Coupling Different Traffic Models

Francesca Marcellini

Department of Mathematics and Applications
University of Milano–Bicocca, Italy

September 28th - October 2nd, 2015

Mathematical Foundations of Traffic
IPAM, UCLA, Los Angeles, USA

Introduction

A Two-Phase Model

A Discrete–Continuous Description

A Traffic Model Aware of Real Time Data

Two–Phase Model

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho\, V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho\, V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

Different drivers have different maximal speeds

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

Different drivers have different maximal speeds

$$\begin{cases} \partial_t \rho + \partial_x (\rho V) = 0 \\ V = w\, \psi(\rho) \end{cases}$$

$\rho = \rho(t,x)$	traffic density
$V = V(w,\rho)$	traffic speed
$w = w(t,x) > 0$	maximal traffic speed

$\psi \in \mathbf{C}^2([0,R])$, $\psi(0) = 1$, $\psi(R) = 0$, $\psi'(\rho) \leq 0$, describes the attitude of drivers to choose their speed depending on the traffic density at their location

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

Different drivers have different maximal speeds

$$\begin{cases} \partial_t \rho + \partial_x (\rho V) = 0 \\ \partial_t w + v(\rho, w)\, \partial_x w = 0 \end{cases}$$

$\rho = \rho(t, x)$	traffic density
$V = w\, \psi(\rho)$	traffic speed
$w = w(t, x) > 0$	maximal traffic speed

w is a specific feature of every single driver

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

We assume that there exists an overall maximal speed $V_{\max}$:

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right) = 0 \\ \partial_t w + v(\rho, w)\, \partial_x w = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max}, w\, \psi(\rho)\right\}$$

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

We assume that there exists an overall maximal speed $V_{\max}$:

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right) = 0 \\ \partial_t w + v(\rho, w)\, \partial_x w = 0 \end{cases} \qquad \text{with} \qquad v = \min \left\{ V_{\max}, w\, \psi(\rho) \right\}$$

Since

$$\partial_t(\rho\, w) + \partial_x \left(\rho\, w\, v(\rho, w)\right) =$$

$$\rho \left(\partial_t w + v(\rho, w)\, \partial_x w\right) + w \left(\partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right)\right)$$

we obtain an equivalent form of the system:

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

A 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ flow:

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right) = 0 \\ \partial_t(\rho\, w) + \partial_x \left(\rho\, w\, v(\rho, w)\right) = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max}, w\,\psi(\rho)\right\}$$

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho\, V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

A 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ flow:

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, w)\right) = 0 \\ \partial_t (\rho\, w) + \partial_x \left(\rho\, w\, v(\rho, w)\right) = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max}, w\, \psi(\rho)\right\}$$

S. Blandin, D. Work, P. Goatin, B. Piccoli, A. Bayen: SIAM J.Appl.Math. 2011

R.M. Colombo, F. Marcellini, M. Rascle: SIAM J.Appl.Math. 2010

Two–Phase Model

The Classical LWR Model

$$\partial_t \rho + \partial_x (\rho V) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	V	$V = V(\rho)$ traffic speed

With $\eta = \rho w$:

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho\, v(\rho, w)\big) = 0 \\ \partial_t \eta + \partial_x \big(\eta\, v(\rho, \eta)\big) = 0 \end{cases} \quad \text{with} \quad v = \min \left\{ V_{\max}, \frac{\eta}{\rho}\, \psi(\rho) \right\}$$

Two–Phase Model

The speed bound $V_{\max}$ induces two distinct *phases*:

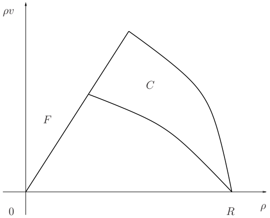
free phase	$F = \{v = V_{\max}\}$	high speed	low density
congested phase	$C = \{v = w\,\psi(\rho)\}$	low speed	high density

Two–Phase Model

The speed bound V_{max} induces two distinct *phases*:

free phase	$F = \{v = V_{max}\}$	high speed	low density
congested phase	$C = \{v = w\,\psi(\rho)\}$	low speed	high density

The Fundamental Diagram

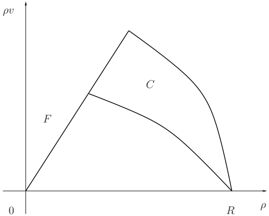


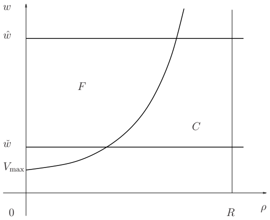
Two–Phase Model

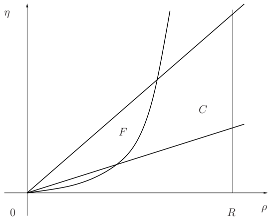
The speed bound $V_{\max}$ induces two distinct *phases*:

free phase	$F = \{v = V_{\max}\}$	high speed	low density
congested phase	$C = \left\{v = w\,\psi(\rho)\right\}$	low speed	high density

The Fundamental Diagram







Two–Phase Model – The Riemann Problem

For all states (ρ^l, η^l), $(\rho^r, \eta^r) \in F \cup C$, the Riemann problem of

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{array}\right. \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$

with initial data

$$\rho(0,x) = \left\{\begin{array}{lll} \rho^l & \text{if } x & < & 0 \\ \rho^r & \text{if } x & > & 0 \end{array}\right. \qquad \eta(0,x) = \left\{\begin{array}{lll} \eta^l & \text{if } x & < & 0 \\ \eta^r & \text{if } x & > & 0 \end{array}\right.$$

Theorem (R.M.Colombo, F.Marcellini, M.Rascle: SIAM J.Appl.Math. 2010)
The Riemann problem admits a unique self similar weak entropy solution (ρ, η)

Two–Phase Model – The Riemann Problem

For all states (ρ^l, η^l), $(\rho^r, \eta^r) \in F \cup C$, the Riemann problem of

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} \quad \text{with} \quad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$

with initial data

$$\rho(0,x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases} \qquad \eta(0,x) = \begin{cases} \eta^l & \text{if } x < 0 \\ \eta^r & \text{if } x > 0 \end{cases}$$

$$\text{If } \left.\begin{matrix} (\rho^l,\eta^l) \in F & (\rho^r,\eta^r) \in C \\ (\rho^l,\eta^l) \in C & (\rho^r,\eta^r) \in F \end{matrix}\right\} \Rightarrow \boxed{\boxed{\text{Phase Transitions}}}$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left(\left(t, p(t)\right)\right)\right) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\, w\,\psi(\rho)\right\},$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left(\left(t, p(t)\right)\right)\right) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\, w\,\psi(\rho)\right\},$$

$$w \text{ is specific of every single driver, } w\left(t, p(t)\right) = w(0, p_o)$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left((t, p(t)\right)\right) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\ w\,\psi(\rho)\right\},$$

$$w \text{ is specific of every single driver, } w\left(t, p(t)\right) = w(0, p_o)$$

$$n \text{ drivers, } \ell \text{ standard car length}$$

$$\Rightarrow\ \rho(t, x) \simeq \sum_i \frac{\ell}{p_{i+1}(t) - p_i(t)}\ \chi_{]p_i(t), p_{i+1}(t)]}(x)$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot{p} = v\left(\rho\left(t, p(t)\right), w\left(\left(t, p(t)\right)\right)\right) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\, w\,\psi(\rho)\right\},$$

the follow-the-leader model
$$\begin{cases} \dot{p}_i = v\left(\frac{\ell}{p_{i+1} - p_i}, w_i\right) \\ \dot{p}_{n+1} = V_{\max} \\ p_i(0) = p_{o,i} \end{cases}$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot{p} = v\Big(\rho\left(t, p(t)\right), w\left((t, p(t)\right)\Big) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\ w\,\psi(\rho)\right\},$$

If the follow-the-leader model $\begin{cases} \dot{p}_i = v\left(\frac{\ell}{p_{i+1}-p_i}, w_i\right) \\ \dot{p}_{n+1} = V_{\max} \\ p_i(0) = p_{o,i} \end{cases}$ is such that

$$\sum_i \frac{\ell}{p_{i+1}(t) - p_i(t)}\, \chi_{]p_i(t), p_{i+1}(t)]}(x) \to \rho(t, x) \text{ a.e.}$$

Follow-The-Leader Model

A single driver starting from p_o at $t = 0$ follows the particle path $p = p(t)$ that solves

$$\begin{cases} \dot p = v\left(\rho\left(t, p(t)\right), w\left((t, p(t)\right)\right) \\ p(0) = p_o \end{cases} \qquad v(\rho, w) = \min\left\{V_{\max},\ w\,\psi(\rho)\right\},$$

If the follow-the-leader model $\begin{cases} \dot p_i = v\left(\frac{\ell}{p_{i+1}-p_i}, w_i\right) \\ \dot p_{n+1} = V_{\max} \\ p_i(0) = p_{o,i} \end{cases}$ is such that

$$\sum_i \frac{\ell}{p_{i+1}(t) - p_i(t)}\, \chi_{]p_i(t), p_{i+1}(t)]}(x) \to \rho(t, x) \text{ a.e.}$$

then ρ solves the macroscopic model.

Follow-The-Leader Model

Scheme

$$\begin{array}{ccc} (\tilde{\rho}, \tilde{w}) & & \big(\rho(t), w(t)\big) \\ \downarrow & & \uparrow \\ (\tilde{p}_i^n, \tilde{w}_i^n) & \rightarrow & \big(p_i^n(t), w_i^n(t)\big) \end{array}$$

Follow-The-Leader Model

Scheme

$$
\begin{array}{cccc}
 & (\tilde{\rho}, \tilde{w}) & & \big(\rho(t), w(t)\big) \\
\text{discretize} & \downarrow & & \uparrow \\
 & (\tilde{p}_i^n, \tilde{w}_i^n) & \rightarrow & \big(p_i^n(t), w_i^n(t)\big)
\end{array}
$$

Follow-The-Leader Model

$$
\begin{array}{cccc}
 & & \text{Scheme} & \\
 & (\tilde{\rho}, \tilde{w}) & & \big(\rho(t), w(t)\big) \\
\text{discretize} & \downarrow & & \uparrow \\
 & (\tilde{p}_i^n, \tilde{w}_i^n) & \underset{\text{Cauchy Theorem}}{\longrightarrow} & \big(p_i^n(t), w_i^n(t)\big)
\end{array}
$$

Follow-The-Leader Model

Scheme

$$
\begin{array}{ccccc}
 & (\tilde{\rho}, \tilde{w}) & & \big(\rho(t), w(t)\big) & \\
\text{discretize} & \downarrow & & \uparrow & n \to +\infty \\
 & (\tilde{p}_i^n, \tilde{w}_i^n) & \underset{\text{Cauchy Theorem}}{\longrightarrow} & \big(p_i^n(t), w_i^n(t)\big) &
\end{array}
$$

Follow-The-Leader Model

$$
\begin{array}{ccccc}
 & & \text{Scheme} & & \\
 & (\tilde{\rho}, \tilde{w}) & \color{red}{\longrightarrow} & \big(\rho(t), w(t)\big) & \\
\text{discretize} & \downarrow & & \uparrow & n \to +\infty \\
 & (\tilde{p}_i^n, \tilde{w}_i^n) & \longrightarrow & \big(p_i^n(t), w_i^n(t)\big) & \\
 & & \text{Cauchy Theorem} & &
\end{array}
$$

Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, \eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho, \eta)\right) = 0 \end{cases}$$

$$v(\rho, \eta) = \min \left\{ V_{\max}, \tfrac{\eta}{\rho}\, \psi(\rho) \right\}$$

Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

$$v(\rho,\eta) = \min \left\{ V_{\max}, \tfrac{\eta}{\rho}\, \psi(\rho) \right\}$$

- A.Aw, M.Rascle: SIAM J.Appl.Math., 2000
- M.Zhang: Transportation Research, 2002
- P.Goatin: Math. Comput. Modelling, 2006

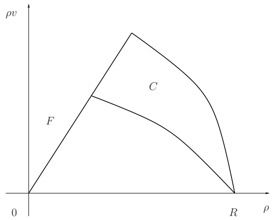
$$\begin{cases} \partial_t \rho + \partial_x \left[\rho\, v(\rho,y)\right] = 0 \\ \partial_t y + \partial_x \left[y\, v(\rho,y)\right] = 0 \end{cases}$$

$$v(\rho,y) = \tfrac{y}{\rho} - p(\rho)$$

Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

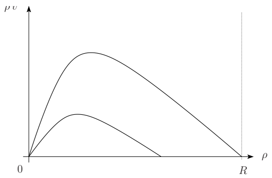
$$v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$



- A.Aw, M.Rascle: SIAM J.Appl.Math., 2000
- M.Zhang: Transportation Research, 2002
- P.Goatin: Math. Comput. Modelling, 2006

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho\, v(\rho,y)\right] = 0 \\ \partial_t y + \partial_x \left[y\, v(\rho,y)\right] = 0 \end{cases}$$

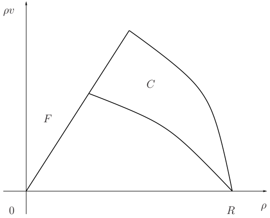
$$v(\rho,y) = \frac{y}{\rho} - p(\rho)$$



Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

$$v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$



R.M.Colombo: SIAM J. Appl. Math., 2002

Free flow: $(\rho, q) \in F$,

$$\partial_t \rho + \partial_x \left[\rho \cdot v_F(\rho)\right] = 0,$$

$$v_F(\rho) = \left(1 - \frac{\rho}{R}\right) \cdot V$$

Congested flow: $(\rho, q) \in C$,

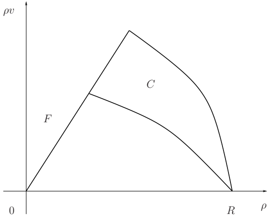
$$\begin{cases} \partial_t \rho + \partial_x \left[\rho \cdot v_C(\rho, q)\right] = 0 \\ \partial_t q + \partial_x \left[(q - q_*) \cdot v_C(\rho, q)\right] = 0 \end{cases}$$

$$v_C(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}$$

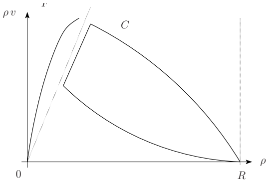
Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

$$v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$



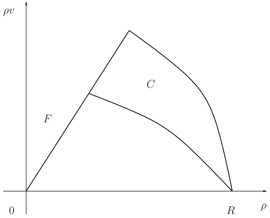
R.M.Colombo: SIAM J. Appl. Math., 2002



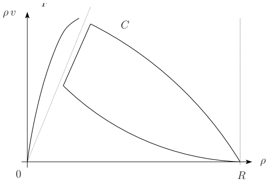
Traffic Modeling – Other Macroscopic Models

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho, \eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho, \eta)\right) = 0 \end{cases}$$

$$v(\rho, \eta) = \min \left\{ V_{\max}, \frac{\eta}{\rho}\, \psi(\rho) \right\}$$



R.M.Colombo: SIAM J. Appl. Math., 2002



- J.P.Lebacque, S.Mammar, H.Haj Salem: ISTTT, 2007

A Kinetic Model

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

$$v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}$$

S.Benzoni-Gavage, R.M.Colombo, P.Gwiazda: Proc. R. Soc. London, 2006

$$\partial_t r(t,x;w) + \partial_x \left[w\, r(t,x;w)\, \psi\left(\int_{\check{w}}^{\hat{w}} r(t,x;w')\, dw'\right)\right] = 0\,.$$

The unknown $r = r(t,x,w)$ is the probability density of vehicles having maximal speed w that at time t are at point x

A Kinetic Model

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x \left(\eta\, v(\rho,\eta)\right) = 0 \end{cases}$$

$$v(\rho,\eta) = \min\left\{V_{\max}, \tfrac{\eta}{\rho}\,\psi(\rho)\right\}$$

S.Benzoni-Gavage, R.M.Colombo, P.Gwiazda: Proc. R. Soc. London, 2006

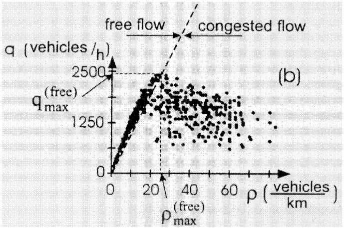
$$\partial_t r(t,x;w) + \partial_x \left[w\, r(t,x;w)\, \psi\left(\int_{\check w}^{\hat w} r(t,x;w')\, dw' \right) \right] = 0\,.$$

If the measure r solves the kinetic model and is such that

$$r(t,x;\cdot) = \rho(t,x)\, \delta_{w(t,x)}$$

than (ρ, w) solves the 2−phase model

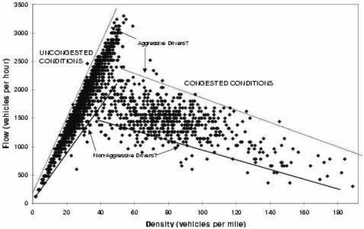
Experimental Fundamental Diagrams



B.Kerner

Traffic and Granular Flow

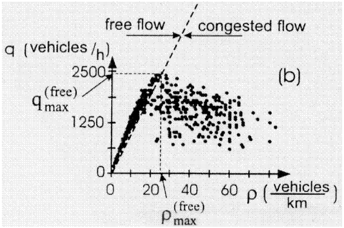
Springer Verlag, 2000



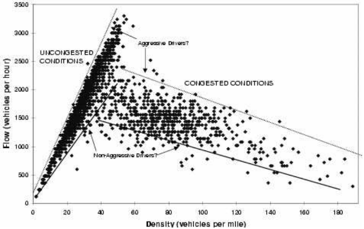
K.M.Kockelman

Transportation, 2001

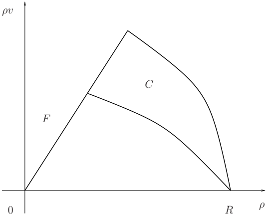
Experimental Fundamental Diagrams



B.Kerner
Traffic and Granular Flow
Springer Verlag, 2000



K.M.Kockelman
Transportation, 2001



Fundamental Diagram by
R.M. Colombo, F. Marcellini, M. Rascle
SIAM J. Appl. Math. 2010

Discrete–Continuous Description

Discrete–Continuous Description

The Macroscopic Part: LWR Model

$$\partial_t \rho + \partial_x (\rho\, v) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	v	$v = v(\rho)$ traffic speed

Discrete–Continuous Description

The Macroscopic Part: LWR Model

$$\partial_t \rho + \partial_x (\rho\, v) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	v	$v = v(\rho)$ traffic speed

The Microscopic Part: Follow-the-Leader Model

$$\dot{p}_i = v\left(\frac{\ell}{p_{i+1} - p_i}\right)$$

$p_i = p_i(t)$	position of the $i-$th driver, for $i = 1, \ldots, n$
$\ell > 0$	vehicles' lenght and

$$p_{i+1} - p_i \geq \ell$$

Discrete–Continuous Description

The Case LWR-FtL

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x < p_1(t) \\ \dot p_i = v\left(\frac{\ell}{p_{i+1} - p_i}\right) & i = 1, \ldots, n-1 \\ \dot p_n = w(t) & \\ \rho(0, x) = \bar\rho(x) & x \leq \bar p_1 \\ p(0) = \bar p & \end{cases}$$

Discrete–Continuous Description

The Case LWR-FtL

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x < p_1(t) \\ \dot p_i = v\left(\frac{\ell}{p_{i+1}-p_i}\right) & i = 1, \ldots, n-1 \\ \dot p_n = w(t) & \\ \rho(0,x) = \bar\rho(x) & x \leq \bar p_1 \\ p(0) = \bar p & \end{cases}$$

Theorem (R.M. Colombo, F. Marcellini: Math. Meth. Appl. Sci., 2014)

There exists a unique solution (ρ, p) *such that*

$$\rho \in \mathbf{C^0}\left([0,T];(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1])\right) \qquad \rho(t,x) = 0 \text{ for } x > p_1(t)$$

$$p \in \mathbf{W^{1,\infty}}([0,T];\mathbb{R}^n)$$

Discrete–Continuous Description

The Case LWR-FtL

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x < p_1(t) \\ \dot p_i = v\left(\frac{\ell}{p_{i+1}-p_i}\right) & i = 1, \ldots, n-1 \\ \dot p_n = w(t) & \\ \rho(0,x) = \bar\rho(x) & x \leq \bar p_1 \\ p(0) = \bar p & \end{cases}$$

Theorem (R.M. Colombo, F. Marcellini: Math. Meth. Appl. Sci., 2014)

There exists a constant $C > 0$ such that

$$\begin{aligned} \left\|\rho(t) - \rho'(t)\right\|_{\mathbf{L}^1} &\leq C\left\|\bar\rho - \bar\rho'\right\|_{\mathbf{L}^1} \\ &+ C(1+t)\left[\left\|\bar p - \bar p'\right\| + \left\|w - w'\right\|_{\mathbf{L}^1([0,t])}\right] e^{2\frac{\mathbf{Lip}(v)}{\ell}t} \\ \left\|p(t) - p'(t)\right\| &\leq \left[\left\|\bar p - \bar p'\right\| + \left\|w - w'\right\|_{\mathbf{L}^1([0,t])}\right] e^{2\frac{\mathbf{Lip}(v)}{\ell}t}. \end{aligned}$$

Discrete–Continuous Description

The Case FtL-LWR

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x > p_n(t) \\ \dot{p}_i = v\left(\frac{\ell}{p_{i+1}-p_i}\right) & i = 1, \ldots, n-1 \\ \dot{p}_n = v\left(\rho(t, p_n(t))\right) & \\ \rho(0,x) = \bar{\rho}(x) & x \geq \bar{p}_n \\ p(0) = \bar{p} & \end{cases}$$

- Well posedness result as before

Discrete–Continuous Description

The Case FtL-LWR

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x > p_n(t) \\ \dot{p}_i = v\left(\frac{\ell}{p_{i+1}-p_i}\right) & i = 1, \ldots, n-1 \\ \dot{p}_n = v\left(\rho(t, p_n(t))\right) & \\ \rho(0,x) = \bar{\rho}(x) & x \geq \bar{p}_n \\ p(0) = \bar{p} & \end{cases}$$

- Well posedness result as before
- Extension to FtL – LWR – FtL – LWR . . .

Discrete–Continuous Description

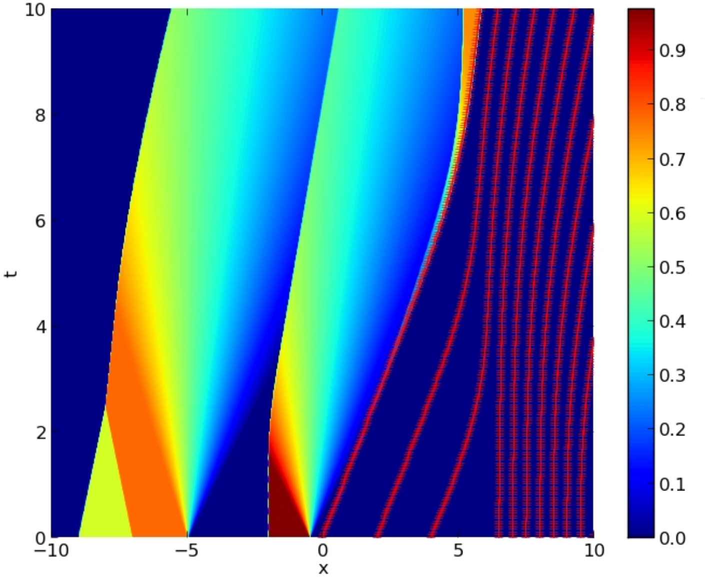
The Case FtL-LWR

$$
\begin{cases}
\partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 & x > p_n(t) \\
\dot p_i = v\left(\frac{\ell}{p_{i+1}-p_i}\right) & i = 1, \ldots, n-1 \\
\dot p_n = v\left(\rho(t, p_n(t))\right) & \\
\rho(0,x) = \bar\rho(x) & x \geq \bar p_n \\
p(0) = \bar p &
\end{cases}
$$

- Well posedness result as before
- Extension to FtL – LWR – FtL – LWR ...
- With fixed boundary and micro-macro transitions
  C. Lattanzio, B. Piccoli: Math. Mod. Meth. Appl. Sci., 2010

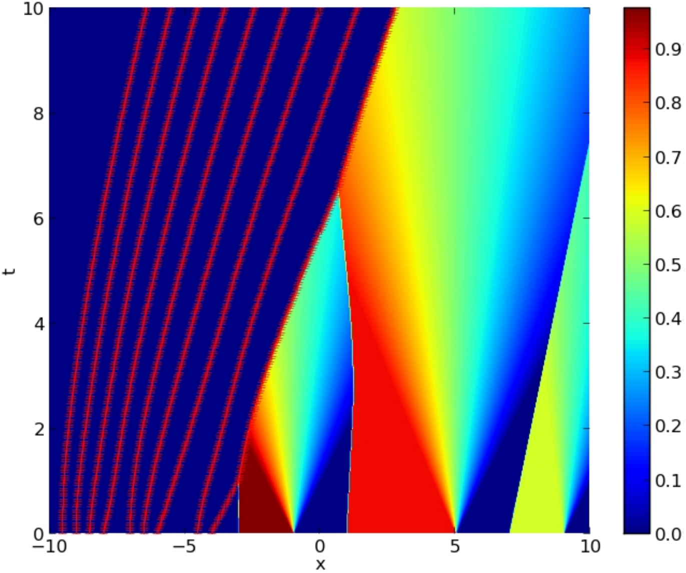
Discrete–Continuous Description

The Case LWR-FtL

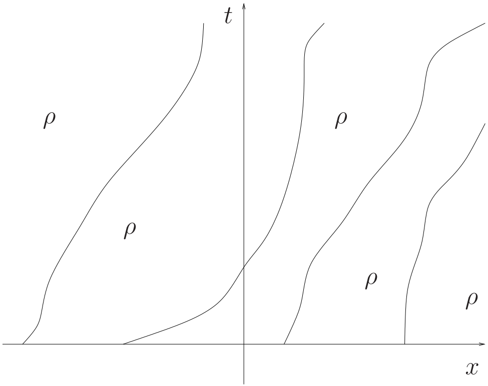


Discrete–Continuous Description

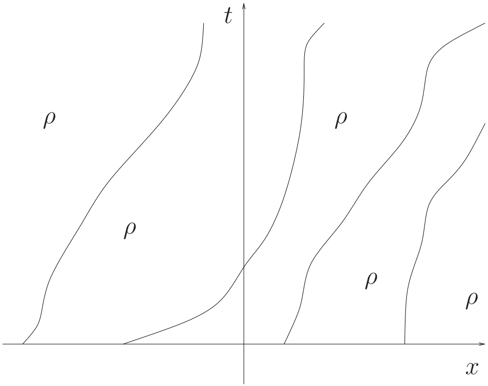
The Case FtL-LWR



The Next Step

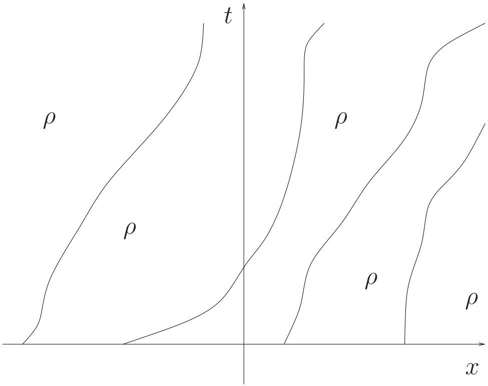


The Next Step



- Millennium Project
- D.B. Work, S. Blandin, O. Tossavainen, B. Piccoli, A. Bayen: Appl. Math. Res. Express, 2010
- C. Lattanzio, A. Maurizi, B. Piccoli: SIAM J. Math. Anal., 2011
- A. Cabassi, P. Goatin: Res. Report INRIA, 2013

The Next Step



- A. Alessandri, R. Bolla, M. Repetto: Proc. Amer. Control Conf., 2003
- P. Cheng, Z. Qiu, B. Ran: Proc. IEEE ITSC'06, 2006
- J.-C. Herrera, D. Work, R. Herring, J.Ban, Q. Jacobson, A. Bayen: Transp. Res. C, 2009

An Inverse Problem for 1D Scalar Conservation Laws

Given the initial density $\rho_o = \rho_o(x)$

and the measured trajectory $p = p(t)$

find the best speed law $v = v(\rho)$, i.e., so that

$$\int_0^T \left| \dot{p}(t) - v(\rho\left(t, p(t)+\right) \right| \mathrm{d}t \text{ is minimal}$$

$$\text{where } \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases}$$

An Inverse Problem for 1D Scalar Conservation Laws

Given the initial density $\rho_o = \rho_o(x)$

and the measured trajectory $p = p(t)$

find the best speed law $v = v(\rho)$, i.e., so that

$$\int_0^T \left| \dot{p}(t) - v(\rho\left(t, p(t)+\right) \right| \mathrm{d}t \text{ is minimal}$$

$$\text{where } \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \rho_0(x) \end{cases}$$

- S. Fan, M. Herty, B. Seibold: Netw. Heterog. Media, 2014

An Inverse Problem for 1D Scalar Conservation Laws

Given the initial density $\rho_o = \rho_o(x)$

and the measured trajectory $p = p(t)$

find the best speed law $v = v(\rho)$, i.e., so that

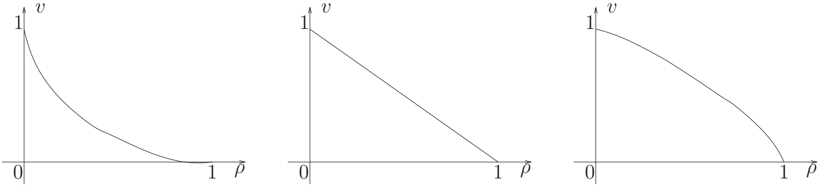
$$\int_0^T \left|\dot{p}(t) - v(\rho\left(t, p(t)+\right)\right| \mathrm{d}t \text{ is minimal}$$

$$\text{where } \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0, x) = \rho_0(x) \end{cases}$$

A solution may fail to exist

Inverse Problem – A Negative Example

Class of speed laws: $\quad v_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho)$



Inverse Problem – A Negative Example

Class of speed laws: $v_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho)$

Measured trajectory: $p(t) = t/2$

ρ_ε solves:
$$\begin{cases} \partial_t \rho + \partial_x \big(\rho\, v(\rho)\big) = 0 \\ \rho(0,x) = \begin{cases} 1/8 & x<0 \\ 3/8 & x>0 \end{cases} \end{cases}$$

Inverse Problem – A Negative Example

$$\text{Class of speed laws:} \quad v_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho)$$

$$\text{Measured trajectory:} \quad p(t) = t/2$$

$$\rho_\varepsilon \text{ solves:} \quad \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \begin{cases} 1/8 & x < 0 \\ 3/8 & x > 0 \end{cases} \end{cases}$$

$$\begin{aligned} \varphi \colon [-1/3,\, 1/3] \quad &\to \quad \mathbb{R} \\ \varepsilon \quad &\to \quad \int_0^T \left| \dot{p}(t) - v_\varepsilon\left(\rho_\varepsilon\left(t, p(t)+\right)\right)\right| \mathrm{d}t \end{aligned}$$

Inverse Problem – A Negative Example

$$\text{Class of speed laws:} \quad v_\varepsilon(\rho) = (1 + \varepsilon\rho)(1 - \rho)$$

$$\text{Measured trajectory:} \quad p(t) = t/2$$

$$\rho_\varepsilon \text{ solves:} \quad \begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0, x) = \begin{cases} 1/8 & x < 0 \\ 3/8 & x > 0 \end{cases} \end{cases}$$

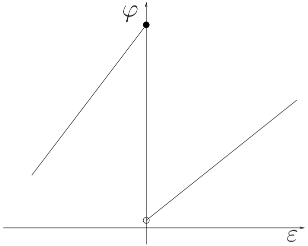
$$\begin{aligned} \varphi\colon [-1/3,\ 1/3] &\to \mathbb{R} \\ \varepsilon &\to \begin{cases} \left(\frac{3}{8} + \frac{7\varepsilon}{64}\right) T & \text{if } \varepsilon > 0 \\ \left(\frac{9}{8} + \frac{15\varepsilon}{64}\right) T & \text{if } \varepsilon \leq 0 \end{cases} \end{aligned}$$

Inverse Problem – A Negative Example

Class of speed laws: $v_\varepsilon(\rho) = (1+\varepsilon\rho)(1-\rho)$

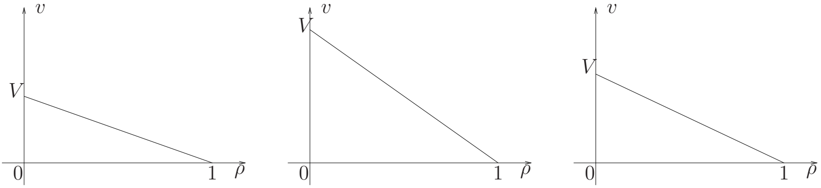
Measured trajectory: $p(t) = t/2$

ρ_ε solves: $$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v(\rho)\right) = 0 \\ \rho(0,x) = \begin{cases} 1/8 & x<0 \\ 3/8 & x>0 \end{cases} \end{cases}$$



Inverse Problem – A Positive Result

Class of Speed Laws: $v_V(\rho) = V(1-\rho)$ for $V \in [\check{V}, \hat{V}]$.



Inverse Problem – A Positive Result

Class of Speed Laws: $v_V(\rho) = V(1-\rho)$ for $V \in [\check{V}, \hat{V}]$.

Theorem (R.M.Colombo, F.Marcellini: Math. Mod. Meth. Appl. Sci., 2015)

Let $\rho_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}; [0,1])$ *and* $p \in \mathbf{W^{1,\infty}}([0,T];\mathbb{R})$.

Let ρ_V *solve* $\begin{cases} \partial_t \rho_V + \partial_x \left(V\, \rho_V (1-\rho_V)\right) = 0 \\ \rho_V(0,x) = \rho_o(x)\,. \end{cases}$

Then, the map

$$\begin{array}{rcl} \varphi \colon [\check{V}, \hat{V}] & \to & \mathbb{R} \\ V & \to & \displaystyle\int_0^T \left| \dot p(t) - v_V\Big(\rho_V\left(t, p(t)+\right)\Big) \right| \mathrm{d}t \end{array}$$

is continuous, provided

$$\mathrm{essinf}_{x\in\mathbb{R}}\, \rho_o > \check{\rho} \quad \text{and} \quad \mathrm{essinf}_{t\in[0,T]}\, \dot p \geq \hat{V}(1-2\check{\rho})\,.$$

A Traffic Model Aware of Real Time Data

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

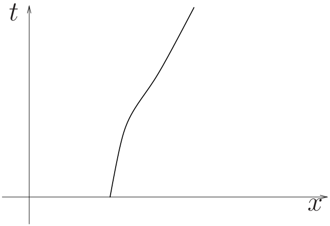
t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed



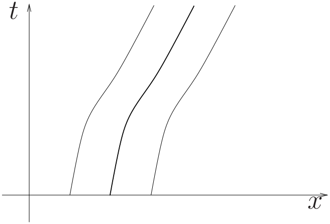
$$\mathcal{V}(t, x, \rho) = ?$$

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed



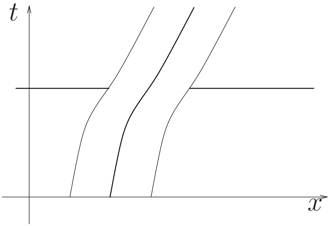
$$\mathcal{V}(t, x, \rho) = ?$$

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed



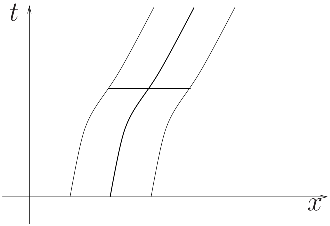
$$\mathcal{V}(t, x, \rho) = v(\rho)$$

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed



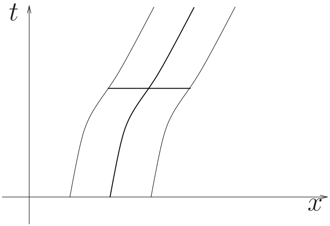
$$\mathcal{V}(t, x, \rho) = \frac{2}{\frac{1}{\dot{p}(t)} + \frac{1}{v(\rho)}}$$

A Traffic Model Aware of Real Time Data

Macroscopic Model

$$\partial_t \rho + \partial_x (\rho \mathcal{V}) = 0$$

t	$t \in [0, +\infty[$ time	x	$x \in \mathbb{R}$ space coordinate
ρ	traffic density	$\mathcal{V}$	$\mathcal{V} = \mathcal{V}(t, x, \rho)$ traffic speed



$$\mathcal{V}(t, x, \rho) = \chi\left(x - p(t)\right) \frac{2}{\frac{1}{\dot{p}(t)} + \frac{1}{v(\rho)}} + \left(1 - \chi\left(x - p(t)\right)\right) v(\rho)$$

A Traffic Model Aware of Real Time Data

Theorem (R.M.Colombo, F.Marcellini: Math. Mod. Meth. Appl. Sci., 2015)

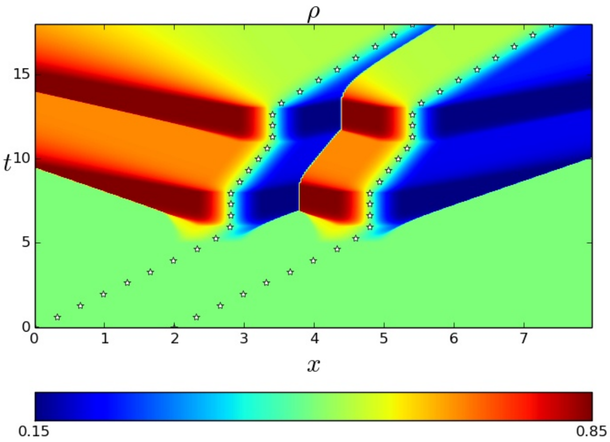
The Cauchy Problem

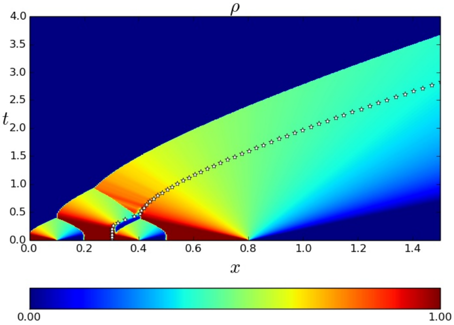
$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, \mathcal{V}(t,x,\rho)\right) = 0 \\ \mathcal{V}(t,x,\rho) = \chi\left(x - p(t)\right) \frac{2\dot{p}(t)\, v(\rho)}{\dot{p}(t)+v(\rho)} + \Big(1 - \chi\left(x - p(t)\right)\Big) v(\rho) \\ \rho(0,x) = \rho_o(x) \end{cases}$$

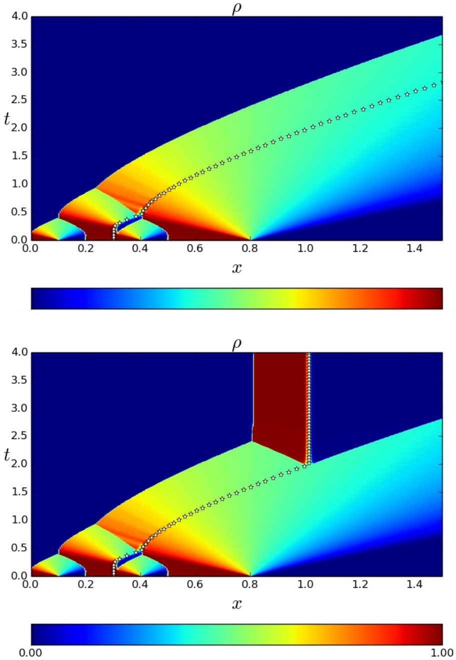
admits a unique solution.

- E.Y. Panov: Arch. Rat. Mech. An., 2010
- K.H. Karlsen, N.H. Risebro: Discr. Cont. Dyn. Syst. – A, 2003
- B. Andreianov, P. Bénilan, S.N. Kružkov: J. Func. An., 2000

A Traffic Model Aware of Real Time Data




ρ
t
x
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.00
1.00



ρ
4.0
3.5
3.0
2.5
t
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
x
ρ
4.0
3.5
3.0
2.5
t
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
x
0.00
1.00