On a Class of Nonlocal Traffic Flow Models with Multivalued Fundamental Diagram

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Motivation

PDE Models for Traffic Flow

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$$\begin{cases} \partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

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Questions:

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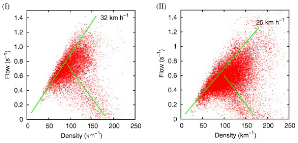

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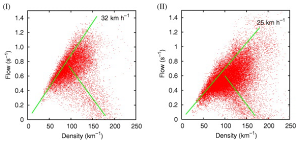

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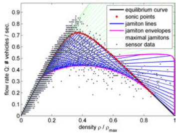

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The Considered Problem

Given: Initial data ρ_0

Wanted:

s.t.
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$*$ $\partial_t \rho + \operatorname{div}_x\big(v(\eta * \rho)\, V(t,x) \;\; \rho\big) = 0,$ see, e.g., [Colombo, Herty & Mercier 2011]

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Goal: Existence of weak solutions ρ and distributional solutions μ

Sufficient conditions for their uniqueness

Continuous dependence on data (i.e., initial data and coeff.)

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$$\begin{cases} \partial_t\,\rho + \operatorname{div}_x\big(\mathcal{G}(t,\rho,\mu)\ \rho\big) = \mathcal{U}(t,\rho,\mu)\,\rho + \mathcal{W}(t,\rho,\mu) \\ \partial_t\,\mu + \operatorname{div}_x\big(\mathcal{B}(t,\rho,\mu)\ \mu\big) = \mathcal{C}(t,\rho,\mu)\,\mu \\ \rho(0) = \rho_0 \\ \mu(0) = \mu_0 \end{cases}$$

Goal 1: Existence of weak solutions ρ and distributional solutions μ

Sufficient conditions for their uniqueness

Continuous dependence on data (i.e., initial data and coeff.)

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

$$\begin{aligned} \text{Coeff. } \mathcal{G} &: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U} &: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W} &: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{aligned}$$

Wanted: Density

$$\begin{aligned} \rho &: [0,T] \longrightarrow L^2(\mathbb{R}^n) \\ \mu &: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)' \end{aligned}$$

s.t.

$$\begin{cases} \partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho,\mu)\ \rho\big) = \mathcal{U}(t,\rho,\mu)\,\rho + \mathcal{W}(t,\rho,\mu) \\ \partial_t \mu + \mathrm{div}_x\big(\mathcal{B}(t,\rho,\mu)\ \mu\big) = \mathcal{C}(t,\rho,\mu)\,\mu \\ \rho(0) = \rho_0 \\ \mu(0) = \mu_0 \end{cases}$$

Goal 2: Existence of weak solutions ρ and distributional solutions μ if coefficient maps are set-valued (e.g., due to imprecision)

i.e., for each tuple (t,ρ,μ), the model provides (possibly) more than one velocity field $\mathcal{G}(t,\rho,\mu)$, $\mathcal{B}(t,\rho,\mu) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ etc.

Theorem (Existence of L^2-valued solutions)

$$\partial_t \rho \; + \; \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) \; = \; \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho),$$

Theorem (Existence of L^2-valued solutions)

$$\partial_t \, \rho \;+\; \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) \;=\; \mathcal{U}(t,\rho)\,\rho \;+\; \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$*, the function* $[0,T] \longrightarrow \mathbb{R},\ t \longmapsto \int_{\mathbb{R}^n} \psi(x)\,\rho(t,x)\;dx$ *is continuous and,* $\rho(\cdot)$ *satisfies for any* $0 \le t_1 < t_2 \le T,\ \varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\,\big(\rho(t_2) - \rho(t_1)\big)\;dx \;=\; \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\;\mathcal{G}\big(s,\,\rho(s)\big)(x)\cdot\nabla_x\,\varphi(x)\;\;dx\;\;ds\;+\; \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\;\varphi(x)\;dx\;ds$$

Theorem (Existence of L^2-valued solutions)

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ *initializes a weak solution* $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ *of*

$$\partial_t \rho \; + \; \mathrm{div}_x\big(\mathcal{G}(t, \rho)\, \rho\big) \;=\; \mathcal{U}(t, \rho)\, \rho + \mathcal{W}(t, \rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$*, the function* $[0, T] \longrightarrow \mathbb{R}$, $t \longmapsto \int_{\mathbb{R}^n} \psi(x)\, \rho(t, x)\; dx$ *is continuous and,* $\rho(\cdot)$ *satisfies for any* $0 \le t_1 < t_2 \le T$, $\varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\; \big(\rho(t_2) - \rho(t_1)\big)\; dx \;=\; \int_{t_1}^{t_2}\!\int_{\mathbb{R}^n} \rho(s, x)\;\; \mathcal{G}\big(s,\; \rho(s)\big)(x) \cdot \nabla_x\, \varphi(x) \;\; dx \;\; ds \; + \int_{t_1}^{t_2}\!\int_{\mathbb{R}^n} \Big(\rho(s, x)\, \mathcal{U}\big(s, \rho(s)\big)(x) + \mathcal{W}\big(s, \rho(s)\big)(x)\Big)\; \varphi(x)\; dx\; ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\begin{aligned} \mathcal{G} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{aligned}$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho)\,\rho) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$, the function $[0,T] \longrightarrow \mathbb{R}$, $t \longmapsto \int_{\mathbb{R}^n} \psi(x)\,\rho(t,x)\,dx$ is continuous and, $\rho(\cdot)$ satisfies for any $0 \le t_1 < t_2 \le T$, $\varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\,(\rho(t_2) - \rho(t_1))\,dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\;\mathcal{G}(s,\rho(s))(x)\cdot\nabla_x\varphi(x)\;dx\;ds\;+ \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}(s,\rho(s))(x) + \mathcal{W}(s,\rho(s))(x)\Big)\,\varphi(x)\,dx\,ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$
$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$
$$\mathcal{W}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\mathrm{div}_x\, \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t\, \rho \;+\; \mathrm{div}_x \big(\mathcal{G}(t,\rho)\, \rho\big) \;=\; \mathcal{U}(t,\rho)\, \rho + \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$, the function $[0,T] \longrightarrow \mathbb{R},\ t \longmapsto \int_{\mathbb{R}^n} \psi(x)\, \rho(t,x)\ dx$ is continuous and, $\rho(\cdot)$ satisfies for any $0 \le t_1 < t_2 \le T$, $\varphi \in C^1_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\ \big(\rho(t_2) - \rho(t_1)\big)\ dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s, \rho(s)\big)(x) \cdot \nabla_x \varphi(x)\ \ dx\ \ ds\ + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x)\, \mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\ \varphi(x)\ dx\ ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\begin{aligned}\mathcal{G}:&\ [0,T]\times L^2(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)\cap L^2\\ \mathcal{U}:&\ [0,T]\times L^2(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n)\cap L^2\\ \mathcal{W}:&\ [0,T]\times L^2(\mathbb{R}^n)\longrightarrow L^2(\mathbb{R}^n)\end{aligned}$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho}\Big(\big\|\mathrm{div}_x\,\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)}+\big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)}+\big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)}\Big)<\infty$$

(2.) *Local a priori bounds: For each $r>0$, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\le r$ implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)}+\big\|\partial_x\,\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})}+\big\|\nabla_x\,\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\le C_r$$

Then every function $\rho_0\in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t\,\rho+\mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big)=\mathcal{U}(t,\rho)\,\rho+\mathcal{W}(t,\rho),$$

i.e., for every $\psi\in L^2(\mathbb{R}^n)$, the function $[0,T]\longrightarrow\mathbb{R}$, $t\longmapsto\int_{\mathbb{R}^n}\psi(x)\,\rho(t,x)\,dx$ is continuous and, $\rho(\cdot)$ satisfies for any $0\le t_1<t_2\le T$, $\varphi\in C_c^1(\mathbb{R}^n)$

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\,\big(\rho(t_2)-\rho(t_1)\big)\,dx=&\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\rho(s,x)\;\mathcal{G}\big(s,\rho(s)\big)(x)\cdot\nabla_x\,\varphi(x)\;dx\;ds\;+\\&\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x)+\mathcal{W}\big(s,\rho(s)\big)(x)\Big)\,\varphi(x)\,dx\,ds\end{aligned}$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$
$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$
$$\mathcal{W}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\mathrm{div}_x\, \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds:* *For each* $r > 0$, *there is* $C_r > 0$ *s.t.* $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ *implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\partial_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \big\|\nabla_x \mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n):\ \mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ *are measurable.*

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ *initializes a weak solution* $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ *of*

$$\partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$, *the function* $[0,T] \longrightarrow \mathbb{R},\ t \longmapsto \int_{\mathbb{R}^n} \psi(x)\,\rho(t,x)\,dx$ *is continuous and,* $\rho(\cdot)$ *satisfies for any* $0 \le t_1 < t_2 \le T$, $\varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\ (\rho(t_2) - \rho(t_1))\ dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s,\rho(s)\big)(x)\cdot\nabla_x\varphi(x)\ dx\ ds + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\ \varphi(x)\ dx\ ds$$

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Suppose for

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(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\mathrm{div}_x\, \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds: For each* $r > 0$, *there is* $C_r > 0$ *s.t.* $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ *implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\partial_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \big\|\nabla_x \mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n):\ \mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ *are measurable.*

(4.) *For a.e.* $t \in [0,T]$, *the functions* $\mathcal{G}(t,\cdot),\ \mathcal{U}(t,\cdot),\ \mathcal{W}(t,\cdot) : L^2(\mathbb{R}^n) \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ *are continuous* .

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ *initializes a weak solution* $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ *of*

$$\partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$, *the function* $[0,T] \longrightarrow \mathbb{R}$, $t \longmapsto \int_{\mathbb{R}^n} \psi(x)\,\rho(t,x)\ dx$ *is continuous and,* $\rho(\cdot)$ *satisfies for any* $0 \le t_1 < t_2 \le T$, $\varphi \in C_c^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\ \big(\rho(t_2) - \rho(t_1)\big)\ dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s,\ \rho(s)\big)(x)\cdot\nabla_x\varphi(x)\ dx\ ds + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\ \varphi(x)\ dx\ ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\begin{aligned} \mathcal{G} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{aligned}$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\mathrm{div}_x\, \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds: For each $r > 0$, there is $C_r > 0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\partial_x\, \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \big\|\nabla_x\, \mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n):\ \mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ *are measurable.*

(4.) *For a.e. $t \in [0,T]$, the functions $\mathcal{G}(t,\cdot),\ \mathcal{U}(t,\cdot),\ \mathcal{W}(t,\cdot) : L^2(\mathbb{R}^n) \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ are continuous (w.r.t. weakly continuous sequences).*

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t\, \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\ \rho\big) = \mathcal{U}(t,\rho)\ \rho + \mathcal{W}(t,\rho),$$

i.e., for every $\psi \in L^2(\mathbb{R}^n)$, the function $[0,T] \longrightarrow \mathbb{R}$, $t \longmapsto \int_{\mathbb{R}^n} \psi(x)\ \rho(t,x)\ dx$ is continuous and, $\rho(\cdot)$ satisfies for any $0 \le t_1 < t_2 \le T$, $\varphi \in C^1_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi\ \big(\rho(t_2) - \rho(t_1)\big)\ dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s,\ \rho(s)\big)(x) \cdot \nabla_x\, \varphi(x)\ dx\ ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x)\, \mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\ \varphi(x)\ dx\ ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$
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$$\mathcal{W}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\|\mathrm{div}_x\, \mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)} + \|\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n)} + \|\mathcal{W}(t,\rho)\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds: For each $r > 0$, there is $C_r > 0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ implies*

$$\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)} + \|\partial_x \mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \|\nabla_x \mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n):\ \mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow (L^2, \|\cdot\|_{L^2})$ *are measurable.*

(4.) *For a.e. $t \in [0,T]$, the functions $\mathcal{G}(t,\cdot),\ \mathcal{U}(t,\cdot),\ \mathcal{W}(t,\cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous (w.r.t. weakly continuous sequences).*

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

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$$\int_{\mathbb{R}^n} \varphi\,\big(\rho(t_2) - \rho(t_1)\big)\,dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s,\rho(s)\big)(x)\cdot\nabla_x\varphi(x)\ dx\ ds + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\,\varphi(x)\,dx\,ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\mathcal{G} : [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$
$$\mathcal{U} : [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$
$$\mathcal{W} : [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\mathrm{div}_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds: For each $r > 0$, there is $C_r > 0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\partial_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \big\|\nabla_x \mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n)$: $\mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ *are measurable.*

(4.) *For a.e. $t \in [0,T]$, the functions $\mathcal{G}(t,\cdot),\ \mathcal{U}(t,\cdot),\ \mathcal{W}(t,\cdot) : L^2(\mathbb{R}^n) \longrightarrow \big(L^2,\ \|\cdot\|_{L^2}\big)$ are continuous (w.r.t. weakly continuous sequences).*

(5.) *For each $r > 0$, there exist $\widehat{w}_r \in L^2(\mathbb{R}^n)$ and a compact subset $K_r \subset \mathbb{R}^n$ s.t.*

$$t \in [0,T],\ \ \|\rho\|_L^2 \le r,\ \ x \in \mathbb{R}^n \setminus K_r \implies \big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}_r(x).$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t\,\rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho),$$

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$$\int_{\mathbb{R}^n} \varphi\ \big(\rho(t_2) - \rho(t_1)\big)\ dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \rho(s,x)\ \mathcal{G}\big(s,\ \rho(s)\big)(x)\cdot \nabla_x\,\varphi(x)\ dx\ ds\ +$$
$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \Big(\rho(s,x)\,\mathcal{U}\big(s,\rho(s)\big)(x) + \mathcal{W}\big(s,\rho(s)\big)(x)\Big)\ \varphi(x)\ dx\ ds$$

Theorem (Existence of L^2-valued solutions)

Suppose for

$$\begin{aligned} \mathcal{G} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2 \\ \mathcal{U} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{aligned}$$

(1.) *Global a priori bounds*

$$\sup_{t,\rho} \Big(\big\|\operatorname{div}_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\mathcal{W}(t,\rho)\big\|_{L^2(\mathbb{R}^n)} \Big) < \infty$$

(2.) *Local a priori bounds: For each $r > 0$, there is $C_r > 0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ implies*

$$\big\|\mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n)} + \big\|\partial_x \mathcal{G}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})} + \big\|\nabla_x \mathcal{U}(t,\rho)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$$

(3.) $\forall\ \rho \in L^2(\mathbb{R}^n):\ \mathcal{G}(\cdot,\rho),\ \mathcal{U}(\cdot,\rho),\ \mathcal{W}(\cdot,\rho) : [0,T] \longrightarrow \big(L^2, \|\cdot\|_{L^2}\big)$ *are measurable.*

(4.) *For a.e. $t \in [0,T]$, the functions $\mathcal{G}(t,\cdot),\ \mathcal{U}(t,\cdot),\ \mathcal{W}(t,\cdot) : L^2(\mathbb{R}^n) \longrightarrow \big(L^2, \|\cdot\|_{L^2}\big)$ are continuous (w.r.t. weakly continuous and tight sequences).*

(5.) *For each $r > 0$, there exist $\widehat{w}_r \in L^2(\mathbb{R}^n)$ and a compact subset $K_r \subset \mathbb{R}^n$ s.t.*

$$t \in [0,T],\ \|\rho\|_L^2 \le r,\ x \in \mathbb{R}^n \setminus K_r \implies \big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}_r(x).$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

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Main Results

Theorem (Uniqueness of L^2-valued solutions)

$$\begin{cases} \partial_t \rho \;+\; \operatorname{div}_x(\mathcal{G}(t,\rho)\,\rho) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho), \\ \rho(0) = \rho_0 \in L^2(\mathbb{R}^n) \end{cases}$$

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in addition:

(6.) (*Locally uniform* Lipschitz *conditions w.r.t. states*)
For every radius $r > 0$, *there exists* $\Lambda_r > 0$ *s.t.* $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)} \leq r$ *imply*

$$\begin{aligned} \| \mathcal{G}(t, \rho_1) - \mathcal{G}(t, \rho_2)\|_{L^2(\mathbb{R}^n, \mathbb{R}^n)} &\leq \Lambda_r \cdot d_{L^2}(\rho_1, \rho_2), \\ \| \mathcal{U}(t, \rho_1) - \mathcal{U}(t, \rho_2)\|_{L^2(\mathbb{R}^n)} &\leq \Lambda_r \cdot d_{L^2}(\rho_1, \rho_2), \\ \| \mathcal{W}(t, \rho_1) - \mathcal{W}(t, \rho_2)\|_{L^2(\mathbb{R}^n)} &\leq \Lambda_r \cdot d_{L^2}(\rho_1, \rho_2). \end{aligned}$$

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is unique.

$$d_{L^2}(\rho_1,\rho_2) := \sup \Big\{ \int_{\mathbb{R}^n} \varphi\, \rho_1\, dx - \int_{\mathbb{R}^n} \varphi\, \rho_2\, dx \;\Big|\; \varphi \in C_c^1(\mathbb{R}^n),\ \|\varphi\|_{L^2} \le 1, \ \|\varphi\|_{L^\infty} \le 1,\ \|\nabla\varphi\|_{L^\infty} \le 1 \Big\}$$

Main Results

Theorem (Continuous dependence on data)

Suppose for

$$\begin{aligned} \widetilde{\mathcal{G}},\ \mathcal{G} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \widetilde{\mathcal{U}},\ \mathcal{U} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \widetilde{\mathcal{W}},\mathcal{W} &: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{aligned}$$

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$$d_{L^2}\big(\rho(t),\ \widetilde{\rho}(t)\big) \ \le\ \Big(d_{L^2}\big(\rho(0),\ \widetilde{\rho}(0)\big) + \int_0^t \Delta(s)\ ds\Big) \cdot e^{c\,t}$$

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with a constant $c = c\big(\|\rho(0)\|_{L^2}, \|\widetilde{\rho}(0)\|_{L^2}, T, \Lambda_r\big)$ and

$$\Delta(s) := \sup_{L^2(\mathbb{R}^n)} \|\mathcal{G}(s,\cdot) - \widetilde{\mathcal{G}}(s,\cdot)\|_{L^2} + \sup_{L^2(\mathbb{R}^n)} \|\mathcal{U}(s,\cdot) - \widetilde{\mathcal{U}}(s,\cdot)\|_{L^2} + \sup_{L^2(\mathbb{R}^n)} \|\mathcal{W}(s,\cdot) - \widetilde{\mathcal{W}}(s,\cdot)\|_{L^2}.$$

Main Results

Theorem (Existence of measure-valued solutions)

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Main Results

Theorem (Existence of measure-valued solutions)

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu : [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

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Suppose for

$$\mathcal{B}: [0, T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$$
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$$\sup_{t,\mu} \Big(\|\mathcal{B}(t,\mu)\|_{L^\infty(\mathbb{R}^n)} + \|\mathcal{C}(t,\mu)\|_{L^\infty(\mathbb{R}^n)} \Big) < \infty$$

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Theorem (Existence of measure-valued solutions)

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$$\sup_{t,\mu} \Big(\|\mathcal{B}(t,\mu)\|_{L^\infty(\mathbb{R}^n)} + \|\mathcal{C}(t,\mu)\|_{L^\infty(\mathbb{R}^n)} \Big) < \infty$$

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(4.) *For a.e. $t \in [0, T]$, the functions $\mathcal{B}(t, \cdot), \ \mathcal{C}(t, \cdot) : \mathcal{M}(\mathbb{R}^n) \longrightarrow \left(W^{1,\infty}, \ \|\cdot\|_{L^\infty}\right)$ are narrowly continuous.*

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Theorem (Uniqueness of measure-valued solutions)

$$\begin{cases} \partial_t \, \mu \; + \; \mathrm{div}_x \big(\mathcal{B}(t, \mu) \, \mu \big) \; = \; \mathcal{C}(t, \mu), \\ \mu(0) \; = \; \mu_0 \; \in \; \mathcal{M}(\mathbb{R}^n) \end{cases}$$

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(6.) (*Locally uniform* LIPSCHITZ *conditions w.r.t. states*)
For every radius $r > 0$, *there exists* $\Lambda_r > 0$ *s.t.* $|\mu_1|(\mathbb{R}^n), |\mu_2|(\mathbb{R}^n) \le r$ *imply*

$$\big\|\mathcal{B}(t,\mu_1) - \mathcal{B}(t,\mu_2)\big\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le \Lambda_r \cdot d_{\mathcal{M}}(\mu_1,\mu_2),$$
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$$d_{\mathcal{M}}(\mu_1,\mu_2) := \sup\left\{ \int_{\mathbb{R}^n} \varphi\, d\mu_1 - \int_{\mathbb{R}^n} \varphi\, d\mu_2 \;\middle|\; \varphi \in C^1(\mathbb{R}^n),\ \|\varphi\|_{L^\infty} \le 1,\ \|\nabla\varphi\|_{L^\infty} \le 1 \right\}$$

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Theorem (Continuous dependence on data)

Suppose for $\widetilde{\mathcal{B}}, \ \mathcal{B} : \ [0, T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$

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with a constant $c=c\big(|\mu_0|(\mathbb{R}^n),\ |\widetilde{\mu}|(\mathbb{R}^n),\ T,\ \Lambda_r\big)$ *and*

$$\Delta(s):=\sup_{\mathcal{M}(\mathbb{R}^n)}\big\|\mathcal{B}(s,\cdot)-\widetilde{\mathcal{B}}(s,\cdot)\big\|_{L^\infty}+\sup_{\mathcal{M}(\mathbb{R}^n)}\big\|\mathcal{C}(s,\cdot)-\widetilde{\mathcal{C}}(s,\cdot)\big\|_{L^\infty}.$$

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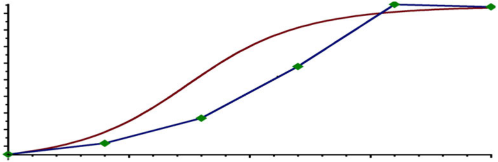

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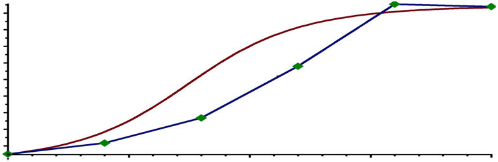

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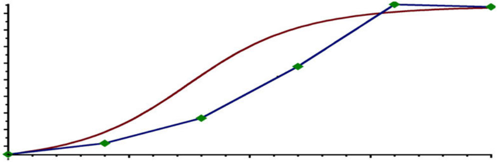

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Established extensions: Evolution equations in Banach spaces

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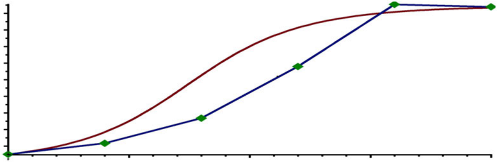

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Established extensions: Evolution equations in Banach spaces

Gist: Extend the notion of EULER method beyond linear spaces.

Mutational Equations

The Step to Metric Spaces

Aim: Extend ordinary differential equation $x' = f(t, x)$ to a metric space (E, d)

Question: Counterpart $x'(t)$ of $x : [0, T] \longrightarrow E$?

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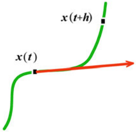

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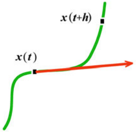

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$v \in \mathbb{R}^n$ induces $[0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ (h, y) \longmapsto y + h\, v$

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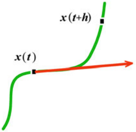

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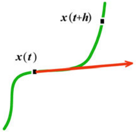

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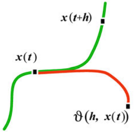


Specify some $\vartheta : [0,1] \times E \longrightarrow E, \quad (h, z) \longmapsto \vartheta(h, z)$.

$$\vartheta = \mathring{x}(t) \quad :\Longleftrightarrow \quad d\big(x(t+h),\ \ \vartheta(h, x(t))\big) \;\leq\; o(h)$$

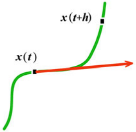
Mutational Equations

The Step to Metric Spaces

Aim: Extend ordinary differential equation $x' = f(t,x)$ to a metric space (E,d)

Question: Counterpart $x'(t)$ of $x : [0,T] \longrightarrow E$?

In $\mathbb{R}^n$:

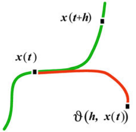


$x'(t) := \lim\limits_{h \to 0} \frac{x(t+h) - x(t)}{h}$ or, equivalently,

$x'(t) = v \quad :\Longleftrightarrow \quad \big| x(t+h) - \big(x(t) + h\, v\big) \big| \leq o(h)$

$v \in \mathbb{R}^n$ induces $[0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ (h,y) \longmapsto y + h\, v$

In (E,d):



Specify some $\vartheta : [0,1] \times E \longrightarrow E, \quad (h,z) \longmapsto \vartheta(h,z)$.

$\vartheta = \mathring{x}(t) \quad :\Longleftrightarrow \quad d\big(x(t+h), \ \vartheta(h, x(t))\big) \leq o(h)$

Given: $\mathcal{F} : [0,T] \times E \longrightarrow \Theta, \ (t,z) \longmapsto \vartheta$

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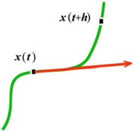

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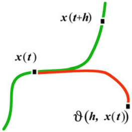

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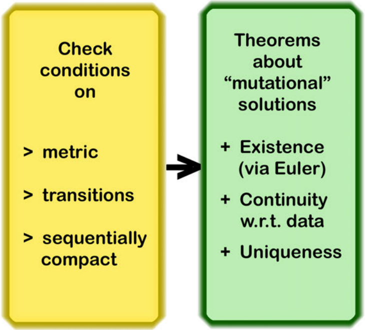
Wanted: cont. $x : [0,T] \longrightarrow E : \ \mathring{x}(t) = \mathcal{F}\big(t,\ x(t)\big)$ for a.e. t

Mutational Equations



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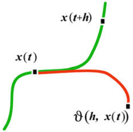

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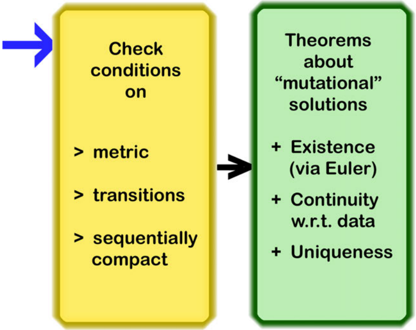

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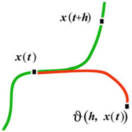

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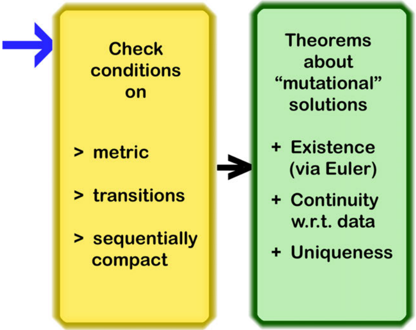
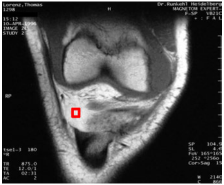
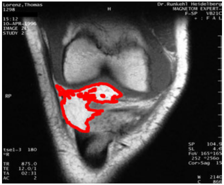




[TL, *Comput.Visual.Sci.* **4**, 2001]



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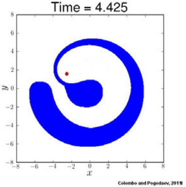


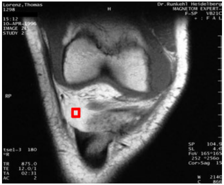
[Colombo, TL & Pogodaev, DCDS-A **35**, 2015]

Mutational Equations

Check conditions on
- > metric
- > transitions
- > sequentially compact

Theorems about "mutational" solutions
- \+ Existence (via Euler)
- \+ Continuity w.r.t. data
- \+ Uniqueness

More popular meaning of "mutational" solution ?
- . classical ?
- . weak ?
- . mild ?
- . link to diff. inclusions ?



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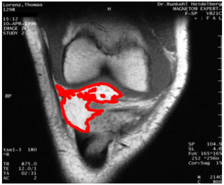

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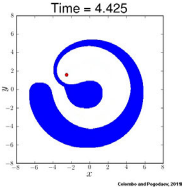

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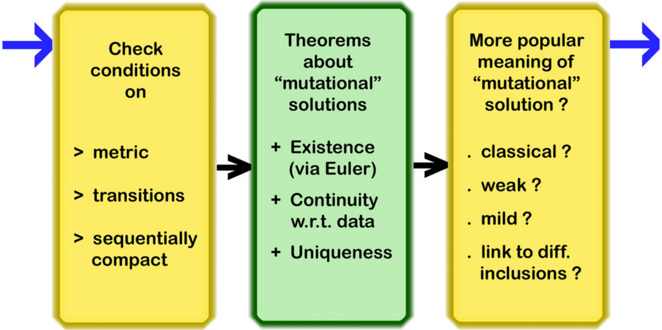
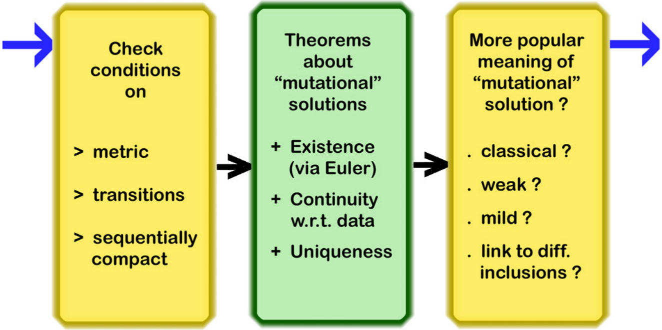
Mutational Equations



$$\partial_t \, \rho \; + \; \mathrm{div}_x\big(\mathcal{G}(t,\rho) \;\; \rho\big) \;\; = \; \mathcal{U}(t,\rho)\,\rho \; + \; \mathcal{W}(t,\rho)$$

$$\partial_t \, \mu \; + \; \mathrm{div}_x\big(\mathcal{B}(t,\mu)\,\mu\big) \;\; = \; \mathcal{C}(t,\mu)\,\mu$$

Mutational Equations



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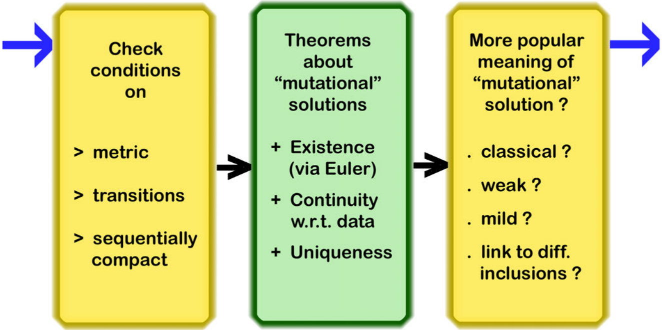

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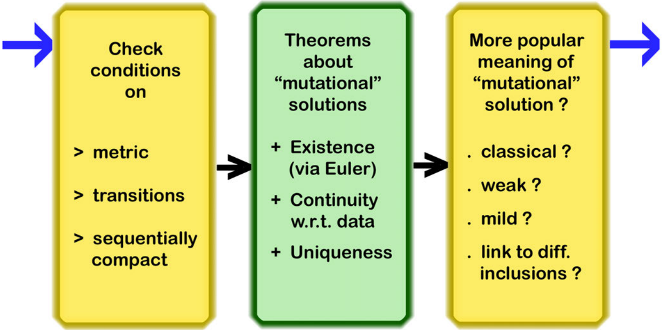

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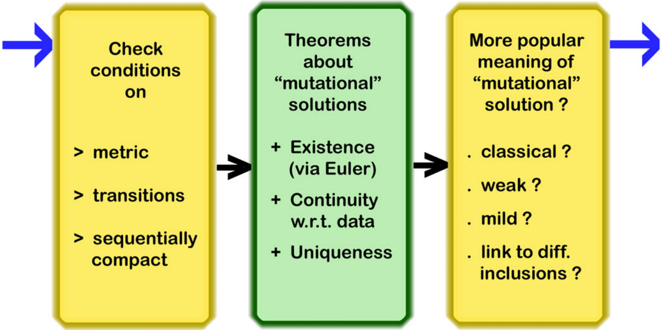

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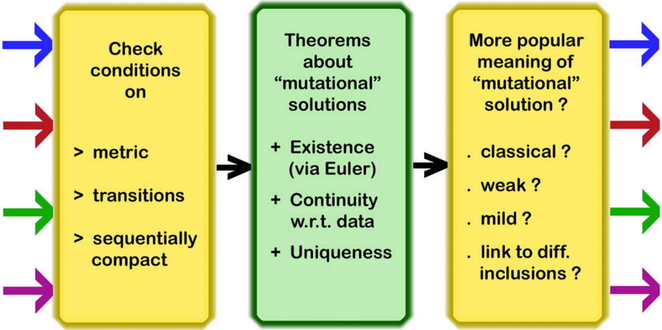

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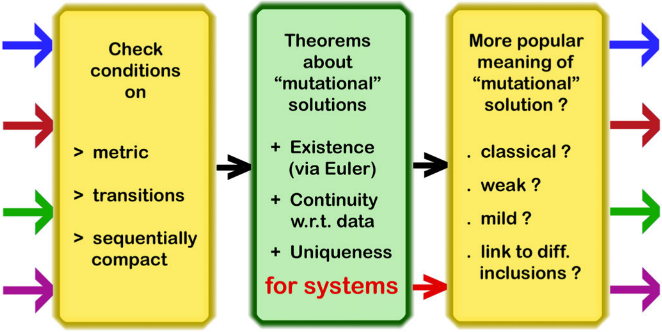

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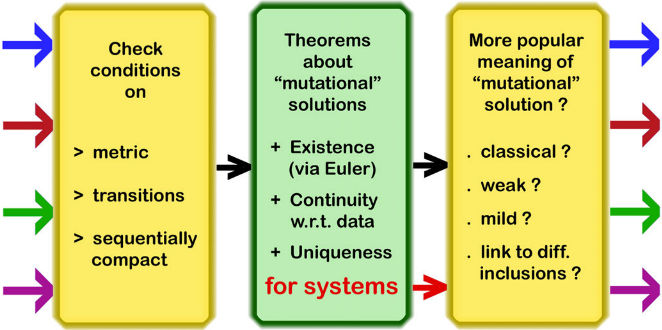

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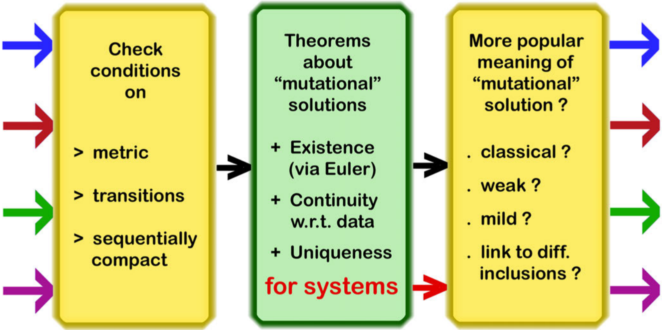

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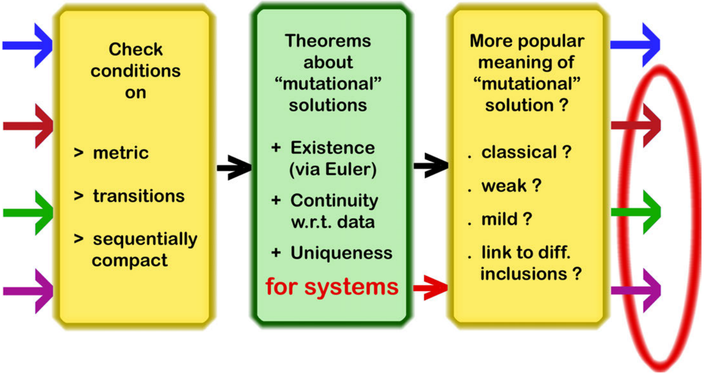

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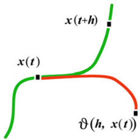
Mutational Equations

Transitions Instead of Affine Maps

Definition (Aubin 1993/99)

$$\vartheta : [0,1] \times E \longrightarrow E, \ \ (h,x) \longmapsto \vartheta(h,x)$$

$$\vartheta \ = \ \mathring{x}(t) \quad :\Longleftrightarrow \qquad \lim_{h \downarrow 0} \ \frac{1}{h} \cdot d\big(\vartheta(h, x(t)), \ x(t+h)\big) = 0$$



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$\vartheta : [0,1] \times E \longrightarrow E, \ (h,x) \longmapsto \vartheta(h,x)$ is called a *transition* if

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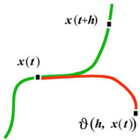

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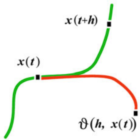

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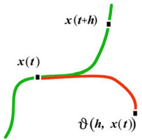

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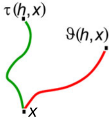

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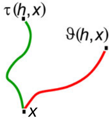

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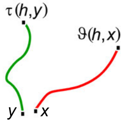

$$d(\vartheta(h,x),\ \tau(h,y)) \le \Big(d(x,y) + h \cdot D(\vartheta,\tau)\Big) \cdot e^{\alpha(\vartheta)\,h}$$

Mutational Equations

Solutions to Initial Value Problems

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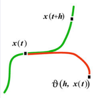
Theorem: Cauchy-Lipschitz (Aubin)

Let $f : (E,d) \longrightarrow (\Theta(E), D)$ be Lipschitz with $\sup \alpha(f(\cdot)) < \infty$.

$$\mathring{x}(\cdot) = f(x(\cdot)) \quad \text{a.e.}$$



Mutational Equations

Solutions to Initial Value Problems

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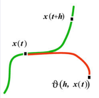
Theorem: CAUCHY-LIPSCHITZ (AUBIN)

Let $f : (E,d) \longrightarrow (\Theta(E), D)$ be LIPSCHITZ with $\sup \alpha(f(\cdot)) < \infty$.
Suppose all bounded closed balls in (E,d) to be complete.

Then in every $x_0 \in E$, there starts a unique continuous solution $x : [0,\infty[\longrightarrow E$ of $\mathring{x}(\cdot) = f(x(\cdot))$ a.e.



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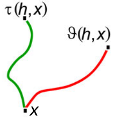

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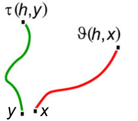

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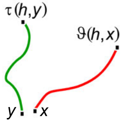

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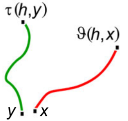

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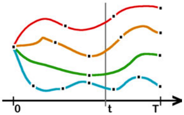

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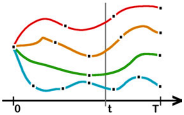
* $\lfloor \cdot \rfloor : E \longrightarrow [0, \infty[$ (just lower semicontinuous w.r.t. d)
* Set of transitions $\Theta(E)$ and its distance $D(\vartheta, \tau)$
* Is (E, d) complete ? "EULER compact" ?



Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho) \quad \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

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The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathbf{g}\;\rho\big) = u(x)\;\;\rho + w(x) \\ \rho(0) = \rho_0 \end{cases}$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

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$$\begin{cases} \partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x\big(\mathbf{g}\ \rho\big) = 0 \\ \rho(0) = \rho_0 \end{cases}$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\mathcal{G}(t,\rho)\, \rho) = \mathcal{U}(t,\rho)\, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\mathbf{g}\ \rho) = 0 \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x)\ \rho_0(x)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E := L^2(\mathbb{R}^n),$

"Abs. value" $\quad \lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

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The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathbf{g}\ \rho\big) = w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x)\ \rho_0(x)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

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Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

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$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\mathbf{g}\ \rho) = w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n): \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\,\rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathcal{G}(t,\rho)\,\rho\big) = \mathcal{U}(t,\rho)\,\rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x\big(\mathbf{g}\ \rho\big) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n): \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\,\rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\,ds\Big)\,dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E := L^2(\mathbb{R}^n),$

"Abs. value" $\quad \lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\quad \mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n), \quad u \in W^{1,\infty}(\mathbb{R}^n), \quad w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x \big(\mathcal{G}(t,\rho)\, \rho\big) = \mathcal{U}(t,\rho)\, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x (\mathbf{g}\ \rho) = u\, \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\, \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\, ds \Big)\, dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E := L^2(\mathbb{R}^n),$

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Transition (?) $\quad \mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n), \quad u \in W^{1,\infty}(\mathbb{R}^n), \quad w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \big(\mathcal{G}(t,\rho)\, \rho\big) = \mathcal{U}(t,\rho)\, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \big(\mathbf{g}\ \rho\big) = u\, \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n): \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E := L^2(\mathbb{R}^n),$

"Abs. value" $\quad \lfloor\rho\rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\quad$ For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathbf{g}\ \rho) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall\ \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\ ds\Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E \;:=\; L^2(\mathbb{R}^n),$

"Abs. value" $\quad \lfloor \rho \rfloor \;:=\; \|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathbf{g}\;\;\rho) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx \;=\; \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 \;+\; w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

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The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t\, \rho + \mathrm{div}_x\big(\mathbf{g}\;\; \rho\big) &= u\, \rho + w \\ \rho(0) &= \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\; \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\; \rho(t)\; dx \;=\; \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\; \rho_0 \;+\; w \int_0^t \psi_{t,\varphi}(s,x)\; ds \Big)\; dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E \ := \ L^2(\mathbb{R}^n),$

"Abs. value" $\lfloor \rho \rfloor \ := \ \|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $d_{L^2}(\rho_1, \rho_2)$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t\, \rho + \operatorname{div}_x(\mathbf{g} \ \ \rho) = u\, \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n) : \ \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx \ = \ \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 \ + \ w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $d_{L^2}(\rho_1, \rho_2)$ $\quad \int_{\mathbb{R}^n} \varphi\, \rho_1\, dx \quad \int_{\mathbb{R}^n} \varphi\, \rho_2\, dx$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

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$$\begin{cases} \partial_t\, \rho + \operatorname{div}_x(\mathbf{g}\;\; \rho) = u\, \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

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Metric $d_{L^2}(\rho_1, \rho_2)$ $\quad \int_{\mathbb{R}^n} \varphi\, \rho_1\, dx - \int_{\mathbb{R}^n} \varphi\, \rho_2\, dx \quad \varphi \in C_c^1(\mathbb{R}^n), \ \|\varphi\|_{L^2} \le 1$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

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$$\begin{cases} \partial_t\, \rho + \mathrm{div}_x\big(\mathbf{g}\ \ \rho\big) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

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Solutions with Values in $L^2(\mathbb{R}^n)$

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Metric $d_{L^2}(\rho_1, \rho_2)$ $\quad \sup \left\{ \int_{\mathbb{R}^n} \varphi \, \rho_1 \, dx - \int_{\mathbb{R}^n} \varphi \, \rho_2 \, dx \;\middle|\; \varphi \in C_c^1(\mathbb{R}^n), \; \|\varphi\|_{L^2} \le 1 \right\}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

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$$\Longleftrightarrow \forall \; \varphi \in C_c^1(\mathbb{R}^n) : \; \int_{\mathbb{R}^n} \varphi \; \rho(t) \; dx \; = \; \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0) \; \rho_0 \; + \; w \int_0^t \psi_{t,\varphi}(s,x) \; ds \Big) \, dx$$

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$\|\varphi\|_{L^\infty} \le 1,\ \|\nabla\varphi\|_{L^\infty} \le 1 \Big\}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

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$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\, \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\, ds \Big)\, dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

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Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

Lemma: For every $\psi \in C^1_c(\mathbb{R}^n)$ and $\rho_1, \rho_2 \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \psi\ (\rho_1 - \rho_2)\ dx$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t\, \rho + \operatorname{div}_x(\mathbf{g}\ \ \rho) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

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Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^\rho_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\, \rho) = u\, \rho + w$.

Lemma: For every $\psi \in C^1_c(\mathbb{R}^n)$ and $\rho_1, \rho_2 \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \psi\, (\rho_1 - \rho_2)\, dx = \left(\|\psi\|_{L^2} + \|\psi\|_{W^{1,\infty}}\right) \cdot \int_{\mathbb{R}^n} \frac{\psi}{\|\psi\|_{L^2} + \|\psi\|_{W^{1,\infty}}}\, (\rho_1 - \rho_2)\, dx$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t\, \rho + \mathrm{div}_x(\mathbf{g}\ \ \rho) = u\, \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\, \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\, ds \Big)\, dx$$

Solutions with Values in $L^2(\mathbb{R}^n)$

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Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

Lemma: For every $\psi \in C_c^1(\mathbb{R}^n)$ and $\rho_1, \rho_2 \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \psi\ (\rho_1 - \rho_2)\ dx \ \le\ \big(\|\psi\|_{L^2} + \|\psi\|_{W^{1,\infty}}\big) \cdot d_{L^2}(\rho_1, \rho_2)$$

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t\,\rho + \mathrm{div}_x(\mathbf{g}\ \ \rho) = u\,\rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall\ \varphi \in C_c^1(\mathbb{R}^n):\ \int_{\mathbb{R}^n} \varphi\ \rho(t)\ dx = \int_{\mathbb{R}^n} \Big(\psi_{t,\varphi}(0)\ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x)\ ds \Big)\ dx$$

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Transition For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^\rho_{\mathbf{g},u,w} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t, \rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \mathrm{div}_x(\mathbf{g}\,\rho) = u\,\rho + w$.

For any $\widetilde{\mathbf{g}}, \mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $\widetilde{u}, u \in W^{1,\infty}(\mathbb{R}^n)$ and $\widetilde{w}, w \in L^2(\mathbb{R}^n)$, it holds for all $\rho_0, \widetilde{\rho}_0 \in L^2(\mathbb{R}^n)$ and $s, t \in [0, T]$ $(s < t)$

(1.) $\vartheta^\rho_{\mathbf{g},u,w}(t, \rho_0) = \vartheta^\rho_{\mathbf{g},u,w}\big(t - s,\ \vartheta^\rho_{\mathbf{g},u,w}(s, \rho_0)\big)$

(2.) $\big\|\vartheta^\rho_{\mathbf{g},u,w}(t, \rho_0)\big\|_{L^2(\mathbb{R}^n)} \le \big(\|\rho_0\|_{L^2(\mathbb{R}^n)} + \|w\|_{L^2(\mathbb{R}^n)}\ t\big) \cdot e^{\mathrm{const}\cdot t}$

(3.) $e_{L^2}\big(\vartheta^\rho_{\mathbf{g},u,w}(s, \rho_0),\ \vartheta^\rho_{\mathbf{g},u,w}(t, \rho_0)\big) \le C \cdot \big(\|\rho_0\|_{L^2} + \|w\|_{L^2}\big) \cdot |t - s|$

(4.) $d_{L^2}\big(\vartheta^\rho_{\mathbf{g},u,w}(t, \rho_0),\ \vartheta^\rho_{\mathbf{g},u,w}(t, \widetilde{\rho}_0)\big) \le d_{L^2}\big(\rho_0,\ \widetilde{\rho}_0\big) \cdot e^{\mathrm{const}\cdot t}$

(5.) $d_{L^2}\big(\vartheta^\rho_{\mathbf{g},u,w}(t, \rho_0),\ \vartheta^\rho_{\widetilde{\mathbf{g}},\widetilde{u},\widetilde{w}}(t, \rho_0)\big) \le C \cdot \big(1 + \|\rho_0\|_{L^2(\mathbb{R}^n)}\big) \cdot \big\|(\mathbf{g}, u, w) - (\widetilde{\mathbf{g}}, \widetilde{u}, \widetilde{w})\big\|_{L^2} \cdot t$

Solutions with Values in $L^2(\mathbb{R}^n)$

Connection Between d_{L^2} and Weak Topology

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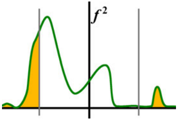

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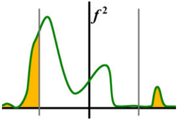

Solutions with Values in $L^2(\mathbb{R}^n)$

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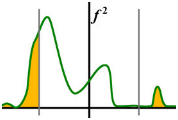

Solutions with Values in $L^2(\mathbb{R}^n)$

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Definition A set $\mathcal{S} \subset L^2(\mathbb{R}^n)$ is called *(uniformly) tight* if

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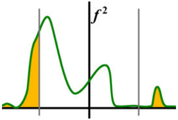

Solutions with Values in $L^2(\mathbb{R}^n)$

Connection Between d_{L^2} and Weak Topology

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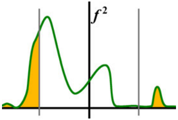

Proposition For every tight sequence $(\rho_k)_{k \in \mathbb{N}}$ in $L^2(\mathbb{R}^n)$ and $\rho \in L^2(\mathbb{R}^n)$,

$$\rho_k \longrightarrow \rho \ \text{ weakly in } L^2(\mathbb{R}^n) \quad \Longleftrightarrow \quad \begin{cases} \sup_{k \in \mathbb{N}} \ \|\rho_k\|_{L^2(\mathbb{R}^n)} < \infty \\ \lim_{k \to \infty} \ d_{L^2}(\rho_k, \rho) = 0 \end{cases}$$

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

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Suppose
$$\begin{aligned}
\mathcal{G} &: [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2,\ \|\cdot\|_{L^2}\big),\\
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Let $\widehat{w} \in L^2(\mathbb{R}^n)$ and a compact set $K \subset \mathbb{R}^n$ be such that $\big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

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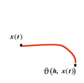

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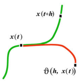

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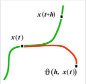
$$\mathcal{U} : [0,T] \times \big(L^2(\mathbb{R}^n), \, d_{L^2}\big) \longrightarrow \big(W^{1,\infty}(\mathbb{R}^n) \cap L^2, \ \|\cdot\|_{L^2}\big),$$
$$\mathcal{W} : [0,T] \times \big(L^2(\mathbb{R}^n), \, d_{L^2}\big) \longrightarrow \big(L^2(\mathbb{R}^n), \ \|\cdot\|_{L^2}\big)$$
to be "bounded" *Carathéodory* functions.

Let $\widehat{w} \in L^2(\mathbb{R}^n)$ and a compact set $K \subset \mathbb{R}^n$ be such that $\big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

Then for every $\rho_0 \in L^2(\mathbb{R}^n)$, there is at least one function $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ s.t.

(1.) ρ is continuous w.r.t. d_{L^2} and bounded w.r.t. $\|\cdot\|_{L^2(\mathbb{R}^n)}$,

(2.) $$\frac{1}{h} \cdot d_{L^2}\Big(\rho(t+h), \ \vartheta^{\rho}_{(\mathcal{G}(t,\rho(t)), \ \mathcal{U}(t,\rho(t)), \ \mathcal{W}(t,\rho(t)))}\big(h, \ \rho(t)\big)\Big) \qquad \text{(a.e. } t\text{)}$$



Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $$d_{L^2}(\rho_1, \rho_2) := \sup \Big\{ \int_{\mathbb{R}^n} \varphi\, \rho_1\, dx - \int_{\mathbb{R}^n} \varphi\, \rho_2\, dx \;\Big|\; \varphi \in C_c^1(\mathbb{R}^n),\ \|\varphi\|_{L^2} \le 1,\ \|\varphi\|_{L^\infty} \le 1,\ \|\nabla \varphi\|_{L^\infty} \le 1 \Big\}$$

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$$\mathcal{G} : [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2,\ \|\cdot\|_{L^2}\big),$$
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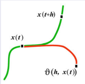

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n)$,

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Suppose

$$\begin{aligned} \mathcal{G} &: [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2,\ \|\cdot\|_{L^2}\big),\\ \mathcal{U} &: [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(W^{1,\infty}(\mathbb{R}^n) \cap L^2,\ \|\cdot\|_{L^2}\big),\\ \mathcal{W} &: [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(L^2(\mathbb{R}^n),\ \|\cdot\|_{L^2}\big) \end{aligned}$$

to be "bounded" *Carathéodory* functions.

Let $\widehat{w} \in L^2(\mathbb{R}^n)$ and a compact set $K \subset \mathbb{R}^n$ be such that $\big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

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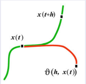
(3.) the image set $\rho([0,T]) \subset L^2(\mathbb{R}^n)$ is tight,



Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

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\mathcal{W} &: [0,T] \times \big(L^2(\mathbb{R}^n),\, d_{L^2}\big) \longrightarrow \big(L^2(\mathbb{R}^n),\ \|\cdot\|_{L^2}\big)
\end{aligned}$$
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Let $\widehat{w} \in L^2(\mathbb{R}^n)$ and a compact set $K \subset \mathbb{R}^n$ be such that $\big|\mathcal{W}(t,\rho)(x)\big| \le \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

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(3.) the image set $\rho([0,T]) \subset L^2(\mathbb{R}^n)$ is tight,

(4.) ρ is a weak solution of $\partial_t \rho + \mathrm{div}_x\big(\mathcal{G}(t,\rho)\, \rho\big) = \mathcal{U}(t,\rho)\, \rho + \mathcal{W}(t,\rho)$.

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

The "full nonlinear" problem

$$\begin{cases} \partial_t \mu + \mathrm{div}_x\big(\mathcal{B}(t,\mu)\,\mu\big) = \mathcal{C}(t,\mu)\,\mu & \text{in } [0,T] \\ \mu(0) = \mu_0 \in \mathcal{M}(\mathbb{R}^n) \end{cases}$$

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := \mathcal{M}(\mathbb{R}^n)$,

"Abs. value" $\lfloor \mu \rfloor := |\mu|(\mathbb{R}^n)$

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The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \mu + \mathrm{div}_x\big(\mathbf{b}\ \mu\big) = c(x)\ \mu \\ \rho(0) = \rho_0 \end{cases}$$

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $\quad E := \mathcal{M}(\mathbb{R}^n),$

"Abs. value" $\quad \lfloor \mu \rfloor := |\mu|(\mathbb{R}^n)$

Transition (?) $\quad \mathbf{b} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ and $c \in W^{1,\infty}(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \, \mu + \mathrm{div}_x \big(\mathcal{B}(t,\mu)\, \mu\big) = \mathcal{C}(t,\mu)\, \mu \quad \text{in } [0,T] \\ \mu(0) = \mu_0 \in \mathcal{M}(\mathbb{R}^n) \end{cases}$$

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

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$$\Longleftrightarrow \qquad \forall\ \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi\ d\mu_t = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0; x)\ d\mu_0(x)$$

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E \;:=\; \mathcal{M}(\mathbb{R}^n)$,

"Abs. value" $\lfloor \mu \rfloor \;:=\; |\mu|(\mathbb{R}^n)$

Transition (?) For $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ and $c \in W^{1,\infty}(\mathbb{R}^n)$, let $\vartheta^{\mu}_{\mathbf{b},c} : [0,1] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathcal{M}(\mathbb{R}^n), \;\; (t, \mu_0) \longrightarrow \mu_t$ denote the unique distributional solution to the autonomous linear equation $\partial_t \mu + \mathrm{div}_x(\mathbf{b}\, \mu) = c\, \mu$.

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := \mathcal{M}(\mathbb{R}^n)$,

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Metric $d_{\mathcal{M}}(\mu_1, \mu_2)$

Transition (?) For $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ and $c \in W^{1,\infty}(\mathbb{R}^n)$, let $\vartheta^{\mu}_{\mathbf{b},c} : [0,1] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathcal{M}(\mathbb{R}^n)$, $(t, \mu_0) \longrightarrow \mu_t$ denote the unique distributional solution to the autonomous linear equation $\partial_t \mu + \mathrm{div}_x(\mathbf{b}\, \mu) = c\, \mu$.

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := \mathcal{M}(\mathbb{R}^n)$,

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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known as *$W^{1,\infty}$ dual metric* [TL, 2010]

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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Existence Theoreom [TL, 2009/10]

Suppose

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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(1.) μ is continuous w.r.t. $d_{\mathcal{M}}$ and bounded w.r.t. $|\cdot|(\mathbb{R}^n)$,

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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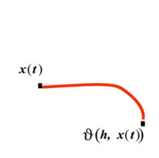

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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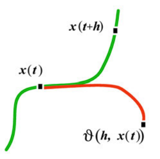

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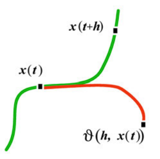

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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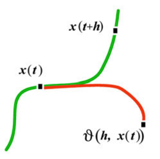

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

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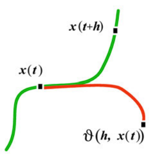
(3.) the image set $\mu([0, T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,



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(3.) the image set $\mu([0,T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,

(4.) μ is narrowly continuous

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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(3.) the image set $\mu([0,T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,

(4.) μ is narrowly continuous and a distributional solution of
$$\partial_t\,\mu + \operatorname{div}_x\big(\mathcal{B}(t,\mu)\;\mu\big) \;=\; \mathcal{C}(t,\mu)\;\mu.$$

Differential Inclusions Without Convexity

Classical Result by ANTOSIEWICZ & CELLINA

Theoreom [ANTOSIEWICZ and CELLINA, 1975]

Suppose for the set-valued map $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$

(1.) each value $F(t, x) \subset \mathbb{R}^n$ is nonempty and compact with $F(t, x) \subset \mathbb{B}_R(0)$,

(2.) $\forall\ x \in \mathbb{R}^n :\ F(\cdot, x) : [0, T] \rightsquigarrow \mathbb{R}^n$ is measurable,

(3.) $\forall\ t \in [0, T] :\ F(t, \cdot) : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is continuous (w.r.t. Hausdorff metric dl).

Then for each $x_0 \in \mathbb{R}^n$, there exists an absolutely cont. solution $x : [0, T] \longrightarrow \mathbb{R}^n$ of

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Differential Inclusions Without Convexity

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Selection Principle [Antosiewicz and Cellina, 1975]

Under the preceding assumptions about the set-valued map $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, there exists a function $g : C^0([0, T], \mathbb{R}^n) \longrightarrow L^1([0, T], \mathbb{R}^n)$ such that

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Extensions

* to separable Banach spaces by Kisielewicz (1982)

Differential Inclusions Without Convexity

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Extensions

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Extensions

* to separable BANACH spaces by KISIELEWICZ (1982)
* to separable metric spaces [TL, 2010]

  e.g., $F : [0, T] \times (E, d) \leadsto (\Theta(E), D)$

Multivalued Nonlocal Balance Laws

Existence of Weak Solutions to Multivalued Balance Laws

Suppose for
$$\mathcal{G} : [0,T] \times L^2(\mathbb{R}^n) \rightsquigarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$
$$\mathcal{U} : [0,T] \times L^2(\mathbb{R}^n) \rightsquigarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$
$$\mathcal{W} : [0,T] \times L^2(\mathbb{R}^n) \rightsquigarrow L^2(\mathbb{R}^n)$$

(1.) Each of the values $\mathcal{G}(t,\rho)$, $\mathcal{U}(t,\rho)$ and $\mathcal{W}(r,\rho)$ is a nonempty set and compact (w.r.t. $\|\cdot\|_{L^2}$).

(2.) The set values of $\mathcal{G}$, $\mathcal{U}$ are globally bounded w.r.t. $\|\cdot\|_{W^{1,\infty}}$ and, the set values of $\mathcal{W}$ are bounded w.r.t. $\|\cdot\|_{L^2}$.

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(3.) $\forall\, \rho \in L^2(\mathbb{R}^n)$: $\mathcal{G}(\cdot,\rho), \mathcal{U}(\cdot,\rho), \mathcal{W}(\cdot,\rho) : [0,T] \rightsquigarrow \left(L^2, \|\cdot\|_{L^2}\right)$ are measurable.

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(4.) For a.e. $t \in [0,T]$: $\mathcal{G}(t,\cdot), \mathcal{U}(t,\cdot), \mathcal{W}(t,\cdot) : \big(L^2(\mathbb{R}^n), d_{L^2}\big) \rightsquigarrow \big(L^2, \|\cdot\|_{L^2}\big)$ are continuous (w.r.t. the HAUSDORFF metric).

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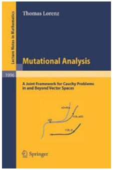

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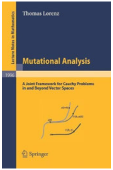

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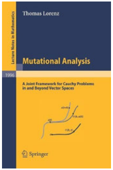

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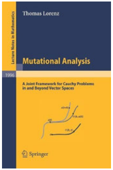

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