t)]\eta_p^{rs^*}(t) \geq 0$$ VI Formulation (infinite dimensional)

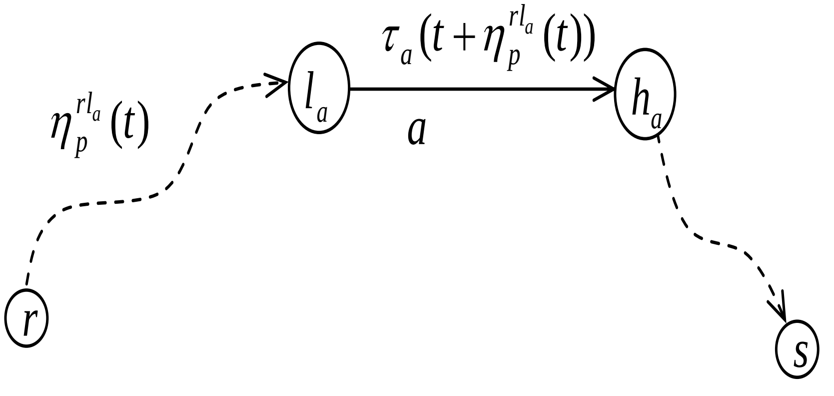
Path Travel Time

- Instantaneous Path Travel Time

$$\psi_p^{rs}(t) = \sum_{a \in p} \tau_a(t)$$

- Ideal (Actual) Path Travel Time

$$\eta_p^{rh_a}(t) = \eta_p^{rl_a}(t) + \tau_a(t + \eta_p^{rl_a}(t))$$



Example

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$$

$$\tau_a(t) = 2 + t$$

$$\tau_b(t) = 3 + 2t$$

$$\tau_c(t) = 4 + 1.5t$$

$$\psi^{14}(t) = \tau_a(t) + \tau_b(t) + \tau_c(t) = 9 + 4.5t$$

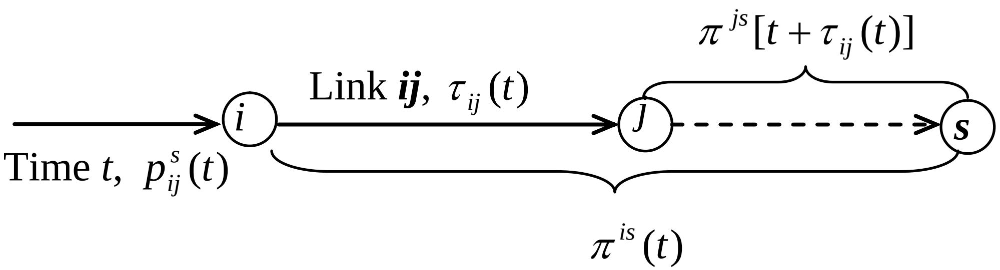
$$\eta^{12}(t) = \tau_a(t) = 2 + t$$

$$\eta^{13}(t) = \tau_a(t) + \tau_b[t + \tau_a(t)] = (2 + t) + \tau_b(2 + 2t) = (2 + t) + [3 + 2(2 + 2t)] = 9 + 5t$$

$$\eta^{14}(t) = \eta^{13}(t) + \tau_c[t + \eta^{13}(t)] = (9 + 5t) + \tau_c(9 + 6t) = (9 + 5t) + [4 + 1.5(9 + 6t)] = 26.5 + 14t$$

Link-Node Based DUE Condition

- If, from each decision node to every destination node at each instant of time, the actual travel times for all the routes that are being used are equal and minimal, then the dynamic traffic flow over the network is in a travel time based dynamic user equilibrium (DUE) state.



$$\begin{cases} \tau_{ij}(t)+\pi^{js}[t+\tau_{ij}(t)]=\pi^{is}(t) \Rightarrow p_{ij}^{s}(t)\geq 0 \\ \tau_{ij}(t)+\pi^{js}[t+\tau_{ij}(t)]>\pi^{is}(t) \Rightarrow p_{ij}^{s}(t)=0 \end{cases}$$

$$\Downarrow$$

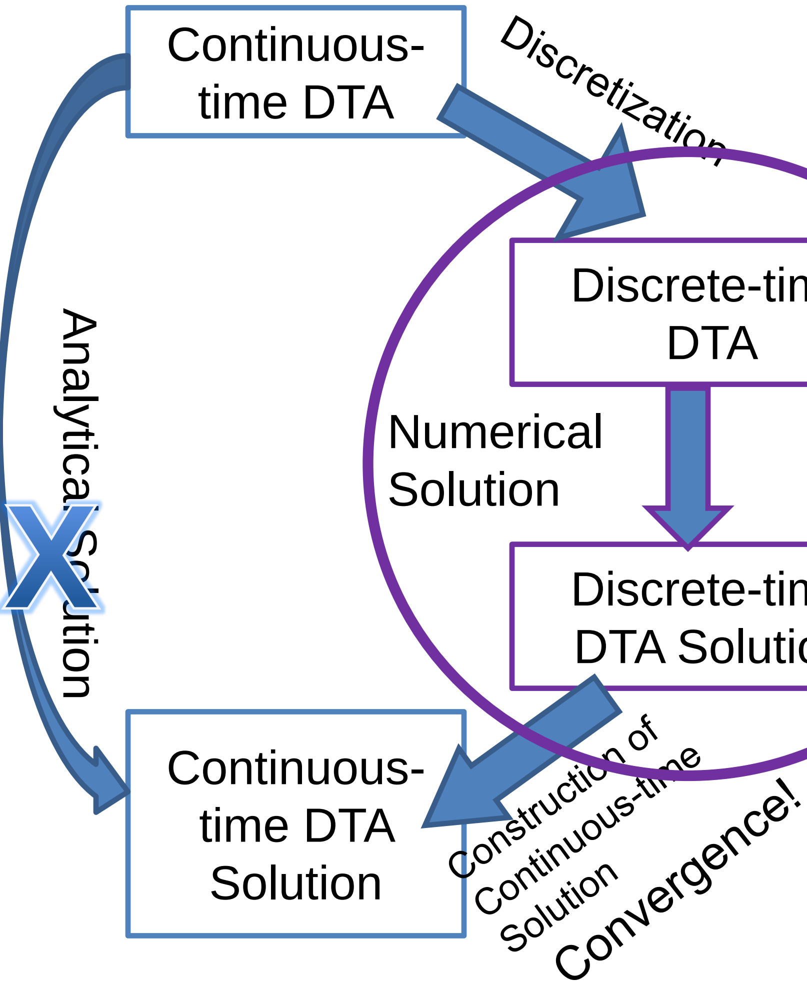
$$0 \leq p_{ij}^{s}(t) \perp \{\tau_{ij}(t)+\pi^{js}[t+\tau_{ij}(t)]-\pi^{is}(t)\} \geq 0 \longleftarrow$$ NCP (infinite dimensional)

Ban, X., Liu, X., Ferris, M.C., and Ran, B., 2008. A link-node complementarity model and solution algorithm for dynamic user equilibria with exact flow propagations. *Transportation Research Part B*, 46(3), 389-408.

Types of DUE

- Simulation-based vs *Analytical* DUE models
- Instantaneous (reactive) vs. actual (predictive) DUE
- Path-based vs *Link-based* DUE models
- *Continuous-time* vs. discrete-time DUE models

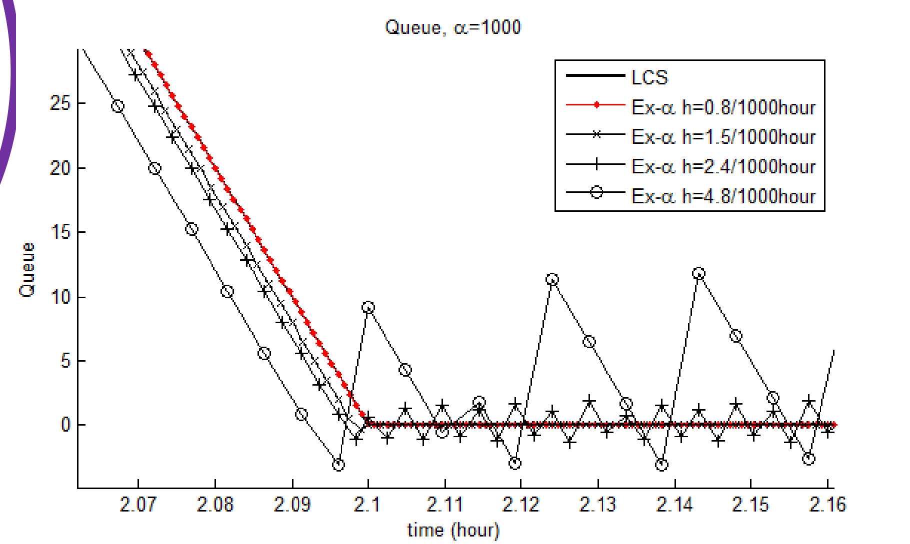
Continuous-time vs. Discrete-time



Eg: α-PQ model: a continuous-time system

$$\dot{q}(t) = \begin{cases} 0 & \text{if } t \in (0, \tau^0) \\ \max\left(p(t-\tau^0) - \overline{C},\, -\alpha\, q(t)\right) & t > \tau^0. \end{cases}$$

Numerical Solutions



Ma, R., Ban, X., Pang, J.S., and Liu, X., 2014. Time discretization of continuous-time dynamic network loading models. *Networks and Spatial Economics,* 1-23.

Modeling DUE

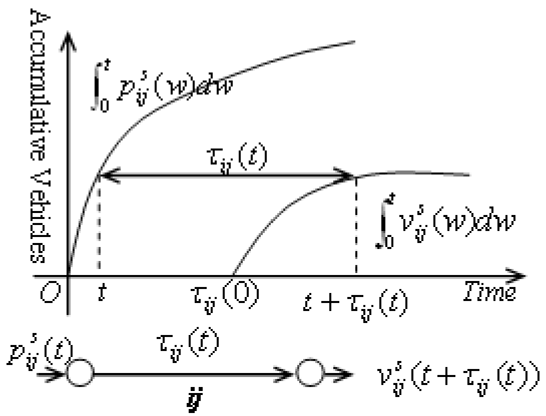
D
U
E

User choice model (behavioral)

User equilibrium

VI/NCP (optimization)

Traffic dynamics (physical)



PDEs or ODEs with time delay

A mathematical framework that can properly capture both?

DUE Literature (Analytical Formulations)

- Most on discrete-time DUE problems
- A handful on continuous-time DUE problems: modeling techniques, solution techniques, among others
  - Optimal control method
    - Friesz et al. (1989), Ran and Shimazaki (1989), Ran et al. (1993) , Lam et al. (1995)
  - Variational inequality (VI) method
    - Friesz et al. (1993), Smith (1993), Ran and Boyce (1994, 1996), Bliemer and Bovy (2003)
- There has been a lack of a proper mathematical framework to capture both aspects of DUE, choice behavior and system dynamics
  - Discretization scheme of continuous-time models
  - Convergence of continuous trajectories constructed from discrete time solutions

Differential Complementarity System (DCS)

- An ODE parameterized by an algebraic variable that is required to be a solution of a finite dimensional state dependent complementarity problem:

$$
\begin{array}{rcll}
\dot{x}(t) \triangleq \dfrac{dx(t)}{dt} & = & f(t, x(t), u(t)) & \text{dynamics} \\
0 \leq u(t) & \perp & F(t, x(t), u(t)) \geq 0 & \text{complementarity constraint} \\
x(0) & = & x^0 & \text{initial condition,}
\end{array}
$$

- DCS is a special case of the Differential Variational Inequality (DVI); see Pang and Stewart (2008) and Friesz (2010) for DVI
- Focus on a special DCS (Pang and Stewart, 2008)
- $\dot{x}(t) = f(t, x) + B(t, x)u$ $\longleftarrow$ Dynamics linear on *u*
- $0 \leq u(t) \perp G(t, x) + F(u(t))$ $\longleftarrow$ Complementarity separable on *u*

- Let Ω=[0, T]xR^n and two conditions

(A) f, B, and G are Lipschitz continuous functions on Ω with Lipschitz constants L_f, L_B, and L_G, respectively;
(B) B is bounded on Ω with $\sigma_B \equiv \sup_{(t,x)\in\Omega} \|B(t,x)\| < \infty$.

- Convergence + Solution existence

Theorem 7.1 *Let $K \subseteq \Re^m$ be a nonempty closed convex set and let (f, B, G) satisfy conditions (A) and (B). Suppose that there exist positive scalars $c_{0,x}$, $c_{1,x}$, $c_{0,u}$, $c_{1,u}$, and $\bar{h}$ such that for all $h \in (0, \bar{h}]$ and all $i = 0, 1, \ldots, N_h$,*

$$\| x^{h,i+1} \| \leq c_{0,x} + c_{1,x} \| x^0 \| \quad \textit{and} \quad \| u^{h,i+1} \| \leq c_{0,u} + c_{1,u} \| x^0 \|. \tag{7.5}$$

There is a sequence $\{h_\nu\} \downarrow 0$ such that the following two limits exist: $\widehat{x}^{h_\nu} \to \widehat{x}$ uniformly on $[0, T]$ and $\widehat{u}^{h_\nu} \to \widehat{u}$ weakly in $L^2(0, T)$. Furthermore, under either one of the following two conditions:

(a) *$F(u) \equiv \Psi(Eu)$, where $E \in \Re^{\ell \times m}$ and $\Psi : \Re^\ell \to \Re^m$ is Lipschitz continuous, and a constant $c_{2,u} > 0$ exists such that for all h sufficiently small,*

$$\| Eu^{h,i+1} - Eu^{h,i} \| \leq h\, c_{2,u}, \tag{7.6}$$

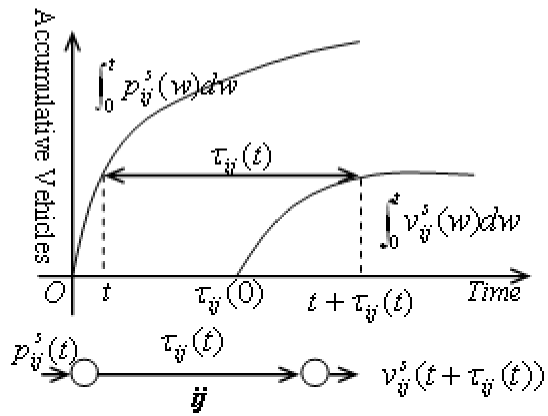
(b) *$F(u) \equiv Du$ for some positive semidefinite matrix D,*

all such limits $(\widehat{x}, \widehat{u})$ are weak solutions of the initial-value DVI (6.2).

DUE

User choice model (behavioral) → User equilibrium → VI* / NCP** → DCS*** with Time delay: A mathematical framework that can properly capture both

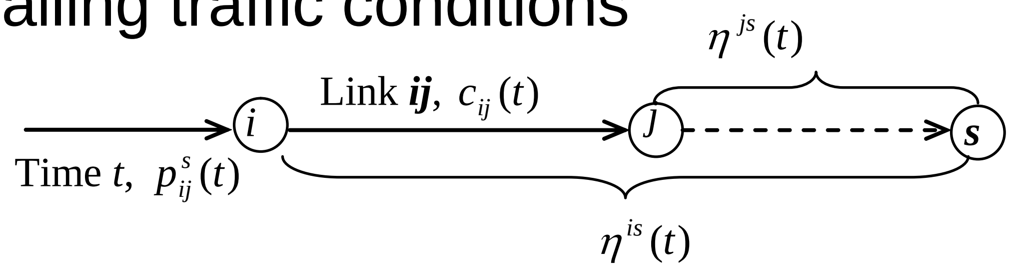
Traffic dynamics (physical) → PDEs / ODEs with time delay → DCS*** with Time delay: A mathematical framework that can properly capture both



*VI: Variational Inequality
**NCP: Nonlinear Complementarity Problem
***DCS: Differential Complementarity System

Instantaneous DUE: Simplification

- Route choice condition of instantaneous DUE: based on prevailing traffic conditions



$$0 \le p_{ij}^{s}(t) \perp \{c_{ij}(t) + \eta^{js}(t) - \eta^{is}(t)\} \ge 0 \longleftarrow$$ **NCP** (infinite dimensional)

- Traffic dynamics: Point Queue Model
- A DCS

$$\dot{q}(t) = u(t) + p(t - t_0) - \overline{C}$$

$$0 \le u(t) \perp q(t) \ge 0,$$

$$v(t) = \overline{C} - u(t)$$

$$c_{ij}(t) = \tau_{ij}^{0} + (\overline{C}_{ij})^{-1} q_{ij}(t)$$

Observations:
Discontinuities may occur when queue length changes from 0 to nonzero or vice-versa.

Ban, X., Pang, J.S., Liu, X., and Ma, R., 2012. Continuous-time Point-Queue Models in Dynamic Network Loading. *Transportation Research Part B*, 46(3), 360-380.

DCS Formulation for IDUE

$$0 \le p_{ij}^s(t) \quad \perp \quad \tau_{ij}^0 + \frac{\sum_{s':(i,j)\in\mathcal{L}_{s'}} q_{ij}^{s'}(t)}{\overline{C}_{ij}} + \eta_j^s(t) - \eta_i^s(t) \ge 0, \qquad \forall\,(i,j) \in \mathcal{L}_s$$

Route Choice

$$0 \le \eta_i^s(t) \quad \perp \quad \sum_{j:(i,j)\in\mathcal{L}_s} p_{ij}^s(t) - \sum_{j:(j,i)\in\mathcal{L}_s} v_{ji}^s(t) - d_i^s(t) \ge 0, \quad \forall i \in \mathcal{N}_s$$

$$0 = \eta_s^s(t),$$

Flow conservation

$$v(t) = \overline{C} - u(t)$$

$$\dot{q}(t) = u(t) + p(t - t_0) - \overline{C}$$

$$0 \le u(t) \quad \perp \quad q(t) \ge 0,$$

$$c_{ij}(t) = \tau_{ij}^0 + (\overline{C}_{ij})^{-1} q_{ij}(t)$$

Queue dynamics

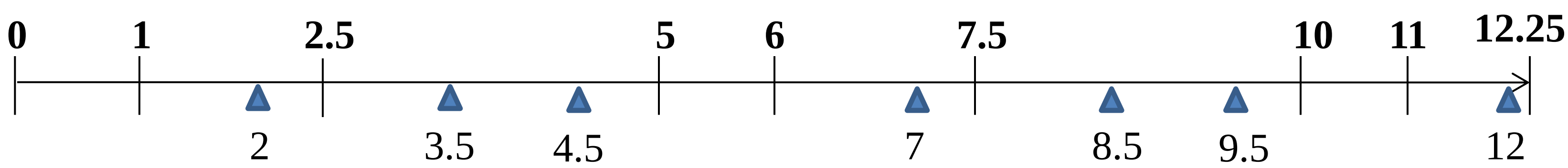
$$v_{ij}^s(t) = \begin{cases} \frac{p_{ij}^s(t')}{p_{ij}(t')} v_{ij}(t) & \text{if } p_{ij}(t') > 0 \\ 0 & \text{if } p_{ij}(t') = 0. \end{cases}$$

$$t' + \tau_{ij}^0 + \frac{q_{ij}(t' + \tau_{ij}^0)}{\overline{C}} = t.$$

Destination-based exit capacity

Solution Method – Time Decomposition

- We apply time decomposition based on free flow link times
  - The DCS model is naturally decomposable to (link, destination pair)
  - Due to instantaneous route choice, the route choice (i.e., to determine the inflow to a link) and traffic dynamics (i.e., to load the inflow to generate queue length and exit flow) can be done separately
  - We divide the entire study period [0, T] to *intervals* based on free flow link travel times, and then divide each interval into *sub-intervals* based on the minimum free flow link travel time; same scheme as in Xu et al. (1999)
  - For the DCS in each sub-interval, the time delayed term (inflow) is already determined in the previous sub-interval and thus known
- Example: three links with $\tau_1^0 = 1; \tau_2^0 = 2.5; \tau_3^0 = 5$; study period [0, 12.25]



Solution Method – Time Stepping

- For each *sub-interval*, IDUE is a DCS without time delay because the delay term (inflow to a link) is already calculated in the previous sub-interval (or known as zero).
- It is solved by the *time stepping method* for ODE (Shampine and Thompson, 2001)
- The traffic dynamics (network loading) and route choice models are solved separately and iteratively
- We apply the *implicit* discretization scheme
- Discrete traffic dynamics model: an NCP that has at least one solution
- Discrete route choice model: an NCP that at least one solution
- NCPs are solved using the PATH solver in GAMS

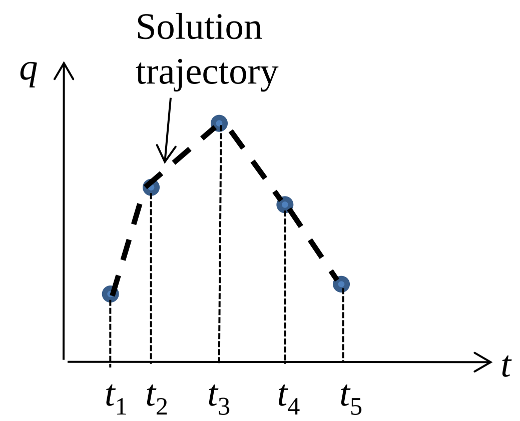
Continuous Time Solution Trajectories

- Construct piece wise linear trajectory based on the discrete solution, for state variables

$$q_{ij}^{s}(t) = q_{ij}^{s,t_i} + \frac{t-t_i}{h}\left(q_{ij}^{s,t_{i+1}} - q_{ij}^{s,t_i}\right), \text{for } t \in [t_i, t_{i+1}]$$

t_i, t_{i+1} : discrete time points

$h = t_{i+1} - t_i$: the length of each discrete time step

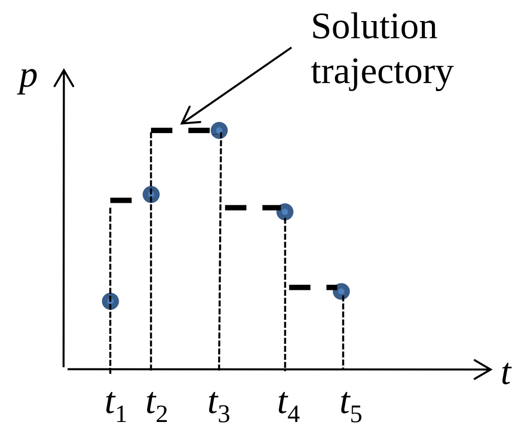


- Construct piece wise constant trajectory based on the discrete solution, for algebraic variables

$$p_{ij}^{s}(t) = p_{ij}^{s,t_i+1}, \text{for } t \in [t_i, t_{i+1}]$$

t_i, t_{i+1} : discrete time points

$h = t_{i+1} - t_i$: the length of each discrete time step



Assumption (A)

- For a *subinterval I*, $\hat{p}^s_{ij}\left(t-\tau^0_{ij}\right)$ is the (known) bounded, nonnegative, integrable inflow rate to link (i,j) for destination s
  $\hat{p}^s_{ij}\left(t-\tau^0_{ij}\right)$ is equal to the weak limit of piecewise constant functior $\hat{p}^{s;h}_{ij}(t)$ as $h \downarrow 0$, i.e., the following conditions hold

  (i) for all $h > 0$ and all $r = 0, 1, \ldots, N_h$,

  $$\hat{p}^{s;h}_{ij}(t) = \hat{p}^{s;h,r+1}_{ij} \triangleq \hat{p}^s_{ij}\left(\tau^{\mathrm{w};h,r+1}_0 - \tau^0_{ij}\right), \quad \forall t \in \left(\tau^{\mathrm{w}}_0 + rh, \tau^{\mathrm{w}}_0 + (r+1)h\right],$$

  (ii) for every continuous function ϕ on I,

  $$\lim_{h\to 0} \int_{\tau^{\mathrm{w}}_0}^{\tau^{\mathrm{w}}_1} \hat{p}^{s;h}_{ij}(t)\,\phi(t)\,dt = \int_{\tau^{\mathrm{w}}_0}^{\tau^{\mathrm{w}}_1} \hat{p}^s_{ij}(t)\,\phi(t)\,dt.$$

Solution Existence and Convergence

Theorem 1. *Suppose that the inflow rate* $\hat{p}_{ij}\left(t-\tau_{ij}^{0}\right)$ *satisfy assumption* (**A**) *on the interval* $I \triangleq \left[\tau_{0}^{w}, \tau_{1}^{w}\right]$. *There exist a sequence of step sizes* $\{h_{v}\} \downarrow 0$, *an absolutely continuous function* $\hat{q}_{ij}(t)$ *on I, and an integral function* $\hat{v}_{ij}(t)$ *on I such that* $\hat{q}_{ij}^{h_{v}} \rightarrow \hat{q}$ *uniformly on I and* $\hat{v}_{ij}^{h_{v}} \rightarrow \hat{v}_{ij}$ *on* $L^{2}(I)$. *Moreover, any such limit function* $\hat{q}_{ij}(t)$ *is a weak solution of the ODE (3) in the sense that for any two times* $t_{2} > t_{1}$ *in the subinterval I,*

$$\hat{q}_{ij}(t_{2}) - \hat{q}_{ij}(t_{1}) = \int_{t_{1}}^{t_{2}} \left[\max\left(\hat{p}_{ij}(s-\tau_{ij}^{0}) - \overline{C}, -\alpha\hat{q}(s)\right)\right] ds = \int_{t_{1}}^{t_{2}} \left(\hat{p}_{ij}\left(s-\tau_{ij}^{0}\right) - \hat{v}_{ij}(s)\right) ds;$$

or equivalently, the ODE (3) and $\hat{v}_{ij}(t) = \hat{p}_{ij}\left(t-\tau_{ij}^{0}\right) - \frac{d\hat{q}_{ij}(t)}{dt}$ *hold for almost all* $t \in I$.

Theorem 3. *Suppose that* $\hat{q}_{ij}^{h} \rightarrow \hat{q}_{ij}$ *uniformly on I. Then a sequence of step sizes* $\{h_{v}\} \downarrow 0$ *exists such that for all* $s \in \mathcal{S}$ *and all* $(i,j) \in \mathcal{L}_{s}$, *the following weak limits exist in* $L^{2}(I)$: $\hat{v}_{ij}^{s;h_{v}} \rightarrow \hat{v}_{ij}^{s}$, $\hat{p}_{ij}^{s;h_{v}} \rightarrow \hat{p}_{ij}^{s}$, *and* $\hat{\eta}_{i}^{s;h_{v}} \rightarrow \hat{\eta}_{i}^{s}$. *Moreover, any such limit* $\left(\hat{v}_{ij}^{s}, \hat{p}_{ij}^{s}, \hat{\eta}_{i}^{s}\right)$ *satisfies for almost all* $t \in I$,

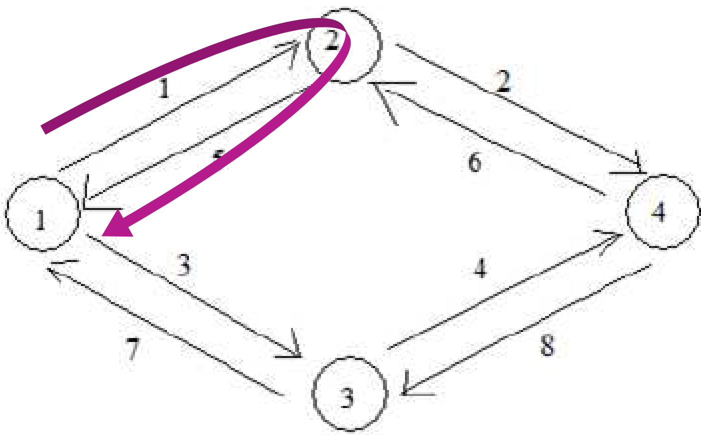
$$\begin{aligned}
&0 \leqslant \hat{p}_{ij}^{s}(t) \quad \text{and} \quad \tau_{ij}^{0} + \frac{\hat{q}_{ij}(t)}{\overline{C}_{ij}} + \hat{\eta}_{j}^{s}(t) - \hat{\eta}_{i}^{s}(t) \geqslant 0, && \forall (i,j) \in \mathcal{L}_{s}\\
&0 \leqslant \hat{\eta}_{i}^{s}(t) \quad \text{and} \quad \sum_{j:(i,j)\in\mathcal{L}_{s}} \hat{p}_{ij}^{s}(t) - \sum_{j:(j,i)\in\mathcal{L}_{s}} \hat{v}_{ji}^{s}(t) - d_{i}^{s}(t) = 0, && \forall i \in \mathcal{N}_{s}\\
&0 = \hat{\eta}_{s}^{s}(t);
\end{aligned} \tag{24}$$

moreover, on any subinterval $I' \subseteq I$ *where* $\hat{p}_{ij}^{s;h_{v}} \rightarrow \hat{p}_{ij}^{s}$ *uniformly, it holds that*

$$\int_{I'} \hat{p}_{ij}^{s}(t) \left[\tau_{ij}^{0} + \frac{\hat{q}_{ij}(t)}{\overline{C}_{ij}} + \hat{\eta}_{j}^{s}(t) - \hat{\eta}_{i}^{s}(t)\right] dt = 0.$$

Numerical results

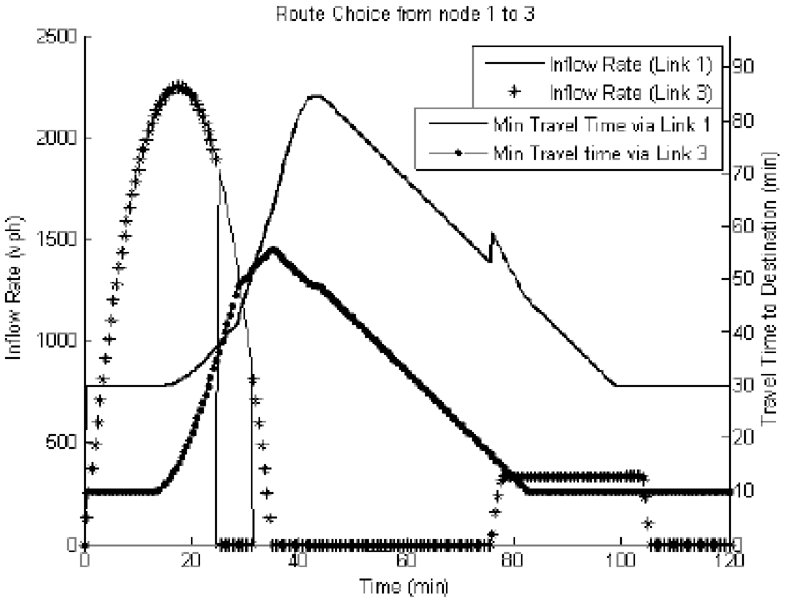
- Observation from the instantaneous route choice behavior:
  - Some drivers from 1 to 3 will select link 1, who will come back to the origin at 1 later on.



Example: a small network

Origin: 1; destinations: 2,3,4;
T= 70 min
Link exit capacity: 500 vph for link 4 and 1000 vph for all other links
Free flow travel time: 17 minutes for links 2 and 6, and 10 minutes for all other links
Demand from node 1 to node 2

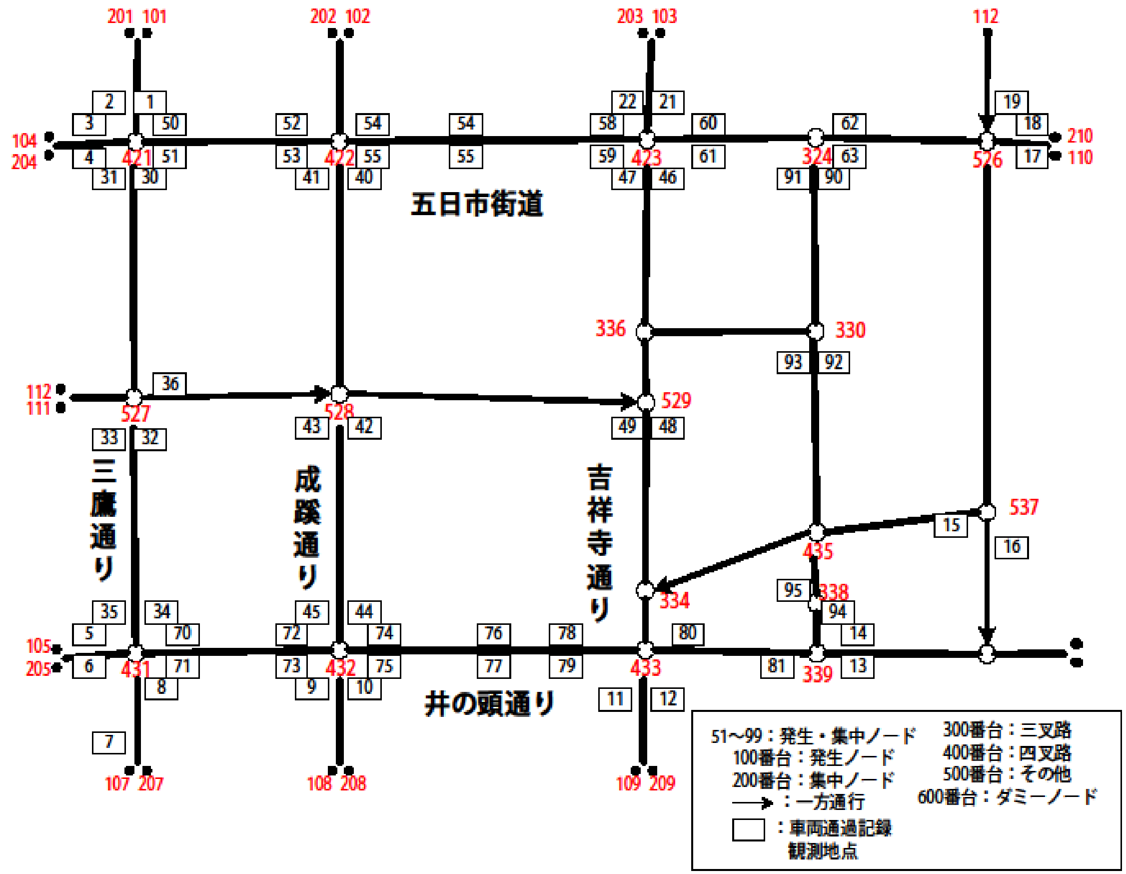
$$D(t) = max(0, 1500 - \frac{24000}{T^2}(t - T/4)^2).$$



Route choice 1 to 3

A Real-World Case Study

- Real network near the Kichijoji Station, Tokyo, Japan
  - Data Collected in 1997
- IDUE model
  - A simplification of the introduced DUE model



IDUE model results ⟷ Real-world data

Evaluation by Micro Simulation Model Calibration Criteria

Results

- Solving IDUE numerically
  - Total time span: 130 minutes
  - Time step length: 5.4 seconds
  - ~100,000 linear complementarity problems (LCP)
- Evaluation by Micro Simulation Model Calibration Criteria (Dowling et al., 2002)
  - Percentages of satisfying links over all observed links

N	1	2	4	5	6	7	13
GEH Statistic	46%	44%	45%	47%	44%	42%	50%
Volume-difference based	39%	43%	42%	42%	40%	40%	44%

<85%

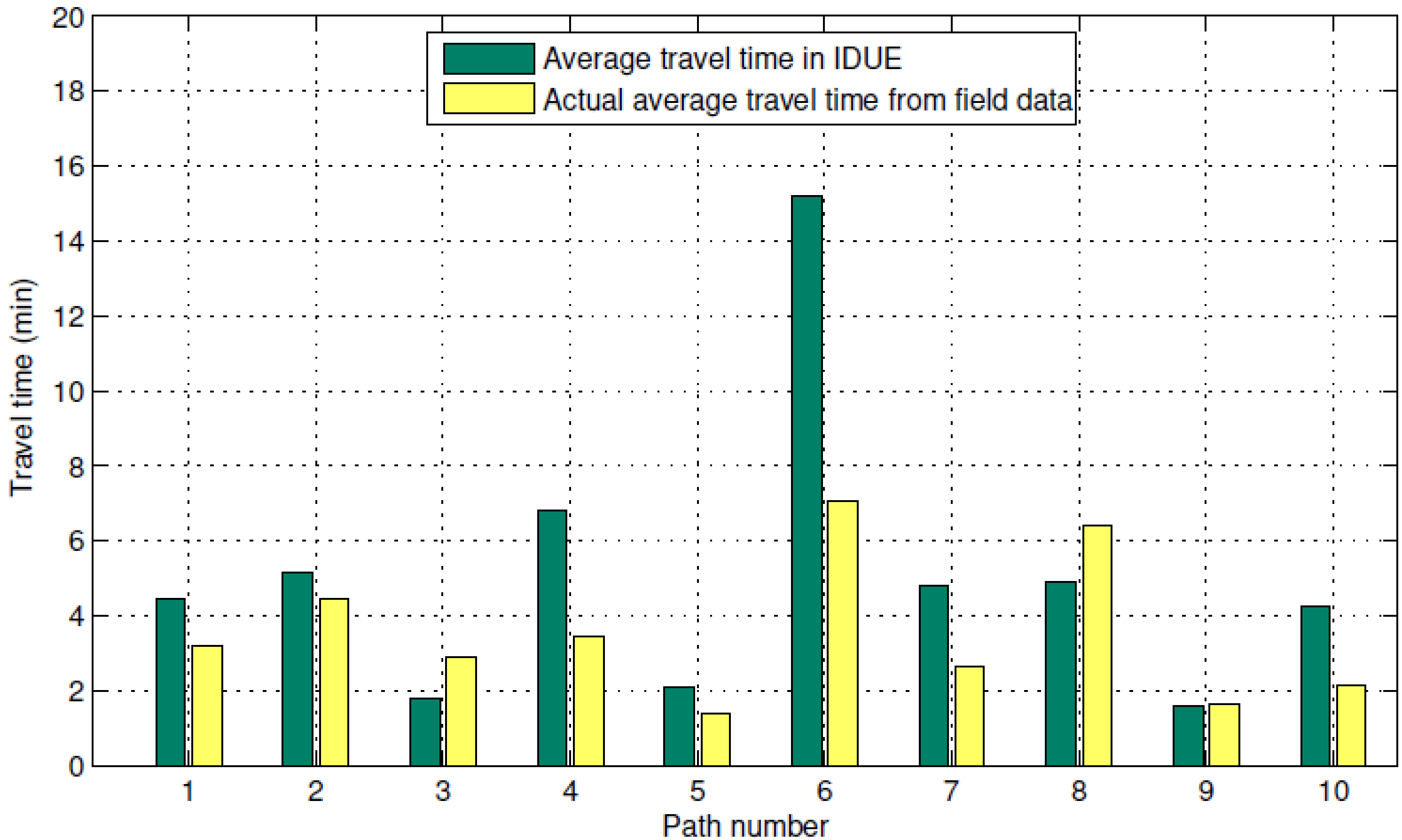
  - Percentages of satisfying links over boundary links

N	1	2	4	5	6	7	13
GEH Statistic	85%	88%	89%	89%	86%	86%	86%
Volume-difference based	90%	88%	86%	85%	82%	81%	83%

>85%

Path Travel Times

- Ten randomly selected paths



Observations

- IDUE is an oversimplified version of UDE
  - Point queue model cannot capture spillback
  - Route choice is based on instantaneous travel times
  - IDUE does not model traffic signals
- Despite all these, IDUE can predict:
  - The volumes of 50% of the links accurately
  - The volumes of 90% of the boundary links accurately
  - The travel times of selected paths to a reasonable extent

Ma, R., Ban, X., 2014. Continuous-time Instantaneous Dynamic User Equilibria on A Real World Traffic Network. In *Proceedings of the 94th Annual Meeting of Transportation Research Board,* Washington, DC.

Predictive DUE

- Uses actual travel times (more complex)
- More realistic traffic dynamics model, especially to capture possible queue spillback
- Queue spillback leads to flow interactions of adjacent links, i.e., the time decomposition (separation) idea for point queue model would not work here

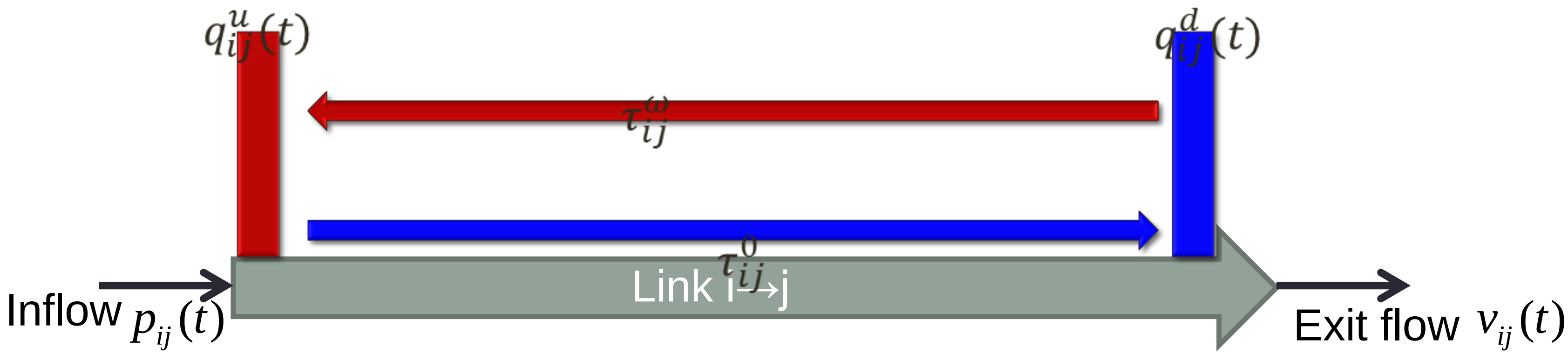
- Traffic dynamics model: capture traffic realism and also be efficient for network (large-scale) applications
- They should focus on capturing inter-link interactions, i.e., link transmission model (LTM) type is preferred

Traffic Dynamics

- Describe how traffic evolves over time and space (surface traffic in particular)
- LWR model (1950s)
- Cell Transmission Model (CTM; Daganzo, 1994)
- Link Transmission Model (LTM; Yperman, 2007)
- Double Queue Model (DQM; Osorio, 2011; *Ma et al., 2014*)

Double Queue Model

$$\dot{q}_{ij}^{u}(t) = p_{ij}(t) - v_{ij}(t - \tau_{ij}^{\omega}),$$
$$\dot{q}_{ij}^{d}(t) = p_{ij}(t - \tau_{ij}^{0}) - v_{ij}(t).$$



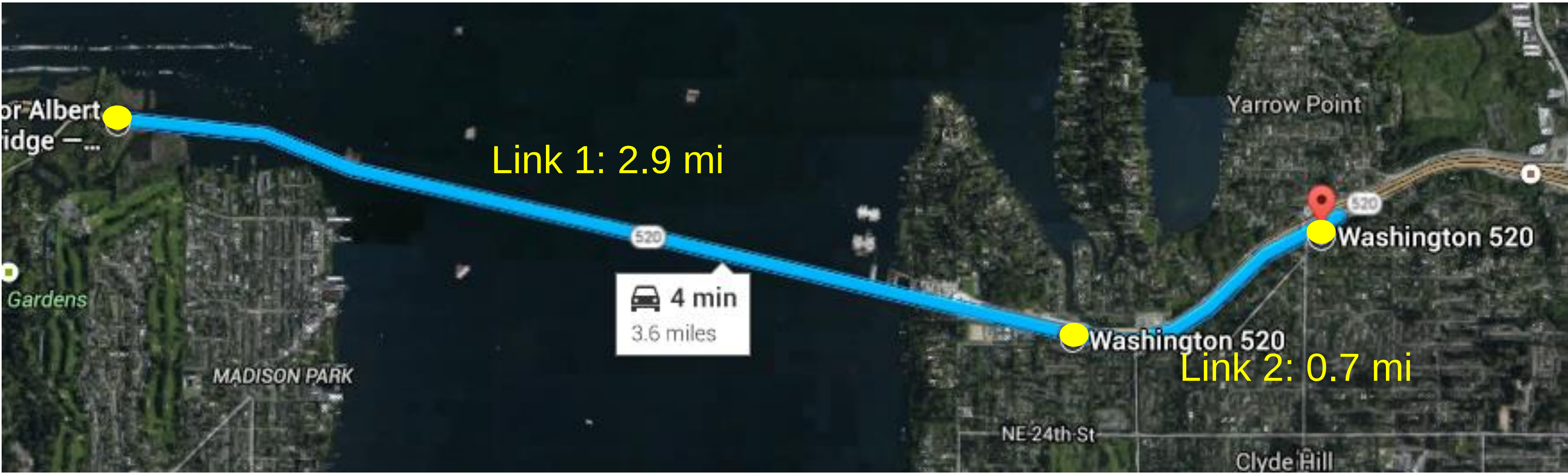
- The two queues are not independent
- Queue capacity: Upper-bound of upstream queue
- Non-negativity: Lower-bound of downstream queue
- The two queues work as “gates” to regulate flow in/out of the link, respecting traffic flow dynamics

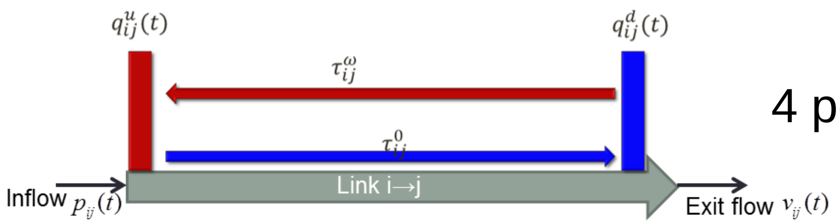
Ma, R., Ban, X., Pang, J.S., 2014. Continuous-time dynamic system optimal for single-destination traffic networks with queue spillbacks. *Transportation Research Part B* 68, 98-122.

Double Queue Model

$$0 \le q_{ij}^{d}(t) \le p_{ij}(t) \le q_{ij}^{u}(t) \le \overline{Q}_{ij}$$

- Spillback happens when $q_{ij}^{u}(t) = \overline{Q}_{ij}$ occurs
- *Spillback* means congestion propagates to the entrance of a link, i.e., the inflow will be determined by the downstream condition (at an early time). This may or may not restrict the inflow rate to the link.
- Spillback does not mean traffic jam (or stoppage)
- Spillback does not necessarily lead to restriction to the inflow, if the inflow is small enough





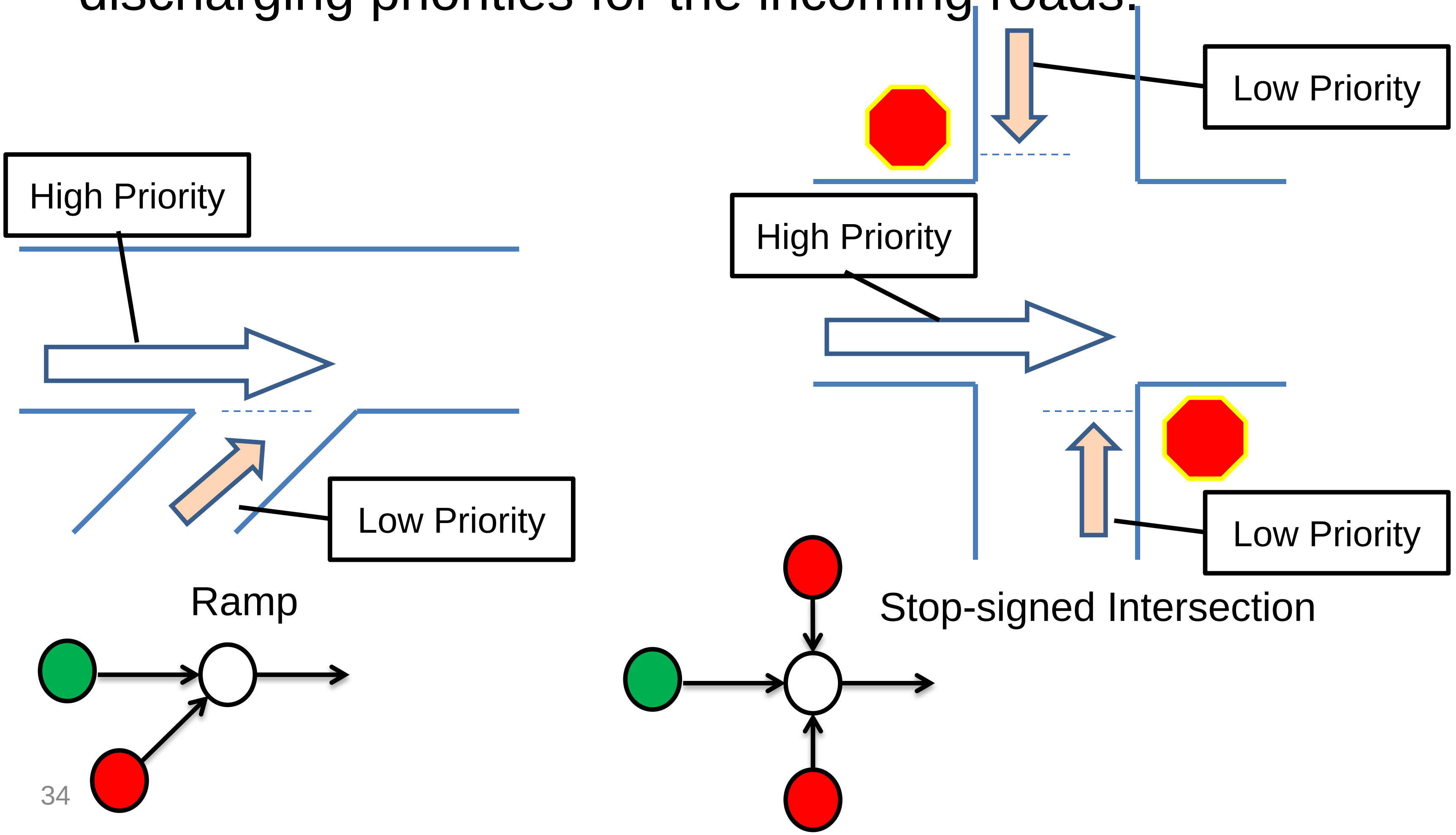
30 cells per mi

Number of variables

	Link 1	**Link 2**
4 per link	**4**	**4**
2 per cell	**156**	**42**

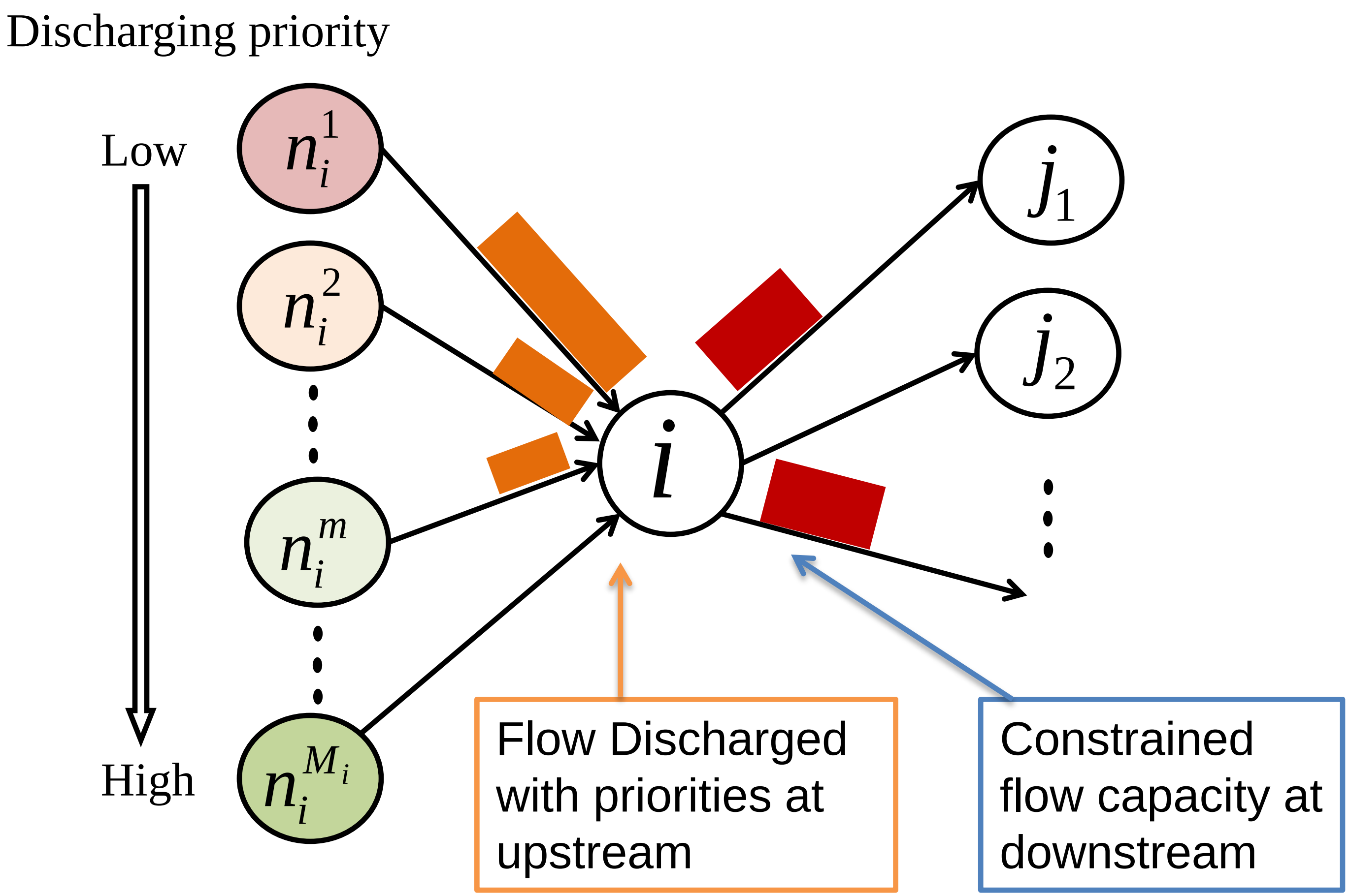
Nodal model with fixed discharging priorities

- In practice, most intersections need to specify the discharging priorities for the incoming roads.



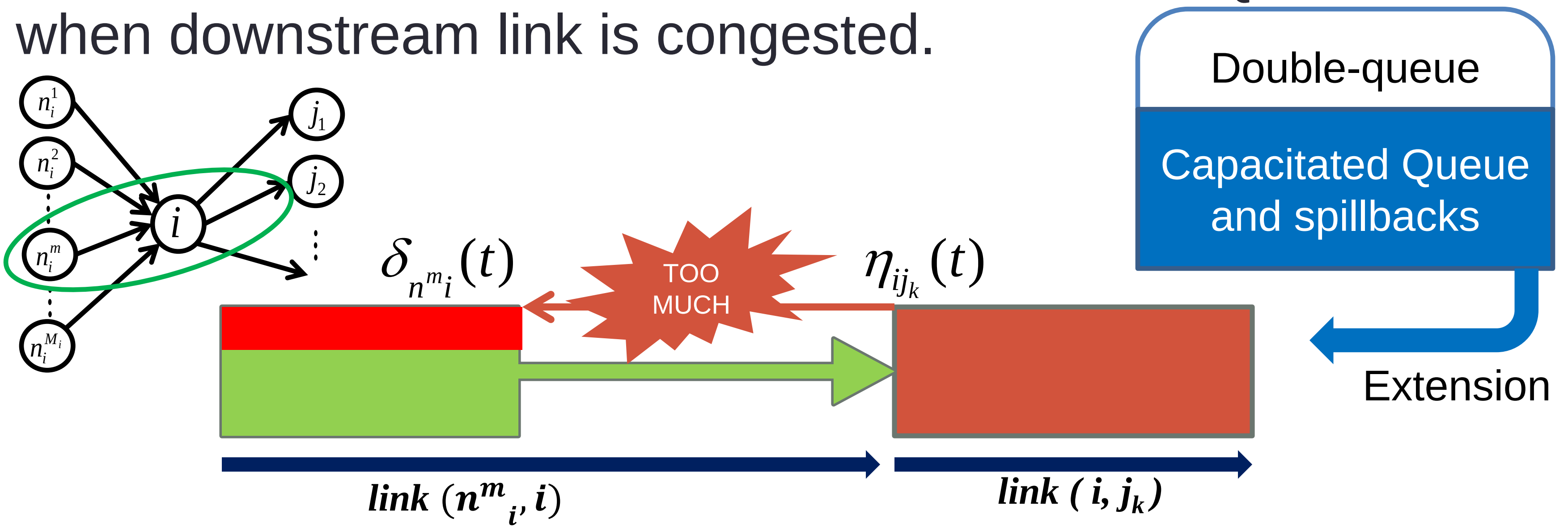
Nodal model with fixed discharging priorities

- Discharging priorities at a general intermediate node



Spillbacks in capacitated queue dynamics

- Exit flow cannot reach the one that defined in PQ model when downstream link is congested.



- $\delta_{ij}(t)$: exit flow 'withheld' due to the congestion at the downstream link(s)
- $\eta_{ij}(t)$: inflow that needs to be 'withheld' at a link because the upstream queue reaches its queue storage capacity

DCS-based Nodal Model

- "If-then-else" rules can be reformulated as complementarity conditions; see Ban et al. (2012)
- Min/max operators can be reformulated as complementarity conditions, e.g.:

$$x = \min(a, b) \Leftrightarrow 0 \le a - x \perp b - x \ge 0$$

- Using complementarity to explicitly express the relationship of state variables (*q*'s) and auxiliary variables (δ and η)

Ban, X., Pang, J.S., Liu, X., and Ma, R., 2012. Continuous-time Point-Queue Models in Dynamic Network Loading. *Transportation Research Part B*, 46(3), 360-380.

Ma, R., Ban, X., Pang, J.S., 2015. Continuous-time dynamic user equilibria for single-destination traffic networks with queue spillbacks. Submitted to *Transportation Science (2nd revision)*.

DCS-based Nodal Model

$$\underline{\delta}_{n_i^m,i}(t) \triangleq \min\left(\overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t),\ \underbrace{\sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t)}_{\substack{\text{total withheld inflow}\\ \text{to downstream links}\\ \text{at node } i}} - \underbrace{\sum_{1\le\widehat{m}\le m-1} \delta_{n_i^{\widehat{m}},i}(t)}_{\substack{\text{total withheld exit flow}\\ \text{from upstream links}\\ \text{with lower priority}}} \right),\quad m = 1,\cdots,M_i$$

$$\delta_{n_i^m,i}(t) = \max\left(0, \underline{\delta}_{n_i^m,i}(t)\right),\qquad m = 1,\cdots,M_i.$$

$$0 \le \quad v_{n_i^m,i}(t) \perp \sum_{j:(i,j)\in\mathcal{L}'} \eta_{ij}(t) - \sum_{1\le\widehat{m}\le m-1} \delta_{n_i^{\widehat{m}},i}(t) - \underline{\delta}_{n_i^m,i}(t) \ge 0,\quad m = 1,\cdots,M_i.$$

$$0 \le \quad \delta_{n_i^m,i}(t) \perp \delta_{n_i^m,i}(t) - \underline{\delta}_{n_i^m,i}(t) \ge 0 \qquad m = 1,\cdots,M_i.$$

where $\underline{\delta}_{n_i^m,i}(t) = \overline{C}_{n_i^m,i} - \mu_{n_i^m,i}(t) - v_{n_i^m,i}(t)$.

$$0 \le q_{ij}^d(t) \perp \mu_{ij}(t) \ge 0$$

$$0 \le \eta_{ij}(t) \perp \overline{Q}_{ij} - q_{ij}^u(t) \ge 0.$$

The DUE Model

- The double queue dynamics (link-level, *constant delays*)
- The nodal model (network level)

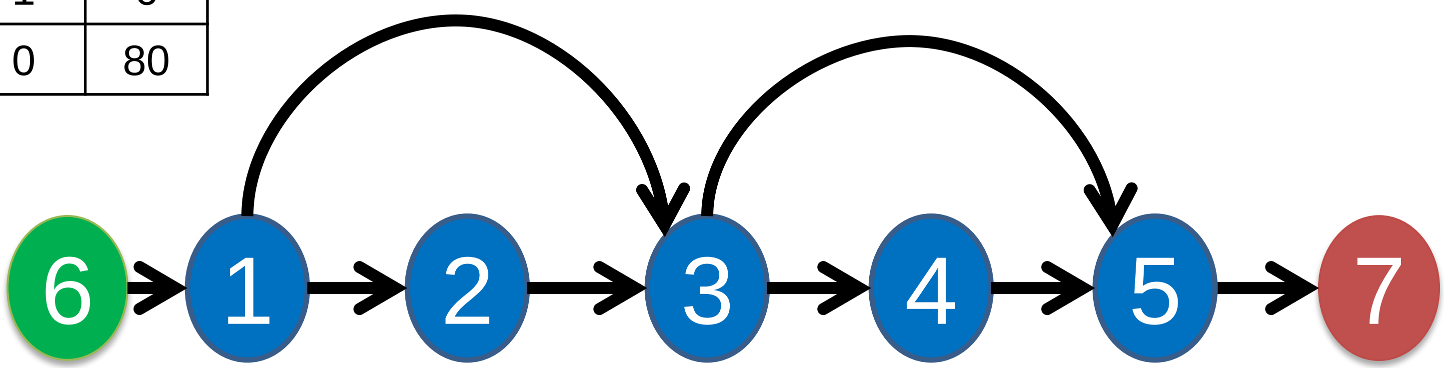
- Flow conservation
- Route choice (*time-varying, state-dependent delays*)
- Departure time choice

- A DCS with time-varying, state-dependent delays
- Approximation via constant-delays

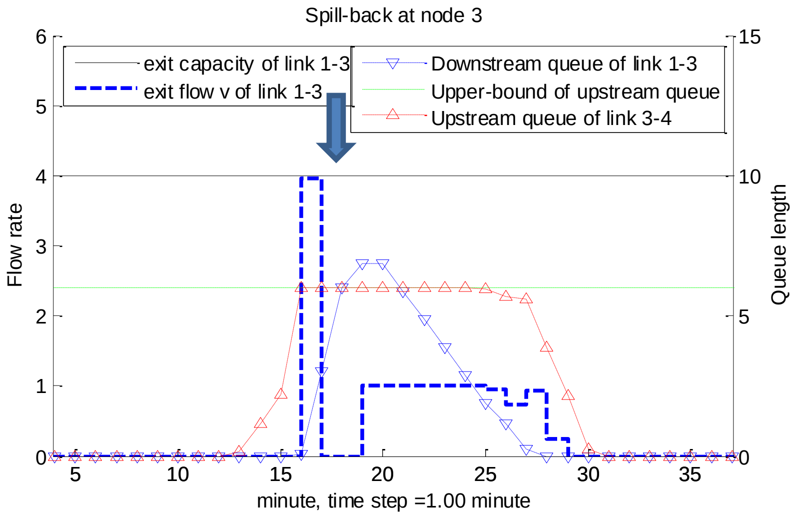
Numerical example

link	τ_{ij}^{0}	τ_{ij}^{ω}	$\overline{C}_{ij}$	$\overline{Q}_{ij}$
6-1	0	0	14	40
1-2	4	8	3	36
2-3	1	2	2	6
1-3	6	12	4	72
3-4	1	2	2	6
4-5	1	2	2	6
3-5	2	4	1	6
5-7	0	0	0	80

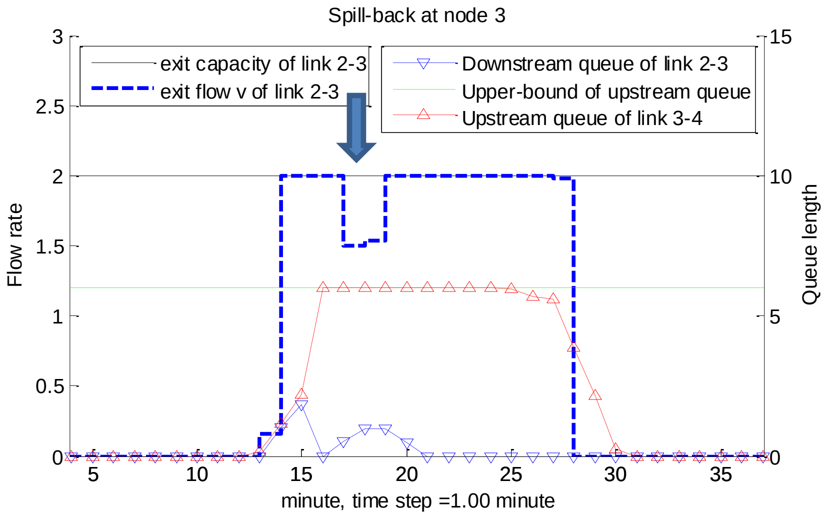
- A simple network with one OD pair, node 1 to 5
- Dummy origin: node 6
- Dummy destination: node 7



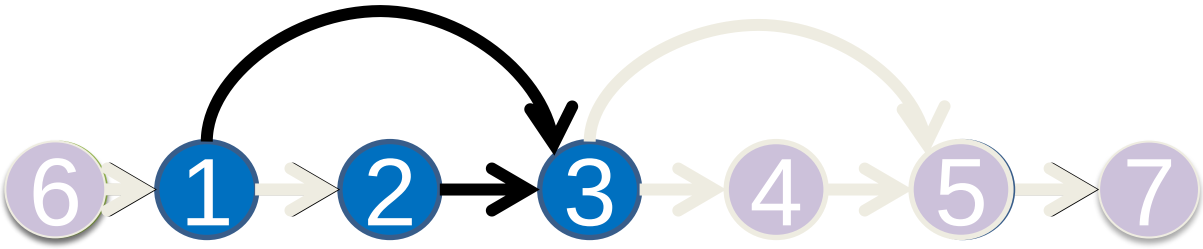
Discharging priorities with spillbacks



Exit flow and queues of link 1-3

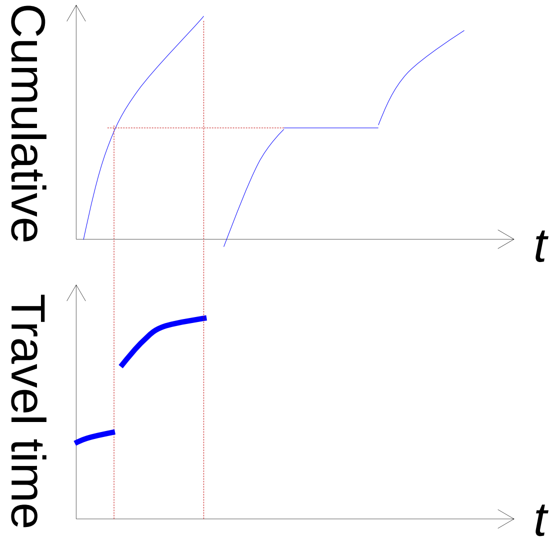


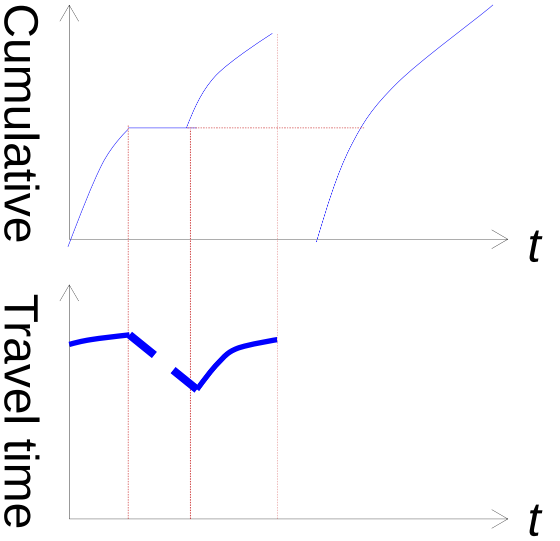
Exit flow and queues of link 2-3



Open Questions

- Travel time discontinuities





- Stability and convergence of the model
- Integration with real time data / decisions
  - Beyond model parameter estimation and periodic updating
  - Serves as network structure for statistical estimation/prediction
  - Fundamental changes to the modeling framework

Thank You

- Questions?
- Contact Information:

Email: banx@rpi.edu

Web: www.rpi.edu/~banx