A Boltzmann-type kinetic approach to the modeling of vehicular traffic

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The spatially homogeneous model

1. Continuous velocity model
2. From continuous to discrete velocity model
3. Fundamental diagrams and the phase transition
4. Traffic safety

The spatially non-homogeneous model

1. Model on a single road
2. Model on road networks
The spatially homogeneous model

Continuous velocity model
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The spatially non-homogeneous model

Model on a single road
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Basics of kinetic modeling

- Microscopic state of the vehicles: speed $v \in [0, 1]$
- Kinetic distribution function: $f = f(t, v)$ s.t.
  
  $$f(t, v) \, dv = \text{fraction of vehicles with speed in } [v, v + dv] \text{ at time } t \geq 0$$

The Boltzmann-type kinetic equation

$$\partial_t f = Q(f, f) := \int_0^1 \int_0^1 P(v_* \rightarrow v|v^*, \rho)f(t, v_*)f(t, v^*) \, dv_* \, dv^* - \rho f \quad (1)$$

- $P(v_* \rightarrow v|v^*, \rho)$ probability distribution of speed transitions due to pairwise (binary) interactions among the vehicles:
  
  $$\int_0^1 P(v_* \rightarrow v|v^*, \rho) \, dv = 1 \quad \forall v_*, v^*, \rho \in [0, 1] \quad (2)$$

- Mass conservation: $\rho(t) := \int_0^1 f(t, v) \, dv$ is constant, in fact from (1)-(2):

  $$\frac{d}{dt} \int_0^1 f(t, v) \, dv = \int_0^1 Q(f, f) \, dv = 0$$
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**The Boltzmann-type kinetic equation**

$$\partial_t f = Q(f, f) := \int_0^1 \int_0^1 \mathcal{P}(v_* \rightarrow v|v^*, \rho)f(t, v_*)f(t, v^*) \, dv_* \, dv^* - \rho f \quad (1)$$

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Let $\mathcal{P}(v_* \rightarrow |v^*, \rho) \in \mathcal{P}([0, 1])$ for all $v_*, v^*, \rho \in [0, 1]$. We assume

$$W_1(\mathcal{P}(v_* \rightarrow |v^*, \rho), \mathcal{P}(w_* \rightarrow |w^*, \rho)) \leq \operatorname{Lip}(\mathcal{P})(|w_* - v_*| + |w^* - v^*| + |\rho - \rho|),$$

where $W_1$ is the 1-Wasserstein metric for probability measures.

**Theorem (P. Freguglia, A. T., 2015 [4])**

Fix $\rho \in [0, 1]$ and $f(0, v) = : f_0(v) \in \mathcal{M}^\rho_+([0, 1])$. There exists a unique $f \in C([0, +\infty); \mathcal{M}^\rho_+([0, 1]))$ which solves (1) in mild form:

$$f(t, v) = e^{-\rho t} f_0(v) + \int_0^t e^{\rho(s-t)} \int_0^1 \int_0^1 \mathcal{P}(v_* \rightarrow v|v^*, \rho)f(t, v_*)f(t, v^*) dv_* dv^* ds.$$

Given $f_{01}, f_{02} \in \mathcal{M}^\rho_+([0, 1])$, the following continuous dependence estimate holds:

$$\sup_{t \in [0, T]} W_1(f_1(t), f_2(t)) \leq e^{2\max\{1, \operatorname{Lip}(\mathcal{P})\} T} W_1(f_{01}, f_{02})$$

up to an arbitrarily large final time $T < +\infty$. 
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We consider the following probability distribution of speed transitions:

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\mathcal{P}(v_* \rightarrow v | v^*, \rho) = \begin{cases} 
(1 - P)\delta_{v_*}(v) + P\delta_{\min\{v_* + \Delta v, 1\}}(v) & \text{if } v_* \leq v^* \\
(1 - P)\delta_{v^*}(v) + P\delta_{v_*}(v) & \text{if } v_* > v^* 
\end{cases}
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where \(0 < \Delta v < 1\) is given and

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P = P(\rho) = 1 - \rho^\gamma \quad (\gamma > 0)
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is a probability of passing (cf. I. Prigogine, 1961)

The time-asymptotic solution of (1), with transition probabilities (3), concentrates only on speeds which are multiples of \(\Delta v\) (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [6])

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**Discrete velocity model**

- Define a speed lattice
  \[ v_j = (j - 1)\Delta v, \quad j = 1, \ldots, n, \quad \Delta v = \frac{1}{n-1} \]

- Assume that \( P \) is a discrete probability distribution over \( v \):
  \[ P(v_* \rightarrow v | v^*, \rho) = \sum_{j=1}^{n} P^j(v_*, v^*, \rho)\delta_{v_j}(v) \] (4)

- Fix \( \rho \in [0, 1] \) and take an initial condition of the form
  \[ f_0(v) = \sum_{j=1}^{n} f_0^j \delta_{v_j}(v) \quad \text{with} \quad f_0^j \geq 0, \quad \sum_{j=1}^{n} f_0^j = \rho \] (5)

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The unique solution to (1) with transition probabilities (4) and initial condition (5) is
\[ f(t, v) = \sum_{j=1}^{n} f_j(t)\delta_{v_j}(v), \quad \text{where the } f_j \text{'s satisfy} \]
\[ \frac{df_j}{dt} = \sum_{h=1}^{n} \sum_{k=1}^{n} P^j_{hk}(\rho)f_h f_k - \rho f_j, \quad f_j(0) = f_0^j \] (6)

and \( P^j_{hk}(\rho) := P^j(v_h, v_k, \rho) \).
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and \( P^j_{hk}(\rho) := P^j(v_h, v_k, \rho) \).
From (3) we deduce:

$$\mathcal{P}^j_{hk}(\rho) = \begin{cases} (1 - P)\delta_{jh} + P\delta_{j,\min\{h+1, n\}} & \text{if } h \leq k \\ (1 - P)\delta_{jk} + P\delta_{jh} & \text{if } h > k \end{cases}$$

which explicitly reads:

$$\mathcal{P}^j_{hk}(\rho) = \begin{cases} 1 - P & \text{if } j = h \\ P & \text{if } j = h + 1 & \text{if } h \leq k, \ h < n \\ 0 & \text{otherwise} \end{cases}$$

We recall that $P = P(\rho) = 1 - \rho^\gamma$ ($\gamma > 0$)
Asymptotic distributions

- We study the asymptotic speed distributions $f^\infty = \{ f^\infty_j \}_{j=1}^n$ resulting from (6)-(7): $f^\infty_j = \lim_{t \to +\infty} f_j(t)$.

- The $f^\infty_j$’s form a one-parameter family, the parameter being the density $\rho$ which is conserved.

**Theorem (L. Fermo, A. T., 2014 [2])**

For every $n \geq 2$ and every $\rho \in [0, 1]$ there exists a unique stable and attractive equilibrium $f^\infty$ of (6), which satisfies:

$$f^\infty_j \geq 0 \quad \forall \ j = 1, \ldots, n, \quad \sum_{j=1}^n f^\infty_j = \rho$$

- In more detail, setting $\rho_c := \left( \frac{1}{2} \right)^\frac{1}{\gamma}$,
  - for $\rho < \rho_c$ there exists only one stable and attractive equilibrium
  - for $\rho > \rho_c$ the previous equilibrium becomes unstable and a second stable and attractive one appears.

- $\rho_c$ is a critical value for equilibria inducing a supercritical bifurcation.
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$\rho_c$ is a critical value for equilibria inducing a supercritical bifurcation.
We compute the macroscopic flux $q$ and the mean speed $u$ at equilibrium:

$$q(\rho) := \sum_{j=1}^{n} v_j f_j^{\infty}(\rho), \quad u(\rho) := \frac{q(\rho)}{\rho}$$

along with their standard deviations (dashed-red lines in the graphs below).

- Bifurcation $\rightsquigarrow$ phase transition, free ($\rho < \rho_c$) to congested flow ($\rho > \rho_c$)
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Traffic as a mixture of different vehicles

● We consider two populations of vehicles, say cars (C) and trucks (T), with different microscopic characteristics.

● Cars are shorter and faster while trucks are longer and slower.

● Speed grids:

\[ v_j = (j - 1)\Delta v, \quad j = 1, \ldots, n^p, \quad p = C, T \]
\[ \Delta v = \frac{1}{n^C - 1}, \quad n^T < n^C \]

● Characteristic lengths: \( \ell^C = 1, \ell^T > 1 \)

● Fraction of road occupancy:

\[ s := \rho^C \ell^C + \rho^T \ell^T, \]

the admissible pairs of densities \( (\rho^C, \rho^T) \in [0, 1]^2 \) being those s.t. \( s \leq 1 \)
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  the admissible pairs of densities \((\rho^C, \rho^T) \in [0, 1]^2\) being those s.t. \( s \leq 1 \)
Traffic as a mixture of different vehicles

- We consider two populations of vehicles, say cars (C) and trucks (T), with different microscopic characteristics.

- Cars are shorter and faster while trucks are longer and slower.

- Speed grids:
  \[ v_j = (j - 1)\Delta v, \quad j = 1, \ldots, n^p, \quad p = C, T \]
  \[ \Delta v = \frac{1}{n^C - 1}, \quad n^T < n^C \]

- Characteristic lengths: \( \ell^C = 1, \ell^T > 1 \)

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**Multi-population model**

- **Model with self- and cross-interactions:**

  \[
  \frac{df_{j}^{p}}{dt} = \sum_{h,k=1}^{n_{p}} P_{hk}^{p,j}(\rho)f_{h}^{p}f_{k}^{p} + \sum_{h=1}^{n_{p}} \sum_{k=1}^{n_{q}} Q_{hk}^{pq,j}(\rho)f_{h}^{p}f_{k}^{q} - (\rho^{C} + \rho^{T})f_{j}^{p}
  \]

  - *self-interactions*
  - *cross-interactions*

- In the transition probabilities \(P_{hk}^{p,j}, Q_{hk}^{pq,j}\) the density \(\rho\) is replaced by the fraction of road occupancy \(s\), i.e., the probability of passing is now:

  \[
  P = P(s) = 1 - s^{\gamma}
  \]

  \(\gamma > 0\)

---

**Theorem (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [5])**

If \(n^{C} = n^{T}\) and \(\ell^{C} = \ell^{T} = 1\) the total kinetic distribution function \(f_{j}(t) := f_{j}^{C}(t) + f_{j}^{T}(t)\) solves the single-population model (6).

- \(s_{c} := \left(\frac{1}{2}\right)^{\frac{1}{\gamma}}\) is again a critical value for equilibria. For \(s = s_{c}\) the maximum flux is attained (G. Puppo, M. Semplice, A. T., G. Visconti, 2015 [5])

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A fully-discrete-state kinetic theory approach to modeling vehicular traffic.

Fundamental diagrams for kinetic equations of traffic flow.

A fully-discrete-state kinetic theory approach to traffic flow on road networks.


Fundamental diagrams in traffic flow: the case of heterogeneous kinetic models.
Accepted (Preprint: arXiv:1411.4988).

Kinetic models for traffic flow resulting in a reduced space of microscopic velocities.
**Transition probabilities**

**Figure:** The asymptotic distribution function concentrates on multiples of $\Delta v$. 

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Figure: Left: fundamental diagram from experimental data (Minnesota Department of Transportation, 2003). Right: fundamental diagram from the multi-population model with $\gamma = 0.5$. 

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