On a Class of Nonlocal Traffic Flow Models with Multivalued Fundamental Diagram

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UCLA, September 29, 2015

M-value

Mutational

C.....

Motivation

PDE Models for Traffic Flow

"Standard" situation in traffic flow models:

* Density $\rho = \rho(t, x)$ (of vehicles on a lane)

PDE Models for Traffic Flow

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- * Density $\rho = \rho(t, x)$ (of vehicles on a lane)
- * Flow \bar{f} (due to the underlying velocities of the vehicles)

Standard

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued

M-valued

Mutationa

C.....

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-value

Mutational

C.....

Motivation

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- * Density $\rho = \rho(t, x)$ (of vehicles on a lane)
- * Flow \bar{f} (due to the underlying velocities of the vehicles)
- * First laws due to conservation princples:

$$\partial_t \rho + \operatorname{div}_x \bar{f}(t, x, \rho) = 0$$

Summa

Motivation

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 - * Flow-density diagram $\bar{f} = \bar{f}(\rho)$

Summai

Motivation

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Summa

Motivation

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Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Mutationa

Summar

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Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued solutions

Mutationa

Summai

Motivation

PDE Models for Traffic Flow

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$$\begin{cases} \partial_t \rho + \operatorname{div}_{\times} (\mathcal{G}(t, \rho) \rho) = \mathcal{W}(t, \rho) & \text{in } [0, T] \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued

M-value

Mutational

Summar

Motivation

PDE Models for Traffic Flow

Questions:

- * What type of solution $\rho: [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$?
- * Which function space for $\rho = \rho(t, \cdot)$?

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\times} (\mathcal{G}(t, \rho) \rho) = \mathcal{W}(t, \rho) & \text{in } [0, T] \\ \rho(0) = \rho_0 \end{cases}$$

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Nonlocal models considered here

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- * What type of solution $\rho: [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$?
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$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}} (\mathcal{G}(t, \rho) \rho) = \mathcal{W}(t, \rho) & \text{in } [0, T] \\ \rho(0) = \rho_0 \end{cases}$$

PDE Models for Traffic Flow

Standard

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

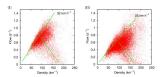
M-value solutions

Mutationa

Summary

Questions:

- * What type of solution $\rho: [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$?
- * Which function space for $\rho = \rho(t, \cdot)$?
- * How to choose the coefficients $\mathcal{G},\,\mathcal{W}$?
- * How to handle the "imprecision" of the fundamental diagram ?



[Nakayama et al. 2009]

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \, \rho) = \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

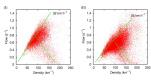
L²-valued solutions

Mutationa

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Questions:

- * What type of solution $\rho:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$?
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[Nakayama et al. 2009]



[Seibold et al. 2013]

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

situation

Nonlocal models

considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Summary

Motivation

The Considered Problem

Given: Initial data ρ_0

Wanted:

s.t.

$$\left\{ \begin{array}{rcl} \partial_t \, \rho \, + \, \mathrm{div}_x \big(\mathcal{G}(t,\rho) & \, \rho \big) & = \, \, \mathcal{U}(t,\rho) & \, \rho + \mathcal{W}(t,\rho) \\ \\ & \rho(0) \, \, = \, \, \rho_0 \end{array} \right.$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}} (\mathcal{G}(t, \rho) \rho) = \mathcal{W}(t, \rho) & \text{in } [0, T] \\ \rho(0) = \rho_0 \end{cases}$$

Standard

Given: Initial data ρ_0

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued

solutions

 \mathcal{M} -valued solutions

Mutationa inclusions

C.....

Wanted: Density $\rho: [0,T] \longrightarrow L^1(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$

s.t.
$$\begin{cases} \partial_t \, \rho \, + \, \mathrm{div}_x \big(\mathcal{G}(t,\rho) \quad \, \rho \big) \, = \, \mathcal{U}(t,\rho) \quad \, \rho + \mathcal{W}(t,\rho) \\ \\ \rho(0) \, = \, \rho_0 \end{cases}$$

Standard

Given: Initial data ρ_0

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued

solutions

M-valued solutions

Mutationa

C.....

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t, \rho) & \rho) = \mathcal{U}(t, \rho) & \rho + \mathcal{W}(t, \rho) \\ \\ \rho(0) = \rho_0 \end{cases}$$

Standard

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$

Nonlocal models considered here

Main results

Ordin. diff. eqns.

L²-valued solutions

M-valued solutions

Mutational

Summar

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t, \rho) & \rho) = \mathcal{U}(t, \rho) & \rho + \mathcal{W}(t, \rho) \\ \\ \rho(0) = \rho_0 \end{cases}$$

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa

Summar

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$

Coeff. $\mathcal{G}:[0,T]\times L^2(\mathbb{R}^n)$

 $\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)\cap L^2$

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

s.t.

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) & \rho) = \mathcal{U}(t,\rho) & \rho + \mathcal{W}(t,\rho) \\ \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

s.t.

L²-valued solutions

M-valued solutions

Mutationa

Summai

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n)$$
 $\longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$
 $\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n)$ $\longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$
 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n)$ $\longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

$$\begin{cases} \partial_t \rho \, + \, \mathrm{div}_x \big(\mathcal{G}(t,\rho) \quad \, \rho \big) \; = \; \mathcal{U}(t,\rho) \quad \, \rho + \mathcal{W}(t,\rho) \\ \\ \rho(0) \; = \; \rho_0 \end{cases}$$

 $\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)\cap L^2$

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa

Summar

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n)$$

$$\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n)$$

 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n)$

$$egin{array}{lll} P' & \longrightarrow & W^{1,\infty}(\mathbb{R}^n) \cap L^2 \ \longrightarrow & L^2(\mathbb{R}^n) \end{array}$$

Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

Wanted:

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}} (\mathcal{G}(t, \rho) & \rho) = \mathcal{U}(t, \rho) & \rho + \mathcal{W}(t, \rho) \\ \\ \rho(0) = \rho_0 \end{cases}$$

Example:

* $\partial_t \rho + \operatorname{div}_x (v(\eta * \rho) \ V(t,x) \ \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]

Nonlocal models considered here

Motivation

The Considered Problem

 $\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)\cap L^2$

Initial data $\rho_0 \in L^2(\mathbb{R}^n)$ Given:

Coeff.
$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n)$$

$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n)$$

$$0 \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$
$$\longrightarrow L^2(\mathbb{R}^n)$$

$$W: [0,T] \times L^2(\mathbb{R}^n)$$

$$\longrightarrow L^2(\mathbb{R}^n)$$

Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ Wanted:

s.t.

$$\begin{cases} \partial_t \, \rho \, + \, \mathrm{div}_x \big(\mathcal{G}(t,\rho) \quad \, \rho \big) \; = \; \mathcal{U}(t,\rho) \quad \, \rho + \mathcal{W}(t,\rho) \\ \\ \rho(0) \; = \; \rho_0 \end{cases}$$

- - * $\partial_t \rho + \operatorname{div}_x(v(\eta * \rho) V(t, x) \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
 - * $\partial_t \rho + \operatorname{div}_x \left(v(\eta * \rho) \left(V(x) \frac{\kappa \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_v (\eta * \rho)|^2}} \right) \rho \right) = 0$, compare [Colombo et al.]

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-value solutions

Mutationa inclusions

Summar

Motivation

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Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$

$$\mu: [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$$

s.t.
$$\begin{cases} \partial_t \rho \, + \, \mathrm{div}_x \big(\mathcal{G}(t,\rho) \quad \, \rho \big) \, = \, \mathcal{U}(t,\rho) \quad \, \rho + \mathcal{W}(t,\rho) \\ \\ \rho(0) \, = \, \rho_0 \end{cases}$$

- * $\partial_t
 ho \,+\, {
 m div}_{
 m x}ig(v(\eta*
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 hoig) \,\,=\,\,0,\,\,\,$ see, e.g., [Colombo, Herty & Mercier 2011]
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Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa

Summar

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n)$$
 $\longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$ $\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n)$ $\longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$ $\longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

s.t. $(\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) - \rho))$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) & \rho) = \mathcal{U}(t,\rho) & \rho + \mathcal{W}(t,\rho) \\ \partial_t \mu + \operatorname{div}_x(\mathcal{B}(t,\mu) & \mu) = \mathcal{C}(t,\mu) & \mu \\ & \rho(0) = \rho_0 \\ & \mu(0) = \mu_0 \end{cases}$$

- * $\partial_t \rho + \operatorname{div}_x ig(v(\eta *
 ho) \ V(t,x) \
 ho ig) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
- * $\partial_t \rho \, + \, \mathrm{div}_x \Big(v(\eta * \rho) \, \Big(V(x) \frac{\kappa \, \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_v (\eta * \rho)|^2}} \Big) \, \, \rho \Big) \, = \, 0, \quad \text{compare [COLOMBO et al.]}$

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summar

Motivation

The Considered Problem

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Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

s.t.
$$(\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho, \mu)) \mu)$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_x (\mathcal{B}(t, \mu) \ \mu) &= \ \mathcal{C}(t, \mu) \ \mu \end{cases}$$

$$\rho(0) &= \rho_0$$

$$\mu(0) &= \mu_0$$

- * $\partial_t \rho + \operatorname{div}_x (v(\eta * \rho) \ V(t,x) \ \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
- * $\partial_t \rho + \operatorname{div}_x \left(v(\eta * \rho) \left(V(x) \frac{\kappa \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_x (\eta * \rho)|^2}} \right) \rho \right) = 0$, compare [Colombo et al.]

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summar

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$

 $\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$
 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

s.t.
$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_x(\mathcal{B}(t, \mu) \ \mu) &= \ \mathcal{C}(t, \mu) \ \mu \end{cases}$$
$$\rho(0) = \rho_0$$
$$\mu(0) = \mu_0$$

- * $\partial_t \rho + \operatorname{div}_x (v(\eta * \rho) \ V(t,x) \ \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
- * $\partial_t \rho \, + \, \mathrm{div}_x \Big(v(\eta * \rho) \, \Big(V(x) \frac{\kappa \, \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_v (\eta * \rho)|^2}} \Big) \, \, \rho \Big) \, = \, 0, \quad \text{compare [COLOMBO et al.]}$

Nonlocal models considered here

Motivation

The Considered Problem

Initial data $\rho_0 \in L^2(\mathbb{R}^n), \ \mu_0 \in \mathcal{M}(\mathbb{R}^n)$ Given:

Coeff.
$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$

 $\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$
 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

s.t.
$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_x(\mathcal{B}(t, \rho, \mu) \ \mu) &= \ \mathcal{C}(t, \rho, \mu) \ \mu \end{cases} \\ \rho(0) &= \rho_0 \\ \mu(0) &= \mu_0 \end{cases}$$

$$\mu(0) = \mu_0$$

$$\iota(0) = \iota$$

- * $\partial_t \rho + \operatorname{div}_x(v(\eta * \rho) V(t, x) \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
- * $\partial_t \rho + \operatorname{div}_x \left(v(\eta * \rho) \left(V(x) \frac{\kappa \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_v (\eta * \rho)|^2}} \right) \rho \right) = 0$, compare [Colombo et al.]

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summar

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$

 $\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$
 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

s.t.
$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_x(\mathcal{B}(t, \rho, \mu) \ \mu) &= \ \mathcal{C}(t, \rho, \mu) \ \mu \end{cases} \\ \rho(0) &= \rho_0 \\ \mu(0) &= \mu_0 \end{cases}$$

- * $\partial_t \rho + \operatorname{div}_x (v(\eta * \rho) \ V(t,x) \ \rho) = 0$, see, e.g., [Colombo, Herty & Mercier 2011]
- * $\partial_t \rho \, + \, \mathrm{div}_x \Big(v(\eta * \rho) \, \Big(V(x) \frac{\kappa \, \nabla_x (\eta * \rho)}{\sqrt{1 + |\nabla_v (\eta * \rho)|^2}} \Big) \, \, \rho \Big) \, = \, 0, \quad \text{compare [COLOMBO et al.]}$

Standard situation

Nonlocal models considered here

iviain results

Ordin. diff. eqns. in a metric space

s.t.

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summary

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}} (\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_{\mathsf{x}} (\mathcal{B}(t, \rho, \mu) \ \mu) &= \ \mathcal{C}(t, \rho, \mu) \ \mu \end{cases}$$

$$\rho(0) &= \rho_0$$

$$\mu(0) &= \mu_0$$

Goal: Existence of weak solutions ρ and distributional solutions μ Sufficient conditions for their uniqueness

Continuous dependence on data (i.e., initial data and coeff.)

Summar

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

 $\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$

$$W: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

Wanted: Density $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$

$$\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$$

s.t. $(\partial_t a + \operatorname{div} (C(t, a, u), a))$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_{\mathsf{x}}(\mathcal{B}(t, \rho, \mu) \ \mu) &= \ \mathcal{C}(t, \rho, \mu) \ \mu \\ \rho(0) &= \ \rho_0 \\ \mu(0) &= \ \mu_0 \end{cases}$$

Goal 1: Existence of weak solutions ρ and distributional solutions μ Sufficient conditions for their uniqueness

Continuous dependence on data (i.e., initial data and coeff.)

Mastrostin

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

s.t.

 L^2 -valued solutions

M-valued solutions

Mutationa

Summar

Motivation

The Considered Problem

Given: Initial data $\rho_0 \in L^2(\mathbb{R}^n)$, $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$

Coeff.
$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

 $\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$
 $\mathcal{W}: [0,T] \times L^2(\mathbb{R}^n) \times \mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

Wanted: Density $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$ $\mu: [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n) = C_0^0(\mathbb{R}^n)'$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t, \rho, \mu) \ \rho) &= \ \mathcal{U}(t, \rho, \mu) \ \rho + \mathcal{W}(t, \rho, \mu) \\ \partial_t \mu + \operatorname{div}_x (\mathcal{B}(t, \rho, \mu) \ \mu) &= \ \mathcal{C}(t, \rho, \mu) \ \mu \end{cases}$$

$$\rho(0) &= \rho_0$$

$$\mu(0) &= \mu_0$$

more than one velocity field $\mathcal{G}(t,\rho,\mu)$, $\mathcal{B}(t,\rho,\mu):\mathbb{R}^n\longrightarrow\mathbb{R}^n$ etc.

Goal 2: Existence of weak solutions ρ and distributional solutions μ if coefficient maps are set-valued (e.g., due to imprecision) i.e., for each tuple (t,ρ,μ) , the model provides (possibly)

Motivatio

Standard situation

Nonlocal models considered here

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Summar

Motivation

2 The Main Results about Nonlocal Traffic Flow Models

3 Ordinary Differential Equations – But in a Metric Space: Mutational Equations

4 $L^2(\mathbb{R}^n)$ -valued Solutions to Nonlocal Balance Laws

 $\mathfrak{S} \mathcal{M}(\mathbb{R}^n)$ -valued Solutions to Nonlocal Balance Laws

6 Differential Inclusions in a Metric Space: Extending the Results of Antosiewicz & Cellina

Summary

Mastrostin

Main results

L²-valued solutions

M-valued solutions

Ordin. diff. eqns

L²-value

M-valued solutions

Mutational

Summar

Theorem (Existence of L^2 -valued solutions)

$$\partial_t \, \rho \ + \ \mathrm{div}_x ig(\mathcal{G}(t,
ho) \,
ho ig) \ = \ \mathcal{U}(t,
ho) \,
ho \, + \, \mathcal{W}(t,
ho),$$

L²-valued solutions

M-valued solutions

Ordin. diff. eqn

L²-valued

M-valued solutions

Mutational inclusions

Summary

Theorem (Existence of L^2 -valued solutions)

$$\partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho),$$

$$\begin{split} \int_{\mathbb{R}^n} \varphi \ \left(\rho(t_2) - \rho(t_1) \right) \ dx \ = \ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x) \ \mathcal{G} \big(s, \ \rho(s) \big)(x) \cdot \nabla_x \, \varphi(x) \quad dx \quad ds \ + \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x) \, \mathcal{U} \big(s, \rho(s) \big)(x) + \mathcal{W} \big(s, \rho(s) \big)(x) \Big) \ \varphi(x) \ dx \ ds \end{split}$$

M-valued solutions

Ordin. diff. eqns

L²-valued

M-valued solutions

Mutational inclusions

Summary

Theorem (Existence of L^2 -valued solutions)

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho),$

$$\int_{\mathbb{R}^n} \varphi \left(\rho(t_2) - \rho(t_1) \right) \ dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s, x) \ \mathcal{G}(s, \rho(s))(x) \cdot \nabla_x \varphi(x) \ dx \ ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left(\rho(s, x) \mathcal{U}(s, \rho(s))(x) + \mathcal{W}(s, \rho(s))(x) \right) \varphi(x) \ dx \ ds$$

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

$$W: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\int_{\mathbb{R}^{n}} \varphi \left(\rho(t_{2}) - \rho(t_{1})\right) dx = \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \rho(s, x) \mathcal{G}(s, \rho(s))(x) \cdot \nabla_{x} \varphi(x) dx ds + \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \left(\rho(s, x) \mathcal{U}(s, \rho(s))(x) + \mathcal{W}(s, \rho(s))(x)\right) \varphi(x) dx ds$$

M-valued solutions

in a metric space

L²-valued solutions

Mutational inclusions

Summary

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

 $W: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

(1.) Global a priori bounds

$$\sup_{t,\rho} \left(\left\| \operatorname{div}_{x} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) < \infty$$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho),$$

$$\int_{\mathbb{R}^n} \varphi \left(\rho(t_2) - \rho(t_1) \right) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s, x) \, \mathcal{G} \big(s, \, \rho(s) \big)(x) \cdot \nabla_x \, \varphi(x) \, dx \, ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left(\rho(s, x) \, \mathcal{U} \big(s, \rho(s) \big)(x) + \mathcal{W} \big(s, \rho(s) \big)(x) \right) \, \varphi(x) \, dx \, ds$$

M-valued solutions

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Summary

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

 $\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

(1.) Global a priori bounds

$$\mathsf{sup}_{\mathsf{t},\rho} \ \left(\left\| \mathrm{div}_{\mathsf{x}} \, \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) \ < \ \infty$$

(2) Local a priori bounds: For each r > 0, there is $C_r > 0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)} \le r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)} + \|\partial_x \mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)} + \|\nabla_x \mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$

Then every function
$$\rho_0 \in L^2(\mathbb{R}^n)$$
 initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\int_{\mathbb{R}^n} \varphi \ \left(\rho(t_2) - \rho(t_1) \right) \ dx \ = \ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x) \ \mathcal{G} \big(s, \ \rho(s) \big)(x) \cdot \nabla_x \, \varphi(x) \ dx \ ds \ + \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x) \, \mathcal{U} \big(s, \rho(s) \big)(x) + \mathcal{W} \big(s, \rho(s) \big)(x) \Big) \ \varphi(x) \ dx \ ds$$

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{l} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\mathsf{sup}_{t,\rho} \ \left(\left\| \operatorname{div}_{\mathsf{x}} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) \ < \ \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\ \|\partial_x\,\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\ \|\nabla_x\,\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\begin{split} \int_{\mathbb{R}^n} \varphi \ \left(\rho(t_2) - \rho(t_1) \right) \ dx \ = \ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x) \ \mathcal{G} \big(s, \ \rho(s) \big)(x) \cdot \nabla_x \, \varphi(x) \ dx \ ds \ + \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x) \, \mathcal{U} \big(s, \rho(s) \big)(x) + \mathcal{W} \big(s, \rho(s) \big)(x) \Big) \ \varphi(x) \ dx \ ds \end{split}$$

L²-valued solutions

Mutationa inclusions

Summary

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{l} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\sup_{t,\rho} \left(\left\| \operatorname{div}_{x} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) < \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\|\partial_x\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\|\nabla_x\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4) For a.e. $t \in [0, T]$, the functions $\mathcal{G}(t, \cdot)$, $\mathcal{U}(t, \cdot)$, $\mathcal{W}(t, \cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_X(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\int_{\mathbb{R}^{n}} \varphi \left(\rho(t_{2}) - \rho(t_{1})\right) dx = \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \rho(s, x) \mathcal{G}(s, \rho(s))(x) \cdot \nabla_{x} \varphi(x) dx ds + \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \left(\rho(s, x) \mathcal{U}(s, \rho(s))(x) + \mathcal{W}(s, \rho(s))(x)\right) \varphi(x) dx ds$$

Mutationa

Summary

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{ll} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\mathsf{sup}_{t,\rho} \ \left(\left\| \operatorname{div}_{\scriptscriptstyle X} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^n)} \right) \ < \ \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\ \|\partial_x\,\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\ \|\nabla_x\,\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4.) For a.e. $t \in [0, T]$, the functions $\mathcal{G}(t, \cdot)$, $\mathcal{U}(t, \cdot)$, $\mathcal{W}(t, \cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous (w.r.t. weakly continuous sequences).

Then every function $\rho_0\in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho),$$

$$\int_{\mathbb{R}^n} \varphi \; \left(\rho(t_2) - \rho(t_1) \right) \; dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s, x) \; \mathcal{G} \left(s, \; \rho(s) \right) (x) \cdot \nabla_x \, \varphi(x) \; \; dx \; \; ds \; + \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left(\rho(s, x) \; \mathcal{U} \left(s, \; \rho(s) \right) (x) + \mathcal{W} \left(s, \; \rho(s) \right) (x) \right) \; \varphi(x) \; dx \; ds$$

solutions

Ordin. diff. eqns. in a metric space

L²-valued solutions

 \mathcal{M} -valued solutions

Mutationa

Summary

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{split} \mathcal{G} : & [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow \mathcal{W}^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U} : & [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow \mathcal{W}^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W} : & [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{split}$$

(1.) Global a priori bounds

$$\mathsf{sup}_{t,\rho} \ \left(\left\| \operatorname{div}_{\scriptscriptstyle X} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^n)} \right) \ < \ \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\ \|\partial_x\,\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\ \|\nabla_x\,\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4.) For a.e. $t \in [0, T]$, the functions $\mathcal{G}(t, \cdot)$, $\mathcal{U}(t, \cdot)$, $\mathcal{W}(t, \cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous (w.r.t. weakly continuous sequences).

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho: [0,T] \longrightarrow L^2(\mathbb{R}^n)$ of

$$\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho),$$

$$\int_{\mathbb{R}^{n}} \varphi \left(\rho(t_{2}) - \rho(t_{1}) \right) dx = \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \rho(s, x) \, \mathcal{G}(s, \, \rho(s))(x) \cdot \nabla_{x} \, \varphi(x) \, dx \, ds + \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \left(\rho(s, x) \, \mathcal{U}(s, \, \rho(s))(x) + \mathcal{W}(s, \, \rho(s))(x) \right) \, \varphi(x) \, dx \, ds$$

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{l} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\mathsf{sup}_{t,\rho} \ \left(\left\| \operatorname{div}_{\mathsf{x}} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) < \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\ \|\partial_x\,\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\ \|\nabla_x\,\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4.) For a.e. $t \in [0,T]$, the functions $\mathcal{G}(t,\cdot), \ \mathcal{U}(t,\cdot), \ \mathcal{W}(t,\cdot): L^2(\mathbb{R}^n) \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right)$ are continuous (w.r.t. weakly continuous sequences).
- (5.) For each r > 0, there exist $\widehat{w}_r \in L^2(\mathbb{R}^n)$ and a compact subset $K_r \subset \mathbb{R}^n$ s.t. $t \in [0,T], \ \|\rho\|_L^2 \le r, \ x \in \mathbb{R}^n \setminus K_r \implies \frac{|\mathcal{W}(t,\rho)(x)|}{|\mathcal{W}(t,\rho)(x)|} \le \frac{\widehat{w}_r(x)}{r}$.

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\int_{\mathbb{R}^{n}} \varphi \left(\rho(t_{2}) - \rho(t_{1}) \right) dx = \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \rho(s, x) \, \mathcal{G}(s, \, \rho(s))(x) \cdot \nabla_{x} \, \varphi(x) \, dx \, ds + \int_{t_{1}}^{t_{2}} \int_{\mathbb{R}^{n}} \left(\rho(s, x) \, \mathcal{U}(s, \, \rho(s))(x) + \mathcal{W}(s, \, \rho(s))(x) \right) \, \varphi(x) \, dx \, ds$$

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{l} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\mathsf{sup}_{t,\rho} \ \left(\left\| \operatorname{div}_{\scriptscriptstyle X} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^n)} \ + \ \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^n)} \right) \ < \ \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\|\partial_x\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\|\nabla_x\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4) For a.e. $t \in [0, T]$, the functions $\mathcal{G}(t, \cdot)$, $\mathcal{U}(t, \cdot)$, $\mathcal{W}(t, \cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous (w.r.t. weakly continuous and tight sequences).
- (5.) For each r>0, there exist $\widehat{w}_r\in L^2(\mathbb{R}^n)$ and a compact subset $K_r\subset \mathbb{R}^n$ s.t. $t\in [0,T], \ \|\rho\|_L^2\leq r, \ x\in \mathbb{R}^n\setminus K_r \implies \left|\mathcal{W}(t,\rho)(x)\right|\leq \widehat{w}_r(x).$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_X(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\int_{\mathbb{R}^n} \varphi \ (\rho(t_2) - \rho(t_1)) \ dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s, x) \ \mathcal{G}\big(s, \ \rho(s)\big)(x) \cdot \nabla_x \, \varphi(x) \ dx \ ds + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s, x) \, \mathcal{U}\big(s, \rho(s)\big)(x) + \mathcal{W}\big(s, \rho(s)\big)(x)\Big) \ \varphi(x) \ dx \ ds$$

Theorem (Existence of L^2 -valued solutions)

Suppose for

$$\begin{array}{l} \mathcal{G}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow \mathcal{W}^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2 \\ \mathcal{U}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow \mathcal{W}^{1,\infty}(\mathbb{R}^n) \cap L^2 \\ \mathcal{W}: \ [0,\,T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \end{array}$$

(1.) Global a priori bounds

$$\sup_{t,\rho} \left(\left\| \operatorname{div}_{\mathsf{x}} \mathcal{G}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{U}(t,\rho) \right\|_{L^{\infty}(\mathbb{R}^{n})} + \left\| \mathcal{W}(t,\rho) \right\|_{L^{2}(\mathbb{R}^{n})} \right) < \infty$$

- (2) Local a priori bounds: For each r>0, there is $C_r>0$ s.t. $\|\rho\|_{L^2(\mathbb{R}^n)}\leq r$ implies $\|\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n)}+\ \|\partial_x\,\mathcal{G}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n\times n)}+\ \|\nabla_x\,\mathcal{U}(t,\rho)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\leq C_r$
- $(3.) \ \forall \ \rho \in L^2(\mathbb{R}^n): \ \mathcal{G}(\cdot,\rho), \ \mathcal{U}(\cdot,\rho), \ \mathcal{W}(\cdot,\rho): [0,T] \longrightarrow \left(L^2, \ \|\cdot\|_{L^2}\right) \ \text{are measurable}.$
- (4) For a.e. $t \in [0, T]$, the functions $\mathcal{G}(t, \cdot)$, $\mathcal{U}(t, \cdot)$, $\mathcal{W}(t, \cdot) : L^2(\mathbb{R}^n) \longrightarrow (L^2, \|\cdot\|_{L^2})$ are continuous (w.r.t. weakly continuous and tight sequences).
- (5.) For each r>0, there exist $\widehat{w}_r\in L^2(\mathbb{R}^n)$ and a compact subset $K_r\subset \mathbb{R}^n$ s.t. $t\in [0,T], \ \|\rho\|_L^2\leq r, \ x\in \mathbb{R}^n\setminus K_r \implies \left|\mathcal{W}(t,\rho)(x)\right|\leq \widehat{w}_r(x).$

Then every function $\rho_0 \in L^2(\mathbb{R}^n)$ initializes a weak solution $\rho : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ of $\partial_t \rho + \operatorname{div}_X(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho),$

$$\begin{split} \int_{\mathbb{R}^n} \varphi \ \left(\rho(t_2) - \rho(t_1) \right) \ dx &= \ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \rho(s,x) \ \mathcal{G} \big(s, \ \rho(s) \big) (x) \cdot \nabla_x \, \varphi(x) \ dx \ ds \ + \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \Big(\rho(s,x) \, \mathcal{U} \big(s, \rho(s) \big) (x) + \mathcal{W} \big(s, \rho(s) \big) (x) \Big) \ \varphi(x) \ dx \ ds \end{split}$$

Motivatio

Main results

L²-valued solutions

M-valued solutions

Ordin. diff. eqns.

L²-valued

solutions

M-valued solutions

Mutationa inclusions

Summar

Theorem (Uniqueness of L^2 -valued solutions)

$$\left\{ \begin{array}{rcl} \partial_t \; \rho \;\; + \;\; \mathrm{div}_x \big(\mathcal{G}(t,\rho) \; \rho \big) \; = \; \mathcal{U}(t,\rho) \; \rho \; + \; \mathcal{W}(t,\rho), \\ \rho(0) \; = \; \rho_0 \; \in \; L^2(\mathbb{R}^n) \end{array} \right.$$

Motivatio

Main results

L²-valued solutions

M-valued solutions

Ordin. diff. eqns.

L²-valued

M-value

Mutationa inclusions

Summar

Theorem (Uniqueness of L^2 -valued solutions)

Then the weak solution $ho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\left\{ \begin{array}{rcl} \partial_t \; \rho \;\; + \;\; \mathrm{div}_x \big(\mathcal{G}(t,\rho) \; \rho \big) \; = \; \mathcal{U}(t,\rho) \; \rho \; + \; \mathcal{W}(t,\rho), \\ \rho(0) \; = \; \rho_0 \; \in \; L^2(\mathbb{R}^n) \end{array} \right.$$

Theorem (Uniqueness of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

$$\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

in addition:

Then the weak solution $ho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\left\{ \begin{array}{rcl} \partial_t \, \rho & + & \mathrm{div}_x \big(\mathcal{G}(t,\rho) \, \rho \big) \, = \, \mathcal{U}(t,\rho) \, \rho \, + \, \mathcal{W}(t,\rho), \\ \rho(0) \, = \, \rho_0 \, \in \, L^2(\mathbb{R}^n) \end{array} \right.$$

Theorem (Uniqueness of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1, \infty}(\mathbb{R}^n, \mathbb{R}^n) \cap L^2$$

$$\mathcal{U}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1, \infty}(\mathbb{R}^n) \cap L^2$$

$$\mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r>0, there exists $\Lambda_r>0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)}\leq r$ imply

$$\begin{split} & \parallel \mathcal{G}(t,\rho_1) - \left. \mathcal{G}(t,\rho_2) \right\|_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \leq \left. \Lambda_r \cdot \frac{d_{L^2}(\rho_1,\rho_2),}{d_{L^2}(\rho_1,\rho_2),} \right. \\ & \parallel \left. \mathcal{U}(t,\rho_1) - \left. \mathcal{U}(t,\rho_2) \right\|_{L^2(\mathbb{R}^n)} \right. \\ & \left. \left\| \left. \mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2) \right\|_{L^2(\mathbb{R}^n)} \right. \\ & \left. \left. \left. \left(\Lambda_r \cdot \frac{d_{L^2}(\rho_1,\rho_2),}{d_{L^2}(\rho_1,\rho_2),} \right) \right. \right. \\ \end{aligned}$$

Then the weak solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho), \\ \rho(0) = \rho_0 \in L^2(\mathbb{R}^n) \end{cases}$$

Theorem (Uniqueness of L^2 -valued solutions)

Suppose for

$$\mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

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For every radius r>0, there exists $\Lambda_r>0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)}\leq r$ imply

$$\begin{aligned} & \| \mathcal{G}(t,\rho_1) - \mathcal{G}(t,\rho_2) \|_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2), \\ & \| \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2) \|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2), \end{aligned}$$

$$\| \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2)\|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\left\| \mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2) \right\|_{L^2(\mathbb{R}^n)} \quad \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2).$$

Then the weak solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ of

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho), \\ \rho(0) = \rho_0 \in L^2(\mathbb{R}^n) \end{cases}$$

$$\begin{array}{l} d_{L^2}\left(\rho_1,\rho_2\right) \,:=\, \sup\Big\{\int_{\mathbb{R}^n}\varphi\;\rho_1\;\mathrm{d}x\;-\;\int_{\mathbb{R}^n}\varphi\;\rho_2\;\mathrm{d}x\;\Big|\;\varphi\in C^1_c(\mathbb{R}^n),\;\;\|\varphi\|_{L^2}\leq 1,\\ \\ \|\varphi\|_{L^\infty}\leq 1,\;\;\|\nabla\varphi\|_{L^\infty}\leq 1\Big\} \end{array}$$

Theorem (Continuous dependence on data)

Suppose for

$$\widetilde{\mathcal{G}}, \ \mathcal{G}: \ [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\widetilde{\mathcal{U}}, \ \mathcal{U}: \ [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$$

$$\widetilde{\mathcal{W}}, \mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r>0, there exists $\Lambda_r>0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)}\leq r$ imply

$$\| \mathcal{G}(t, \rho_1) - \mathcal{G}(t, \rho_2) \|_{L^2(\mathbb{R}^n, \mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1, \rho_2),$$

$$\| \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2) \|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\|\mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2)\|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2).$$

Theorem (Continuous dependence on data)

$$\widetilde{\mathcal{G}}, \ \mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

 $\widetilde{\mathcal{U}}, \ \mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$

$$\widetilde{\mathcal{W}}, \mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

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(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r > 0, there exists $\Lambda_r > 0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)} \le r$ imply

$$\| \mathcal{G}(t,\rho_1) - \mathcal{G}(t,\rho_2) \|_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\| \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2) \|_{L^2(\mathbb{R}^n)} \le \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\|\mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2)\|_{L^2(\mathbb{R}^n)} \le \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2).$$

Then the weak solutions $\rho, \widetilde{\rho} : [0, T] \longrightarrow L^2(\mathbb{R}^n)$ satisfy

Theorem (Continuous dependence on data)

$$\widetilde{\mathcal{G}}, \ \mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

 $\widetilde{\mathcal{U}}, \ \mathcal{U}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n) \cap L^2$

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in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r>0, there exists $\Lambda_r>0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)}\leq r$ imply

$$\begin{aligned} & \parallel \mathcal{G}(t,\rho_1) - \mathcal{G}(t,\rho_2) \rVert_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2), \\ & \parallel \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2) \rVert_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2), \end{aligned}$$

$$\|\mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2)\|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2).$$

Then the weak solutions $\rho, \widetilde{\rho}: [0, T] \longrightarrow L^2(\mathbb{R}^n)$ satisfy

$$d_{L^2}(\rho(t), \ \widetilde{\rho}(t)) \le (d_{L^2}(\rho(0), \ \widetilde{\rho}(0)) + \int_0^t \Delta(s) \ ds) \cdot e^{c t}$$

Theorem (Continuous dependence on data)

Suppose for
$$\widetilde{\mathcal{G}}, \ \mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

$$\widetilde{\mathcal{G}}, \ \mathcal{G}: [0,T] \times L^2(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2$$

$$\widetilde{\mathcal{U}},\ \mathcal{U}:\ [0,T]\times L^2(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n)\cap L^2$$

$$\widetilde{\mathcal{W}}, \mathcal{W}: [0, T] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r>0, there exists $\Lambda_r>0$ s.t. $\|\rho_1\|_{L^2(\mathbb{R}^n)}, \|\rho_2\|_{L^2(\mathbb{R}^n)}\leq r$ imply

$$\| \mathcal{G}(t,\rho_1) - \mathcal{G}(t,\rho_2) \|_{L^2(\mathbb{R}^n,\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\| \mathcal{U}(t,\rho_1) - \mathcal{U}(t,\rho_2) \|_{L^2(\mathbb{R}^n)} \le \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2),$$

$$\|\mathcal{W}(t,\rho_1) - \mathcal{W}(t,\rho_2)\|_{L^2(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{L^2}(\rho_1,\rho_2).$$

Then the weak solutions $ho,\widetilde{
ho}:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ satisfy

$$d_{L^2}(\rho(t), \widetilde{\rho}(t)) \leq (d_{L^2}(\rho(0), \widetilde{\rho}(0)) + \int_0^t \Delta(s) ds) \cdot e^{ct}$$

with a constant
$$c = c(\|\rho(0)\|_{L^2}, \|\widetilde{\rho}(0)\|_{L^2}, T, \Lambda_r)$$
 and

$$\begin{split} \Delta(s) \; := & \sup_{L^2(\mathbb{R}^n)} \left\| \mathcal{G}(s,\cdot) - \left. \widetilde{\mathcal{G}}(s,\cdot) \right\|_{L^2} + \sup_{L^2(\mathbb{R}^n)} \left\| \mathcal{U}(s,\cdot) - \widetilde{\mathcal{U}}(s,\cdot) \right\|_{L^2} + \\ & \sup_{2^2 \in \mathbb{R}^n} \left\| \left. \mathcal{W}(s,\cdot) - \widetilde{\mathcal{W}}(s,\cdot) \right\|_{L^2} \right. \end{split}$$

Thomas Lorenz

Motivatio

NA-To-Contraction

L²-valued solutions

M-valued solutions

Ordin. diff. eqns

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L²-valued solutions

M-value

Mutationa

Summar

Main Results

Theorem (Existence of measure-valued solutions)

$$\partial_t \mu + \operatorname{div}_x(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

L²-valued solutions

 \mathcal{M} -valued solutions

Ordin. diff. eqn

L²-valued

M-value

Mutationa inclusions

Summar

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Theorem (Existence of measure-valued solutions)

$$\partial_t \mu + \operatorname{div}_{\mathsf{x}}(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

$$\int_{\mathbb{R}^n} \varphi \ d\mu_{t_2} \ - \ \int_{\mathbb{R}^n} \varphi \ d\mu_{t_1} \ = \ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \ \mathcal{B}\big(s, \, \mu_s\big)(x) \cdot \nabla_x \, \varphi(x) \ d\mu_s(x) \ ds \ + \\ \int_{t_*} \int_{\mathbb{R}^n} \ \mathcal{C}\big(s, \, \mu_s\big)(x) \ \varphi(x) \ d\mu_s(x) \ ds.$$

Theorem (Existence of measure-valued solutions)

L²-valued solutions

M-valued solutions

Ordin. diff. eqns

L²-valued

M-valued

Mutationa inclusions

Summary

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu:[0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_x(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

$$\int_{\mathbb{R}^n} \varphi \ d\mu_{t_2} \ - \ \int_{\mathbb{R}^n} \varphi \ d\mu_{t_1} \ = \ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{B}\big(s, \ \mu_s\big)(x) \cdot \nabla_x \, \varphi(x) \ d\mu_s(x) \ ds \ + \\ \int_{t_*} \ \int_{\mathbb{R}^n} \ \mathcal{C}\big(s, \ \mu_s\big)(x) \ \varphi(x) \ d\mu_s(x) \ ds.$$

Main Results

Theorem (Existence of measure-valued solutions)

Suppose for

 $\mathcal{B}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathcal{W}^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$

 $\mathcal{C}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n)$

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_x(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

$$\int_{\mathbb{R}^n} \varphi \ d\mu_{t_2} \ - \ \int_{\mathbb{R}^n} \varphi \ d\mu_{t_1} \ = \ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{B}\big(s, \ \mu_s\big)(x) \cdot \nabla_x \, \varphi(x) \ d\mu_s(x) \ ds \ + \\ \int_{t_2}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{C}\big(s, \ \mu_s\big)(x) \ \varphi(x) \ d\mu_s(x) \ ds.$$

Main Results

Theorem (Existence of measure-valued solutions)

Suppose for

 $\mathcal{B}:\; [0,T]\times \mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$

 $\mathcal{C}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n)$

(1.) Global a priori bounds

$$\sup_{t,\mu} \left(\left\| \mathcal{B}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| \mathcal{C}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty$$

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu: [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_x (\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

$$\int_{\mathbb{R}^n} \varphi \ d\mu_{t_2} \ - \ \int_{\mathbb{R}^n} \varphi \ d\mu_{t_1} \ = \ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{B} \big(s, \ \mu_s \big) (x) \cdot \nabla_x \, \varphi(x) \ d\mu_s(x) \ ds \ + \\ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{C} \big(s, \ \mu_s \big) (x) \ \varphi(x) \ d\mu_s(x) \ ds.$$

solutions

in a metric space

L²-valued solutions

Mutationa inclusions

Summar

Main Results

Theorem (Existence of measure-valued solutions)

Suppose for

 $\mathcal{B}:\ [0,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$

 $\mathcal{C}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n)$

(1.) Global a priori bounds

$$\sup_{t,\mu} \left(\left\| \mathcal{B}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| \mathcal{C}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty$$

(2) Local a priori bounds: For each r > 0, there is $C_r > 0$ s.t. $|\mu|(\mathbb{R}^n) \le r$ implies $\|\partial_x \mathcal{B}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n \times n)} + \|\nabla_x \mathcal{C}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$

Then every
$$\mu_0 \in \mathcal{M}(\mathbb{R}^n)$$
 initializes a narrowly continuous distributional solution $\mu:[0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_x (\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

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Main Results

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(1.) Global a priori bounds

$$\sup\nolimits_{t,\mu} \; \left(\left\| \mathcal{B}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^{n})} \; + \; \left\| \mathcal{C}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^{n})} \right) \, < \, \infty$$

- (2) Local a priori bounds: For each r > 0, there is $C_r > 0$ s.t. $|\mu|(\mathbb{R}^n) \le r$ implies $\|\partial_x \mathcal{B}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n \times n)} + \|\nabla_x \mathcal{C}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$
- $(3) \ \forall \ \mu \in \mathcal{M}(\mathbb{R}^n): \ \mathcal{B}(\cdot,\mu), \ \mathcal{C}(\cdot,\mu): [0,T] \longrightarrow \left(W^{1,\infty}, \ \|\cdot\|_{L^\infty}\right) \ \text{are measurable}.$

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_x (\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

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Thomas Lorenz

Motivation

Main result

L²-valued solutions

M-valued solutions

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summar

Main Results

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(1.) Global a priori bounds

$$\sup_{t,\mu} \left(\left\| \mathcal{B}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| \mathcal{C}(t,\mu) \right\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty$$

- (2) Local a priori bounds: For each r > 0, there is $C_r > 0$ s.t. $|\mu|(\mathbb{R}^n) \le r$ implies $\|\partial_x \mathcal{B}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n \times n)} + \|\nabla_x \mathcal{C}(t,\mu)\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le C_r$
- (3) $\forall \ \mu \in \mathcal{M}(\mathbb{R}^n)$: $\mathcal{B}(\cdot,\mu), \ \mathcal{C}(\cdot,\mu): [0,T] \longrightarrow \left(W^{1,\infty}, \ \|\cdot\|_{L^\infty}\right)$ are measurable.
- (4) For a.e. $t \in [0, T]$, the functions $\mathcal{B}(t, \cdot)$, $\mathcal{C}(t, \cdot) : \mathcal{M}(\mathbb{R}^n) \longrightarrow (W^{1, \infty}, \|\cdot\|_{L^{\infty}})$ are narrowly continuous.

Then every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$ initializes a narrowly continuous distributional solution $\mu: [0,T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\partial_t \mu + \operatorname{div}_{\mathsf{x}} (\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu,$$

$$\int_{\mathbb{R}^n} \varphi \ d\mu_{t_2} \ - \ \int_{\mathbb{R}^n} \varphi \ d\mu_{t_1} \ = \ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{B}\big(s, \ \mu_s\big)(x) \cdot \nabla_x \, \varphi(x) \quad d\mu_s(x) \quad ds \ + \\ \int_{t_1}^{t_2} \ \int_{\mathbb{R}^n} \ \mathcal{C}\big(s, \ \mu_s\big)(x) \qquad \varphi(x) \quad d\mu_s(x) \quad ds.$$

Main results

L²-valued solutions

M-valued

solutions

Ordin. diff. eqns in a metric space

L²-valued

M-value

Mutational

Summary

Theorem (Uniqueness of measure-valued solutions)

$$\left\{ \begin{array}{rcl} \partial_t \; \mu & + \; \operatorname{div}_x \big(\mathcal{B}(t,\mu) \; \mu \big) \; = \; \mathcal{C}(t,\mu), \\ & \mu(0) \; = \; \mu_0 \; \in \; \mathcal{M}(\mathbb{R}^n) \end{array} \right.$$

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Main results

L²-valued

M-valued solutions

solutions

Ordin. diff. eqns in a metric space

L²-valued

M-value

solutions

Mutationa inclusions

Summar

Theorem (Uniqueness of measure-valued solutions)

Then the narrowly continuous distributional solution $\mu:[0,T]\longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\left\{ \begin{array}{rl} \partial_t \; \mu \; + \; \operatorname{div}_x \big(\mathcal{B}(t,\mu) \; \mu \big) \; = \; \mathcal{C}(t,\mu), \\ \mu(0) \; = \; \mu_0 \; \in \; \mathcal{M}(\mathbb{R}^n) \end{array} \right.$$

Theorem (Uniqueness of measure-valued solutions)

Suppose for

$$\mathcal{B}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

$$\mathcal{C}:\ [0,\,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\,\infty}(\mathbb{R}^n)$$

in addition

Then the narrowly continuous distributional solution $\mu:[0,T]\longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\begin{cases} \partial_t \mu + \operatorname{div}_x (\mathcal{B}(t, \mu) \mu) = \mathcal{C}(t, \mu), \\ \mu(0) = \mu_0 \in \mathcal{M}(\mathbb{R}^n) \end{cases}$$

Summai

Theorem (Uniqueness of measure-valued solutions)

Suppose for

$$\mathcal{B}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$
$$\mathcal{C}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n)$$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r > 0, there exists $\Lambda_r > 0$ s.t. $|\mu_1|(\mathbb{R}^n), |\mu_2|(\mathbb{R}^n) \le r$ imply

$$\begin{aligned} & \left\| \mathcal{B}(t, \mu_1) - \mathcal{B}(t, \mu_2) \right\|_{L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)} \leq \Lambda_r \cdot \frac{d_{\mathcal{M}}(\mu_1, \mu_2)}{d_{\mathcal{M}}(\mu_1, \mu_2)}, \\ & \left\| \mathcal{C}(t, \mu_1) - \mathcal{C}(t, \mu_2) \right\|_{L^{\infty}(\mathbb{R}^n)} \leq \Lambda_r \cdot \frac{d_{\mathcal{M}}(\mu_1, \mu_2)}{d_{\mathcal{M}}(\mu_1, \mu_2)}. \end{aligned}$$

Then the narrowly continuous distributional solution $\mu:[0,T]\longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\left\{ \begin{array}{rcl} \partial_t \; \mu \; + \; \operatorname{div}_x \big(\mathcal{B}(t,\mu) \; \mu \big) \; = \; \mathcal{C}(t,\mu), \\ \mu(0) \; = \; \mu_0 \; \in \; \mathcal{M}(\mathbb{R}^n) \end{array} \right.$$

Theorem (Uniqueness of measure-valued solutions)

Suppose for

$$\mathcal{B}: [0,T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

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Then the narrowly continuous distributional solution $\mu:[0,T]\longrightarrow \mathcal{M}(\mathbb{R}^n)$ of

$$\left\{ \begin{array}{rl} \partial_t \; \mu \; + \; \operatorname{div}_x \big(\mathcal{B}(t,\mu) \; \mu \big) \; = \; \mathcal{C}(t,\mu), \\ \mu(0) \; = \; \mu_0 \; \in \; \mathcal{M}(\mathbb{R}^n) \end{array} \right.$$

$$d_{\mathcal{M}}\left(\mu_{1},\mu_{2}\right) \,:=\, \sup \Big\{ \int_{\mathbb{R}^{n}} \varphi \; d\mu_{1} - \int_{\mathbb{R}^{n}} \varphi \; d\mu_{2} \; \Big| \; \varphi \in \textit{\textbf{C}}^{1}(\mathbb{R}^{n}), \, \|\varphi\|_{\textit{\textbf{L}}^{\infty}} \, \leq 1, \, \|\nabla \varphi\|_{\textit{\textbf{L}}^{\infty}} \, \leq 1 \Big\}$$

Theorem (Continuous dependence on data)

Suppose for

$$\frac{\widetilde{\mathcal{B}}}{\widetilde{\mathcal{C}}}, \ \mathcal{B}: \ [0, T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1, \infty}(\mathbb{R}^n, \mathbb{R}^n)$$
$$\frac{\widetilde{\mathcal{C}}}{\widetilde{\mathcal{C}}}, \ \mathcal{C}: \ [0, T] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow W^{1, \infty}(\mathbb{R}^n)$$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r > 0, there exists $\Lambda_r > 0$ s.t. $|\mu_1|(\mathbb{R}^n), |\mu_2|(\mathbb{R}^n) \le r$ imply

$$\begin{split} & \left\| \mathcal{B}(t,\mu_1) \, - \mathcal{B}(t,\mu_2) \right\|_{L^{\infty}(\mathbb{R}^{n},\mathbb{R}^{n})} \, \leq \, \Lambda_r \cdot d_{\mathcal{M}} \big(\mu_1,\mu_2 \big), \\ & \left\| \mathcal{C}(t,\mu_1) \, - \mathcal{C}(t,\mu_2) \right\|_{L^{\infty}(\mathbb{R}^{n})} \, \, \leq \, \Lambda_r \cdot d_{\mathcal{M}} \big(\mu_1,\mu_2 \big). \end{split}$$

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Main result

L²-valued solutions

M-valued solutions

Ordin. diff. eqns. in a metric space

L²-valued solutions

A.d. volume

Solutions

inclusions

Summar

Theorem (Continuous dependence on data)

Suppose for

$$\widetilde{\mathcal{B}},\ \mathcal{B}:\ [0,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

 $\widetilde{\mathcal{C}},\ \mathcal{C}:\ [0,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n)$

in addition:

(6.) (Locally uniform LIPSCHITZ conditions w.r.t. states)

For every radius r > 0, there exists $\Lambda_r > 0$ s.t. $|\mu_1|(\mathbb{R}^n), |\mu_2|(\mathbb{R}^n) \le r$ imply

$$\begin{aligned} & \left\| \mathcal{B}(t,\mu_1) - \mathcal{B}(t,\mu_2) \right\|_{L^{\infty}(\mathbb{R}^n,\mathbb{R}^n)} \leq \Lambda_r \cdot d_{\mathcal{M}}(\mu_1,\mu_2), \\ & \left\| \mathcal{C}(t,\mu_1) - \mathcal{C}(t,\mu_2) \right\|_{L^{\infty}(\mathbb{R}^n)} & \leq \Lambda_r \cdot d_{\mathcal{M}}(\mu_1,\mu_2). \end{aligned}$$

Then the weak solutions $\mu,\widetilde{\mu}:[0,T]\longrightarrow \mathcal{M}(\mathbb{R}^n)$ satisfy

Theorem (Continuous dependence on data)

Suppose for

$$\widetilde{\mathcal{B}},\ \mathcal{B}:\ [0,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

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$$\begin{aligned} & \left\| \mathcal{B}(t, \mu_1) - \mathcal{B}(t, \mu_2) \right\|_{L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)} \leq \Lambda_r \cdot d_{\mathcal{M}}(\mu_1, \mu_2), \\ & \left\| \mathcal{C}(t, \mu_1) - \mathcal{C}(t, \mu_2) \right\|_{L^{\infty}(\mathbb{R}^n)} \leq \Lambda_r \cdot d_{\mathcal{M}}(\mu_1, \mu_2). \end{aligned}$$

Then the weak solutions $\mu, \widetilde{\mu}: [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ satisfy

$$d_{\mathcal{M}}(\mu(t), \widetilde{\mu}(t)) \leq \left(d_{\mathcal{M}}(\mu(0), \widetilde{\mu}(0)) + \int_{0}^{t} \Delta(s) ds\right) \cdot e^{c t}$$

Theorem (Continuous dependence on data)

Suppose for

$$\widetilde{\mathcal{B}},\ \mathcal{B}:\ [0,T]\times\mathcal{M}(\mathbb{R}^n)\longrightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$
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Then the weak solutions $\mu, \widetilde{\mu}: [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ satisfy

$$d_{\mathcal{M}}(\mu(t), \, \widetilde{\mu}(t)) \leq \left(d_{\mathcal{M}}(\mu(0), \, \widetilde{\mu}(0)) + \int_0^t \Delta(s) \, ds\right) \cdot e^{c \, t}$$

with a constant $c=cig(|\mu_0|(\mathbb{R}^n),\ |\widetilde{\mu}|(\mathbb{R}^n),\ T,\ \Lambda_rig)$ and

$$\Delta(s) \; := \; \sup_{\mathcal{M}(\mathbb{R}^n)} \, \left\| \mathcal{B}(s,\cdot) \, - \, \widetilde{\mathcal{B}}(s,\cdot) \right\|_{\boldsymbol{L}^{\infty}} \; + \; \sup_{\mathcal{M}(\mathbb{R}^n)} \, \left\| \mathcal{C}(s,\cdot) \, - \, \widetilde{\mathcal{C}}(s,\cdot) \right\|_{\boldsymbol{L}^{\infty}} \, .$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued

M-value

Mutationa

Ordinary Differential Equations

A Very Familar Situation

Solving an ordinary differential equation x' = f(t, x)

* Explicit formula (e.g., variation of constants)

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value

M-value

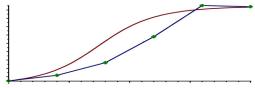
Mutationa

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- * Explicit formula (e.g., variation of constants)
- * Approximation scheme: Euler method



Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value solutions

M-value

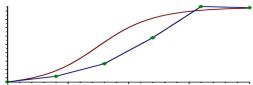
Mutationa inclusions

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- * Explicit formula (e.g., variation of constants)
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- * Standard theorems about well-posedness:
 - * Peano: Existence due to compactness
 - * PICARD-LINDELÖF a.k.a. CAUCHY-LIPSCHITZ: Existence and uniqueness due to completeness

Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value

M-valued

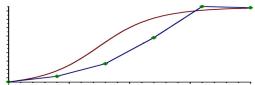
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Established extensions: Evolution equations in Banach spaces

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value

M-valued

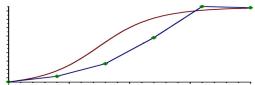
Mutationa inclusions

Ordinary Differential Equations

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 - $* \ \ \, \mathsf{Approximation} \ \, \mathsf{scheme:} \quad \mathrm{EULER} \,\, \boldsymbol{\mathsf{method}}$



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Established extensions: Evolution equations in Banach spaces

Gist: Extend the notion of EULER method beyond linear spaces.

The Step to Metric Spaces

Aim:

Extend ordinary differential equation x' = f(t, x)

to a metric space (E, d)

Question:

Counterpart x'(t) of $x:[0,T] \longrightarrow E$?

In \mathbb{R}^n :

 $x'(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$

"Time derivative"

Tool box for IVPs

in a metric space

Aubin's proposal

Generalization

Key ingredients

L²-valued

M-value

Mutational

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

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$$x'(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$
 or, equivalently,
 $x'(t) = v :\iff |x(t+h) - (x(t) + h v)| \le o(h)$



In \mathbb{R}^n :

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

derivative"

In \mathbb{R}^n :

x(t)

Tool box for IVPs
Aubin's proposal

Generalization

Generalization

Key ingredients

L²-valued solutions

M-value

Mutationa inclusions

Mutational Equations

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 $v \in \mathbb{R}^n \text{ induces } [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ (h,y) \longmapsto y + h v$

The gist "Time

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

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 $v \in \mathbb{R}^n$ induces $[0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(h,y) \longmapsto y + h v$

In (E,d):

x(t)

Specify some $\vartheta: [0,1] \times E \longrightarrow E$, $(h,z) \longmapsto \vartheta(h,z)$.



The Step to Metric Spaces

Aim:

Extend ordinary differential equation x' = f(t, x)to a metric space (E, d)

Question: Counterpart x'(t) of $x:[0,T] \longrightarrow E$?

The gist "Time derivative"

In \mathbb{R}^n :

Tool box for IVPs Aubin's proposal Generalization

 $x'(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$ or, equivalently,

 $x'(t) = v : \iff |x(t+h) - (x(t) + h v)| \le o(h)$

 $v \in \mathbb{R}^n$ induces $[0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(h,v) \longmapsto v + hv$

Key ingredients

In (E,d):

Specify some $\vartheta: [0,1] \times E \longrightarrow E$, $(h,z) \longmapsto \vartheta(h,z)$.

 $\vartheta = \mathring{x}(t) : \iff d(x(t+h), \vartheta(h, x(t))) < o(h)$

x(t) $\vartheta(h, x(t))$

The gist "Time

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

The Step to Metric Spaces

Aim:

Extend ordinary differential equation x' = f(t, x)

to a metric space (E, d)

Question:

Counterpart x'(t) of $x:[0,T] \longrightarrow E$?

In \mathbb{R}^n :

x(t)

$$x'(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$
 or, equivalently,

 $\vartheta(h, x(t))$

$$x'(t) = v :\iff |x(t+h) - (x(t) + h v)| \leq o(h)$$

 $v \in \mathbb{R}^n$ induces $[0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(h,v) \longmapsto v + hv$

In (E,d):

x(t)

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Given: $\mathcal{F}: [0,T] \times E \longrightarrow \Theta, (t,z) \longmapsto \vartheta$

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Tool box for IVPs

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Wanted: cont. x: $[0, T] \longrightarrow E$: $\mathring{x}(t) = \mathcal{F}(t, x(t))$ for a.e. t

The gist

derivative" Tool box for IVPs

Aubin's proposal

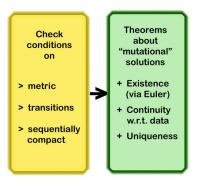
Generalization

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x(t)

 $\vartheta(h, x(t))$

Mutational Equations



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Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

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Key ingredients

x(t+h)

 $\vartheta(h, x(t))$

x(t)

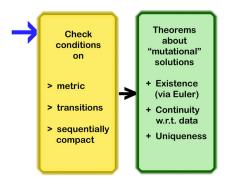
L²-valued solutions

M-valued solutions

Mutationa inclusions

Summary

Mutational Equations



In
$$(E, d)$$
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On a Class of Nonlocal Traffic Flow Models

Thomas Lorenz

in a metric space

The gist

"Time derivative"

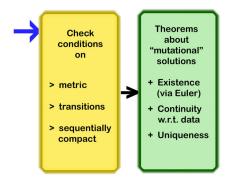
Tool box for IVPs

Aubin's proposal

Generalization

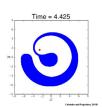
Key ingredients

Mutational Equations









[TL, Comput. Visual. Sci. 4, 2001] [TL, SIAM J. Control Optim. 48, 2010]

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Motivatio

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The gist

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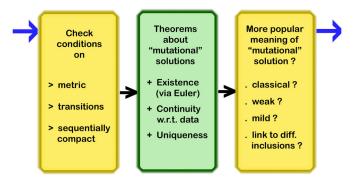
Generalization

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L²-valued

M-valued

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[TL, Comput. Visual. Sci. 4, 2001] [TL, SIAM J. Control Optim. 48, 2010] [COLOMBO, TL & POGODAEV, DCDS-A 35, 2015]

Motivatio

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The gist

"Time

derivative"

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Aubin's proposal

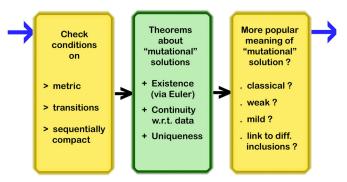
Generalization

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M-valued

Mutationa inclusions



$$\partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho)$$

 $\partial_t \mu + \operatorname{div}_x(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu$

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Main result

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Aubin's proposal

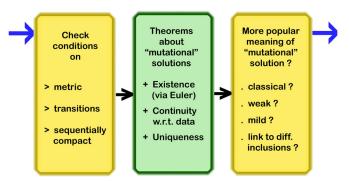
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M-valued

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

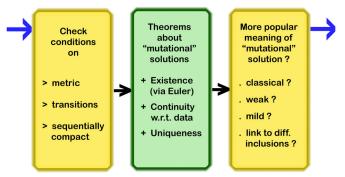
Generalization

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in a metric space

The gist

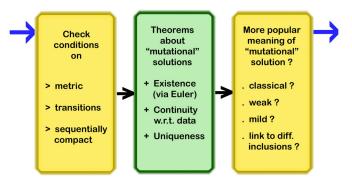
"Time

derivative" Tool box for IVPs

Aubin's proposal

Generalization

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The gist

"Time

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

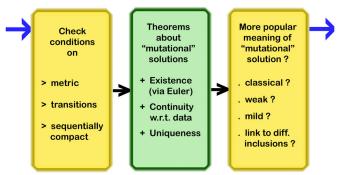
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Key ingredients

L²-value

M-valued solutions

Mutation



$$\mathring{x}(t) = f(t,x) \qquad \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho)$$

$$\mathring{y}(t) = g(t, y)$$
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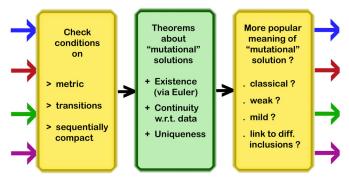
Key ingredients

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The gist

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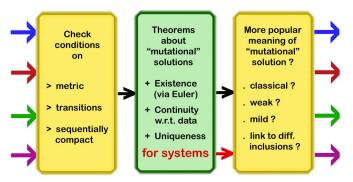
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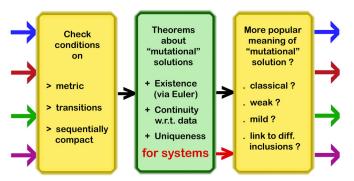
GCITCIAIIEGE

Key ingredients

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$$\begin{cases} \mathring{x}(t) = f(t, x, y) & \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho) \\ \mathring{y}(t) = g(t, x, y) & \partial_t \mu + \operatorname{div}_x(\mathcal{B}(t, \mu) \mu) = \mathcal{C}(t, \mu) \mu \end{cases}$$

Motivatio

Main result:

Ordin. diff. eqns. in a metric space

The gist

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Tool box for IVPs

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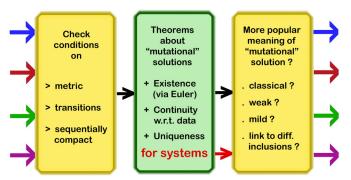
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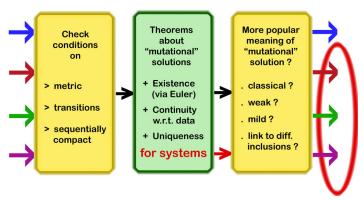
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in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

Transitions Instead of Affine Maps

Definition (Aubin 1993/99)

$$\vartheta: [0,1] \times E \longrightarrow E, \ (h,x) \longmapsto \vartheta(h,x)$$

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

"Time derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-valued solutions

Mutationa inclusions

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Aubin's proposal

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Ordin. diff. eqns. in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

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The gist

derivative"

Tool hox for IVPs

Aubin's proposal

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$$D(\vartheta,\tau) := \sup_{\mathsf{x}\in E} \left(\overline{\lim_{h\downarrow 0}} \quad \frac{1}{h} \cdot d(\vartheta(h,\mathsf{x}),\,\tau(h,\mathsf{x})) \right)$$

The gist

derivative"

Tool box for IVPs

Aubin's proposal Generalization

Key ingredients

Motivatio

Main result

Ordin. diff. eqns in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Key ingredients

/2-valued

solutions

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$$d(\vartheta(h,x), \tau(h,y)) \leq (d(x,y) + h \cdot D(\vartheta,\tau)) \cdot e^{\alpha(\vartheta)h}$$

The gist

derivative"

Tool hox for IVPs

Aubin's proposal

Generalization

Key ingredients

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Solutions to Initial Value Problems

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Theorem: Cauchy-Lipschitz (Aubin)

Let $f:(E,d)\longrightarrow (\Theta(E),D)$ be LIPSCHITZ with sup $\alpha(f(\cdot))<\infty$.

$$\dot{x}(\cdot) = f(x(\cdot))$$
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Theorem: Cauchy-Lipschitz (Aubin)

Let $f:(E,d)\longrightarrow (\Theta(E),D)$ be LIPSCHITZ with sup $\alpha(f(\cdot))<\infty$. Suppose all bounded closed balls in (E, d) to be complete.

Then in every $x_0 \in E$, there starts a unique continuous solution $x:[0,\infty[\longrightarrow E \text{ of } \mathring{x}(\cdot)=f(x(\cdot))]$ a.e.

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The gist

derivative"

Tool hox for IVPs Aubin's proposal

Generalization

Key ingredients

Motivatio

Main result

Ordin. diff. eqns in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

solutions

M-valued solutions

Mutationa

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General results about mutational equations [TL, Springer LNM 1996 (2010)]

- * Existence due to compactness and continuity (PEANO)
- * Existence for equations with finite delay
- * Existence under state constraints $x(t) \in \mathcal{V}$ (NAGUMO)
- * Continuous dependence of solutions on given data

Motivation

Main result

Ordin. diff. eqns in a metric space

The gist

.

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

solutions

M-valued solutions

Mutationa

Summary

Mutational Equations

Solutions to Initial Value Problems

Definition (Aubin 1993/99)

$$\vartheta: [0,1] \times E \longrightarrow E, (h,x) \longmapsto \vartheta(h,x)$$
 is called a transition if

$$(1.) \quad \vartheta(0,x) = x \qquad \forall x \in E,$$

$$(2.) \quad \vartheta(t, x) = \vartheta(t - s, \vartheta(s, x)) \qquad \forall x \in E, s < t$$

(3.)
$$\exists \alpha(\vartheta): d(\vartheta(h,x), \vartheta(h,y)) \leq d(x,y) \cdot e^{\alpha(\vartheta) \cdot h} \forall x,y, h \in [0,1],$$

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General results about mutational equations [TL, Springer LNM 1996 (2010)]

- * Existence due to compactness and continuity (PEANO)
- * Existence for equations with finite delay
- * Existence under state constraints $x(t) \in \mathcal{V}$ (NAGUMO)
- * Continuous dependence of solutions on given data
- Existence for inclusions (i.e., multivalued r.h.s.) (ANTOSIEWICZ & CELLINA)

On a Class of Nonlocal Traffic Flow Models

Thomas Lorenz

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

Weakening the Conditions on Transitions

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Thomas Lorenz

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

Weakening the Conditions on Transitions

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$$(3) \exists \alpha(\vartheta) : d(\vartheta(h \times) \vartheta(h \vee)) < d(\times \vee) \cdot e^{\alpha}(\vartheta) \cdot h \vee \dots \wedge e^{\alpha}(\vartheta) \cdot h$$

$$(3.) \quad \exists \ \alpha(\vartheta): \quad d(\vartheta(h,x), \ \vartheta(h,y)) \leq \ d(x,y) \cdot e^{\alpha \ (\vartheta) \cdot h} \ \ \forall \ x,y, \ \ h \in [0,1],$$

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Thomas Lorenz

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

"Time

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-valued solutions

Mutationa

Summary

Mutational Equations

Weakening the Conditions on Transitions

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$$\exists \alpha(\vartheta)$$
: $d(\vartheta(h,x), \vartheta(h,y)) \leq d(x,y) \cdot e^{\alpha(\vartheta) \cdot h} \forall x,y, h \in [0,1],$

$$(3.) \quad \exists \alpha(v) : \quad u(v(n,x), v(n,y)) \leq u(x,y) \cdot e^{-(x-y)} \quad \forall x,y, \ n \in [v,1]$$

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- Existence for inclusions (i.e., multivalued r.h.s.) (ANTOSIEWICZ & CELLINA)

Thomas Lorenz

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-valued solutions

Mutations inclusions

Summary

Mutational Equations

Weakening the Conditions on Transitions

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$$(3.) \exists \alpha(\theta): \quad d(\theta(n,x), \theta(n,y)) \leq d(x,y) + e^{-tx} \quad \forall x,y, \ n \in [0,1],$$

$$(4.) \exists \beta(\theta): \quad d(\theta(s,x), \theta(t,x)) \leq \beta(\theta) \cdot |t-s| \quad \forall x \in E, s, t \in [0,1]$$

$$(5.) \quad \exists \ \gamma(\vartheta): \qquad \qquad \lfloor \vartheta(h,x) \rfloor \ \leq \ \left(\lfloor x \rfloor + \gamma \ h \right) e^{\gamma \cdot h} \quad \forall \ x \in E, \ h \in [0,1]$$

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- * Continuous dependence of solutions on given data
- Existence for inclusions (i.e., multivalued r.h.s.) (ANTOSIEWICZ & CELLINA)

Thomas Lorenz

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

.....

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued

M-valued solutions

Mutationa

Summary

Mutational Equations

Weakening the Conditions on Transitions

Definition (TL 2010)

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 - Existence for inclusions (i.e., multivalued r.h.s.) (Antosiewicz & Cellina)

Thomas Lorenz

Motivatio

Main result

Ordin. diff. eqns in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-valued solutions

Mutations

Mutational Equations

Weakening the Conditions on Transitions

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 - Existence for inclusions (i.e., multivalued r.h.s.) (ANTOSIEWICZ & CELLINA)

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

1110 811

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-value solutions

Mutationa

Mutational Equations

Key Ingredients for Each Example

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- * Basic set F
- * Distance functions (e.g., metrics) $d, e: E \times E \longrightarrow [0, \infty[$
- $* [\cdot]: E \longrightarrow [0, \infty[$ (just lower semicontinuous w.r.t. d)

Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-valued solutions

M-valued

Mutation

Mutational Equations

Key Ingredients for Each Example

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- * $\lfloor \cdot \rfloor : E \longrightarrow [0, \infty[$ (just lower semicontinuous w.r.t. d)
- * Set of transitions $\Theta(E)$

The gist

derivative" Tool hox for IVPs

Aubin's proposal

Generalization

Key ingredients

Definition (TL 2010)

 $\vartheta: [0,1] \times E \longrightarrow E, (h,x) \longmapsto \vartheta(h,x)$ is called a <u>transition</u> if

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

The gist

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value solutions

M-valued solutions

Mutationa

Mutational Equations

Key Ingredients for Each Example

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- * Set of transitions $\Theta(E)$ and its distance $D(\vartheta, \tau)$ $\rightarrow d(\vartheta(h, x), \tau(h, y)) \leq (d(x, y) + h \cdot D(\vartheta, \tau)) \cdot e^{\alpha_r(\vartheta) h}$



The gist

derivative"

Tool box for IVPs Aubin's proposal

Generalization

Key ingredients

Mutational Equations

Key Ingredients for Each Example

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- * Set of transitions $\Theta(E)$ and its distance $D(\vartheta, \tau)$
 - * Is (E, d) complete?



Motivatio

Main result

Ordin. diff. eqns in a metric space

The gist

. .

derivative"

Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

L²-value solutions

M-valued solutions

Mutationa

nclusions

Mutational Equations

Key Ingredients for Each Example

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The gist

derivative" Tool box for IVPs

Aubin's proposal

Generalization

Key ingredients

Mutational Equations

Key Ingredients for Each Example

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The gist

derivative"

Tool box for IVPs Aubin's proposal

Generalization

Key ingredients

Definition (TL 2010)

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 L^2 -valued

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology

on tight subsets

First conclusions

M-value

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}} (\mathcal{G}(t,\rho) \, \rho) = \mathcal{U}(t,\rho) \, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-value solutions

Autonomous linear problem

The metric on ${}_{1}^{2}(\mathbb{R}^{n})$

Verifying the transition

Weak topology

on tight subsets

First conclusions

M-value

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

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$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology

on tight subsets

M-valued

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

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$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = \mathbf{u}(\mathsf{x}) & \rho + \mathbf{w}(\mathsf{x}) \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-value solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology

on tight subsets

M-value

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_X (\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{X}}(\mathbf{g} \ \rho) = \mathbf{0} \\ \rho(\mathbf{0}) = \rho_0 \end{cases}$$

Motivatio

Main results

Ordin. diff. eqns in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\left\{ \begin{array}{rcl} \partial_t \, \rho \, + \, \mathrm{div}_x \big(\mathbf{g} \; \; \rho \big) \; = \; 0 \\ \rho(0) \; = \; \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \ \rho_0(x) \ dx$$

Motivatio

Main results

Ordin. diff. eqns in a metric space

L²-value solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology

on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?)
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$$\begin{cases} \partial_t \, \rho \, + \, \mathrm{div}_{\mathsf{x}}(\mathbf{g} \; \rho) \, = & \mathbf{w} \\ \rho(0) \, = \, \rho_0 \end{cases}$$

$$\iff \forall \; \varphi \in C^1_c(\mathbb{R}^n) : \; \int_{\mathbb{R}^n} \varphi \; \rho(t) \; dx \, = \; \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \; \rho_0(x) \; dx$$

Motivatio

Main results

Ordin. diff. eqns in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?)
$$\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$
,

 $w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x (\mathcal{G}(t, \rho) \rho) = \mathcal{U}(t, \rho) \rho + \mathcal{W}(t, \rho) & \text{in } [0, T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathbf{g} \ \rho) = w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \ \rho_0(x) \ dx$$

Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $|\rho| := ||\rho||_{L^2(\mathbb{R}^n)}$

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 $\mathbf{g}\in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n),$

 $w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C_c^1(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + \underset{\bullet}{w} \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

i nomas Lorei

Motivatio

Main results

Ordin. diff. eqns

solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology

on tight subsets
First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $|\rho|:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?)
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,

 $w\in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \, \rho) = \mathcal{U}(t,\rho) \, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = \frac{\mathsf{u} \ \rho}{\rho} + \mathsf{w} \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

I homas Loren

Motivatio

Main results

Ordin. diff. eqns

solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology

on tight subsets
First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := L^2(\mathbb{R}^n),$ "Abs. value" $|\rho| := ||\rho||_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n), \ u \in W^{1,\infty}(\mathbb{R}^n), \ w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \, \rho) = \mathcal{U}(t,\rho) \, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\left\{ \begin{array}{rcl} \partial_t \, \rho \, + \, \mathrm{div}_{\times} \big(\mathbf{g} \; \; \rho \big) \; = \; u \; \rho \, + \; w \\ & \rho(0) \; = \; \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C_c^1(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

I homas Loren

Motivatio

Main results

Ordin. diff. eqns

solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=L^2(\mathbb{R}^n),$$
 "Abs. value" $|\rho|:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n), \ u \in W^{1,\infty}(\mathbb{R}^n), \ w \in L^2(\mathbb{R}^n)$

The "full nonlinear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho) \, \rho) = \mathcal{U}(t,\rho) \, \rho + \mathcal{W}(t,\rho) & \text{in } [0,T] \\ \rho(0) = \rho_0 \end{cases}$$

$$\left\{ \begin{array}{rcl} \partial_t \, \rho \, + \, \mathrm{div}_{\mathsf{X}} \big(\mathbf{g} \; \; \rho \big) \; = \; u \; \rho \, + \, w \\ & \rho(0) \; = \; \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C_c^1(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(\mathbf{0}) \ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

I nomas Loren

Motivatio

Main results

Ordin. diff. eqns

solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

 \mathcal{M} -valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E:=L^2(\mathbb{R}^n),$ "Abs. value" $|\rho|:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = u \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\Longleftrightarrow \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology on tight subsets

M-valued

Mutationa

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E:=L^2(\mathbb{R}^n),$ "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w}:[0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathbf{g} \ \rho) = u \ \rho + w$.

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\times}(\mathbf{g} \ \rho) = u \ \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns in a metric space

solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E:=L^2(\mathbb{R}^n),$ "Abs. value" $\lfloor \rho \rfloor:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w}:[0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathbf{g} \ \rho) = u \ \rho + w$.

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = u \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Summary

I homas Loren

Ordin. diff. eqns.

L²-value solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

 $\mathcal{M} ext{-valued}$

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E:=L^2(\mathbb{R}^n),$ "Abs. value" $|\rho|:=\|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $d_{L^2}(\rho_1, \rho_2)$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta_{\mathbf{g},u,w}^{\rho} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathbf{g} \, \rho) = u \, \rho + w$.

The auxiliary "autonomous linear" problem

$$\left\{ \begin{array}{l} \partial_t \, \rho \, + \, \mathrm{div}_x \big(\mathbf{g} \, \, \rho \big) \, = \, u \, \rho \, + \, w \\ \rho(0) \, = \, \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Summary

I homas Loren

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-value solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

M-valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n),$

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $d_{L^2}(\rho_1, \rho_2)$ $\int_{\mathbb{R}^n} \varphi \ \rho_1 \ dx \quad \int_{\mathbb{R}^n} \varphi \ \rho_2 \ dx$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta_{\mathbf{g},u,w}^{\rho} : [0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathbf{g} \, \rho) = u \, \rho + w$.

$$\left\{ \begin{array}{ll} \partial_t \, \rho \, + \, \mathrm{div}_X \big(\mathbf{g} \, \, \rho \big) \, = \, u \, \rho \, + \, w \\ \\ \rho(0) \, = \, \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C_c^1(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology on tight subsets

First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $|\rho| := \|\rho\|_{L^2(\mathbb{R}^n)}$

Metric $d_{L^2}(\rho_1, \rho_2)$

$$\int_{\mathbb{R}^n} \varphi \ \rho_1 \ dx - \int_{\mathbb{R}^n} \varphi \ \rho_2 \ dx \quad \varphi \in C^1_c(\mathbb{R}^n), \ \|\varphi\|_{L^2} \le 1$$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, $\vartheta^{\rho}_{\sigma |_{l}|_{W}}: [0,1] \times L^{2}(\mathbb{R}^{n}) \longrightarrow L^{2}(\mathbb{R}^{n}), (t,\rho_{0}) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \, \rho) = u \, \rho + w$.

$$\left\{ \begin{array}{ll} \partial_t \, \rho \, + \, \mathrm{div}_X \big(\mathbf{g} \, \, \rho \big) \, = \, u \, \rho \, + \, w \\ \\ \rho(0) \, = \, \rho_0 \end{array} \right.$$

$$\Longleftrightarrow \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n),$

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

 $\text{Metric } d_{L^2}(\rho_1,\rho_2) \qquad \sup \Big\{ \int_{\mathbb{R}^n} \varphi \; \rho_1 \; dx - \int_{\mathbb{R}^n} \varphi \; \rho_2 \; dx \; \Big| \; \varphi \in C^1_c(\mathbb{R}^n), \; \|\varphi\|_{L^2} \leq 1 \Big\}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w}:[0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g}\,\rho) = u\,\rho + w$.

The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \ \rho) = u \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Summary

Thomas Lorenz

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology on tight subsets

First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $|\rho| := \|\rho\|_{L^2(\mathbb{R}^n)}$

 $\begin{array}{ll} \mathsf{Metric} \ d_{L^2}(\rho_1,\rho_2) \ := \ \sup \Big\{ \int_{\mathbb{R}^n} \varphi \ \rho_1 \ dx - \int_{\mathbb{R}^n} \varphi \ \rho_2 \ dx \ \Big| \ \varphi \in C^1_c(\mathbb{R}^n), \ \|\varphi\|_{L^2} \le 1 \\ \|\varphi\|_{L^\infty} \le 1, \ \|\nabla \varphi\|_{L^\infty} \le 1 \Big\} \end{array}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, $\vartheta^{\rho}_{\sigma,\mu,\nu}:[0,1]\times L^{2}(\mathbb{R}^{n})\longrightarrow L^{2}(\mathbb{R}^{n}), (t,\rho_{0})\longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathbf{g} \, \rho) = u \, \rho + w$.

$$\left\{ \begin{array}{ll} \partial_t \, \rho \, + \, \mathrm{div}_X \big(\mathbf{g} \, \, \rho \big) \, = \, u \, \rho \, + \, w \\ \\ \rho(0) \, = \, \rho_0 \end{array} \right.$$

$$\iff \forall \varphi \in C^1_c(\mathbb{R}^n): \int_{\mathbb{R}^n} \varphi \, \rho(t) \, dx = \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \, \rho_0 + w \, \int_0^t \psi_{t,\varphi}(s,x) \, ds \right) dx$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

Basic set $E := L^2(\mathbb{R}^n),$

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

 $\begin{array}{ll} \mathsf{Metric} \,\, d_{L^2}(\rho_1,\rho_2) \,\, := \,\, \sup \Big\{ \int_{\mathbb{R}^n} \varphi \,\, \rho_1 \,\, d\mathsf{x} - \int_{\mathbb{R}^n} \varphi \,\, \rho_2 \,\, d\mathsf{x} \, \Big| \, \varphi \in \mathit{C}^1_c(\mathbb{R}^n), \,\, \|\varphi\|_{\mathit{L}^2} \leq 1 \\ & \|\varphi\|_{\mathit{L}^\infty} \leq 1, \,\, \|\nabla \varphi\|_{\mathit{L}^\infty} \leq 1 \Big\} \end{array}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, let $\vartheta^{\rho}_{\mathbf{g},u,w}:[0,1] \times L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(t,\rho_0) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_{\mathbf{x}}(\mathbf{g} \, \rho) = u \, \rho + w$.

Lemma: For every $\psi \in C^1_c(\mathbb{R}^n)$ and $\rho_1, \rho_2 \in L^2(\mathbb{R}^n)$, $\int \psi \left(\rho_1 - \rho_2\right) dx$

$$\begin{cases} \partial_t \rho + \operatorname{div}_X(\mathbf{g} \ \rho) = u \rho + w \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Thomas Lorenz

Autonomous linear problem

The metric on L²(ℝ")

Verifying the

Weak topology on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

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 $\begin{array}{ll} \text{Metric } d_{L^2}(\rho_1,\rho_2) \ := \ \sup \Big\{ \int_{\mathbb{R}^n} \varphi \; \rho_1 \; d\mathbf{x} - \int_{\mathbb{R}^n} \varphi \; \rho_2 \; d\mathbf{x} \; \Big| \; \varphi \in C^1_c(\mathbb{R}^n), \; \|\varphi\|_{L^2} \leq 1 \\ & \|\varphi\|_{L^\infty} \leq 1, \; \; \|\nabla \varphi\|_{L^\infty} \leq 1 \Big\} \end{array}$

Transition (?) For $\mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^n)$, $w \in L^2(\mathbb{R}^n)$, $\vartheta^{\rho}_{\sigma |_{l}|_{W}}: [0,1] \times L^{2}(\mathbb{R}^{n}) \longrightarrow L^{2}(\mathbb{R}^{n}), (t,\rho_{0}) \longrightarrow \rho(t)$ denote the unique weak solution to the autonomous linear equation $\partial_t \rho + \operatorname{div}_x(\mathbf{g} \, \rho) = u \, \rho + w$.

Lemma: For every $\psi \in C_c^1(\mathbb{R}^n)$ and $\rho_1, \rho_2 \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \psi \left(\rho_1 - \rho_2 \right) dx = \left(\|\psi\|_{L^2} + \|\psi\|_{W^{1,\infty}} \right) \cdot \int_{\mathbb{R}^n} \frac{\psi}{\|\psi\|_{L^2} + \|\psi\|_{W^{1,\infty}}} \left(\rho_1 - \rho_2 \right) dx$$

$$\left\{ \begin{array}{ll} \partial_t \, \rho \, + \, \mathrm{div}_{\mathsf{X}} \big(\mathbf{g} \ \, \rho \big) \, \, = \, u \, \rho \, + \, w \\ \rho(0) \, \, = \, \rho_0 \end{array} \right.$$

$$\iff \forall \ \varphi \in C^1_c(\mathbb{R}^n): \ \int_{\mathbb{R}^n} \varphi \ \rho(t) \ dx = \ \int_{\mathbb{R}^n} \left(\psi_{t,\varphi}(0) \ \rho_0 + w \ \int_0^t \psi_{t,\varphi}(s,x) \ ds \right) dx$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

 \mathcal{M} -valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Candidate for the Metric

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-value solutions

Mutation

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $\binom{2}{\mathbb{R}^n}$

Verifying the transition

Weak topology on tight subsets

i ii de concidation

M-valued solutions

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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- $(3.) \ e_{L^{2}} \left(\vartheta^{\rho}_{\mathbf{g},u,w}(s,\rho_{0}), \ \vartheta^{\rho}_{\mathbf{g},u,w}(t,\rho_{0}) \right) \ \leq \ C \cdot \left(\|\rho_{0}\|_{L^{2}} + \|w\|_{L^{2}} \right) \cdot |t-s|$
- $(4.) \ d_{L^2}(\vartheta_{\mathbf{g},u,w}^{\rho}(t,\rho_0), \ \vartheta_{\mathbf{g},u,w}^{\rho}(t,\widetilde{\rho}_0)) \ \leq \ d_{L^2}(\rho_0, \ \widetilde{\rho}_0) \cdot e^{\operatorname{const} \cdot t}$

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutations

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivation

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

i ii st concidsion

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

i ii de concidation

M-valued solutions

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

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For any $\widetilde{\mathbf{g}}, \mathbf{g} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $\widetilde{u}, u \in W^{1,\infty}(\mathbb{R}^n)$ and $\widetilde{w}, w \in L^2(\mathbb{R}^n)$, it holds for all $\rho_0, \widetilde{\rho}_0 \in L^2(\mathbb{R}^n)$ and $s, t \in [0, T]$ (s < t)

$$(1.) \quad \vartheta_{\mathbf{g},u,w}^{\rho}(t,\rho_0) = \vartheta_{\mathbf{g},u,w}^{\rho}(t-s, \vartheta_{\mathbf{g},u,w}^{\rho}(s,\rho_0))$$

$$(2.) \|\vartheta^{\rho}_{\mathbf{g},u,w}(t,\rho_{0})\|_{L^{2}(\mathbb{R}^{n})} \leq (\|\rho_{0}\|_{L^{2}(\mathbb{R}^{n})} + \|w\|_{L^{2}(\mathbb{R}^{n})} t) \cdot e^{\operatorname{const} \cdot t}$$

$$(3.) \ e_{L^{2}}\left(\vartheta_{\mathbf{g},u,w}^{\rho}(s,\rho_{0}), \ \vartheta_{\mathbf{g},u,w}^{\rho}(t,\rho_{0})\right) \ \leq \ C \cdot (\|\rho_{0}\|_{L^{2}} + \|w\|_{L^{2}}) \cdot |t-s|$$

$$(4.) \ d_{L^2}\left(\vartheta^{\rho}_{\mathbf{g},u,w}(t,\rho_0),\ \vartheta^{\rho}_{\mathbf{g},u,w}(t,\widetilde{\rho}_0)\right) \ \leq \ d_{L^2}\left(\rho_0,\ \widetilde{\rho}_0\right) \cdot e^{\operatorname{const} \cdot t}$$

 $(5.) \ d_{L^{2}}\left(\vartheta_{\mathbf{g},u,w}^{\rho}(t,\rho_{0}),\ \vartheta_{\widetilde{\mathbf{g}},\widetilde{u},\widetilde{w}}^{\rho}(t,\rho_{0})\right) \leq C \cdot \left(1 + \|\rho_{0}\|_{L^{2}(\mathbb{R}^{n})}\right) \cdot \left\|(\mathbf{g},u,w) - (\widetilde{\mathbf{g}},\widetilde{u},\widetilde{w})\right\|_{L^{2}} \cdot t$

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Verfiying the Conditions on Transitions

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- $(3.) \ e_{L^{2}}\left(\vartheta_{\mathbf{g},u,w}^{\rho}(s,\rho_{0}), \ \vartheta_{\mathbf{g},u,w}^{\rho}(t,\rho_{0})\right) \ \leq \ C \cdot (\|\rho_{0}\|_{L^{2}} + \|w\|_{L^{2}}) \cdot |t-s|$
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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

Mutations

Solutions with Values in $L^2(\mathbb{R}^n)$

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology on tight subsets

First conclusions

M-value

Mutation

Solutions with Values in $L^2(\mathbb{R}^n)$

Connection Between d_{L^2} and Weak Topology

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology

on tight subsets

M-value

Mutationa

Solutions with Values in $L^2(\mathbb{R}^n)$

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$$\lim_{r\to\infty}$$

$$\int_{\mathbb{R}^n \setminus \mathbb{B}_*(0)} |f(x)|^2 dx = 0.$$



Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

Connection Between d_{12} and Weak Topology

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A set
$$S \subset L^2(\mathbb{R}^n)$$

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Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

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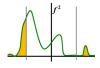
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Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

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Definition A set $S \subset L^2(\mathbb{R}^n)$ is called (*uniformly*) *tight* if

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Autonomous linear problem

The metric on

Verifying the Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

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For every tight sequence $(\rho_k)_{k\in\mathbb{N}}$ in $L^2(\mathbb{R}^n)$ and $\rho\in L^2(\mathbb{R}^n)$, Proposition

$$\rho_k \longrightarrow \rho \ \text{ weakly in } L^2(\mathbb{R}^n) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \sup\limits_{k \, \in \, \mathbb{N}} \, \|\rho_k\|_{L^2(\mathbb{R}^n)} \, < \, \infty \\ \lim\limits_{k \, \to \, \infty} \, d_{L^2}\big(\rho_k, \rho\big) \, = \, 0 \end{array} \right.$$

Thomas Lorenz

Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set
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Suppose
$$\mathcal{G}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}), \mathcal{U}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}),$$

$$\mathcal{U}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n) \cap L^2, \quad \|\cdot\|_{L^2}),$$

$$\mathcal{W}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (L^2(\mathbb{R}^n), \quad \|\cdot\|_{L^2})$$

$$\mathcal{V}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (L^2(\mathbb{R}^n), \|\cdot\|_{L^2})$$

to be "bounded" Carathéodory functions.

Thomas Lorenz

Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set
$$E := L^2(\mathbb{R}^n),$$

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Let
$$\widehat{w} \in L^2(\mathbb{R}^n)$$
 and a compact set $K \subset \mathbb{R}^n$ be such that $|\mathcal{W}(t,\rho)(x)| \leq \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

Thomas Lorenz

Autonomous linear problem

The metric on

Verifying the

Weak topology

on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set
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Then for every $\rho_0 \in L^2(\mathbb{R}^n)$, there is at least one function $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$ s.t.

Thomas Lorenz

Motivation

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set
$$E := L^2(\mathbb{R}^n),$$

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$$\begin{array}{ll} \mathsf{Metric} \,\, d_{L^2}(\rho_1,\rho_2) \,\, := \,\, \sup \Big\{ \int_{\mathbb{R}^n} \varphi \,\, \rho_1 \,\, d\mathsf{x} - \int_{\mathbb{R}^n} \varphi \,\, \rho_2 \,\, d\mathsf{x} \,\, \Big| \,\, \varphi \in C^1_c(\mathbb{R}^n), \,\, \|\varphi\|_{L^2} \leq 1 \\ \qquad \qquad \|\varphi\|_{L^\infty} \leq 1, \,\, \|\nabla \varphi\|_{L^\infty} \leq 1 \Big\} \end{array}$$

Suppose
$$\mathcal{G}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}),$$

 $\mathcal{U}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}),$

$$\mathcal{W}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (L^2(\mathbb{R}^n), \|\cdot\|_{L^2})$$

$$\|\cdot\|_{L^2}$$

$$\mathcal{V}: [0, T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (L^2(\mathbb{R}^n), \|\cdot\|)$$

to be "bounded" Carathéodory functions.

Let
$$\widehat{w} \in L^2(\mathbb{R}^n)$$
 and a compact set $K \subset \mathbb{R}^n$ be such that $|\mathcal{W}(t,\rho)(x)| \leq \widehat{w}(x)$ holds for all $x \in \mathbb{R}^n \setminus K$, $t \in [0,T]$ and $\rho \in L^2(\mathbb{R}^n)$.

Then for every $\rho_0 \in L^2(\mathbb{R}^n)$, there is at least one function $\rho:[0,T] \longrightarrow L^2(\mathbb{R}^n)$ s.t.

(1.)
$$\rho$$
 is continuous w.r.t. d_{L^2} and bounded w.r.t. $\|\cdot\|_{L^2(\mathbb{R}^n)}$,

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n),$

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θ(h, x(t))

Thomas Lorenz

Motivation

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n),$

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Thomas Lorenz

Motivation

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued

Mutational inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n),$

Suppose

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Thomas Lorenz

Motivation

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

M-valued solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n),$

Suppose

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() (h, x(t))

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

Autonomous linear problem

The metric on $L^2(\mathbb{R}^n)$

Verifying the transition

Weak topology on tight subsets

First conclusions

 \mathcal{M} -value solutions

Mutationa inclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set
$$E := L^2(\mathbb{R}^n),$$

Suppose

"Abs. value" $\lfloor
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$$\begin{array}{ll} \mathcal{U}: [0,\,I] \times (L^2(\mathbb{R}^n),\,d_{L^2}) &\longrightarrow (\mathbb{W}^{1,\infty}(\mathbb{R}^n) \cap L^2, & \|\cdot\|_{L^2}), \\ \mathcal{W}: [0,\,T] \times (L^2(\mathbb{R}^n),\,d_{L^2}) &\longrightarrow (L^2(\mathbb{R}^n), & \|\cdot\|_{L^2}) \end{array}$$

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- (3.) the image set $ho([0,T])\subset L^2(\mathbb{R}^n)$ is tight,

On a Class of Nonlocal Traffic Flow Models Thomas Lorenz

Autonomous linear problem Suppose

The metric on

Verifying the

Weak topology on tight subsets First conclusions

Solutions with Values in $L^2(\mathbb{R}^n)$

First Conclusions about Existence

Basic set $E := L^2(\mathbb{R}^n)$,

"Abs. value" $\lfloor \rho \rfloor := \|\rho\|_{L^2(\mathbb{R}^n)}$

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Then for every $\rho_0 \in L^2(\mathbb{R}^n)$, there is at least one function $\rho: [0, T] \longrightarrow L^2(\mathbb{R}^n)$ s.t. (1.) ρ is continuous w.r.t. d_{12} and bounded w.r.t. $\|\cdot\|_{L^{2}(\mathbb{R}^{n})}$,

 $(2.) \lim_{h \downarrow 0} \frac{1}{h} \cdot d_{L^2} \left(\rho(t+h), \, \vartheta^{\rho}_{\left(\mathcal{G}(t, \, \rho(t)), \, \, \mathcal{U}(t, \, \rho(t)), \, \, \mathcal{W}(t, \, \rho(t))\right)} (h, \, \rho(t)) \right) = 0 \quad (\text{a.e. } t)$

(3.) the image set $\rho([0,T]) \subset L^2(\mathbb{R}^n)$ is tight,

(4.) ρ is a weak solution of $\partial_t \rho + \operatorname{div}_X(\mathcal{G}(t,\rho) \rho) = \mathcal{U}(t,\rho) \rho + \mathcal{W}(t,\rho)$.

 $\text{Metric } d_{L^2}(\rho_1,\rho_2) \ := \ \sup \Big\{ \int_{\mathbb{D}^n} \varphi \ \rho_1 \ d\mathbf{x} - \int_{\mathbb{D}^n} \varphi \ \rho_2 \ d\mathbf{x} \ \Big| \ \varphi \in C^1_c(\mathbb{R}^n), \ \|\varphi\|_{L^2} \le 1$

 $\|\varphi\|_{L^{\infty}} \le 1, \ \|\nabla\varphi\|_{L^{\infty}} \le 1$

 $\mathcal{G}: [0,T] \times (L^2(\mathbb{R}^n), d_{L^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}),$ $\mathcal{U}: [0,T] \times (L^2(\mathbb{R}^n), d_{I^2}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n) \cap L^2, \|\cdot\|_{L^2}),$ $\|\cdot\|_{L^{2}}$

L²-valued solutions

M-value

Autonomous linear problem

Conclusions about existence

Mutationa

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

The "full nonlinear" problem

$$\begin{cases} \partial_t \mu + \operatorname{div}_X (\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu & \text{in } [0,T] \\ \mu(0) = \mu_0 \in \mathcal{M}(\mathbb{R}^n) \end{cases}$$

L²-value solutions

M-value solutions

Autonomous linear problem

Conclusions about existence

Mutationa inclusions

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=\mathcal{M}(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \mu \rfloor:= |\mu|(\mathbb{R}^n)$

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-value solutions

M-value solutions

Autonomous linear problem

Conclusions about existence

Mutation

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
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$$\begin{cases} \partial_t \mu + \operatorname{div}_{\mathsf{x}} \big(\mathcal{B}(t, \mu) \, \mu \big) \, = \, \mathcal{C}(t, \mu) \, \mu & \text{in } [0, T] \\ \mu(0) \, = \, \mu_0 \, \in \, \mathcal{M}(\mathbb{R}^n) \end{cases}$$

$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mathbf{b} \ \mu) = \mathbf{c}(\mathbf{x}) \ \mu \\ \rho(0) = \rho_0 \end{cases}$$

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-value solutions

Autonomous linear problem

Conclusions about existence

Mutation

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=\mathcal{M}(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \mu \rfloor:= |\mu|(\mathbb{R}^n)$

Transition (?)
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Motivatio

Main results

Ordin. diff. eqns in a metric space

solutions

M-value

Autonomous linear problem

Conclusions about existence

Mutationa

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=\mathcal{M}(\mathbb{R}^n),$$
 "Abs. value" $|\mu|:=|\mu|(\mathbb{R}^n)$

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$$\begin{cases} \partial_t \, \mu \, + \, \mathrm{div}_{\mathsf{x}} \big(\mathbf{b} \ \mu \big) \, = \, c(\mathsf{x}) \ \mu \\ \rho(\mathsf{0}) \, = \, \rho_\mathsf{0} \\ \\ \iff \quad \forall \ \varphi \in C^1_c(\mathbb{R}^n) : \, \int_{\mathbb{R}^n} \varphi \ d\mu_t \, = \, \int_{\mathbb{R}^n} \psi_{t,\varphi}(\mathsf{0};\mathsf{x}) \ d\mu_\mathsf{0}(\mathsf{x}) \end{cases}$$

Motivatio

Main results

Ordin. diff. eqns in a metric space

L²-value solutions

Autonomous linear problem

Conclusions about existence

Mutation

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set
$$E:=\mathcal{M}(\mathbb{R}^n),$$
 "Abs. value" $\lfloor \mu \rfloor:= |\mu|(\mathbb{R}^n)$

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$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mathbf{b} \ \mu) = c(x) \ \mu \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi \ d\mu_t = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \ d\mu_0(x)$$

Motivation

Main results

Ordin. diff. eqns in a metric space

L²-value solutions

M-valued solutions

Autonomous linear problem

Conclusions about existence

Mutation

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E:=\mathcal{M}(\mathbb{R}^n),$ "Abs. value" $|\mu|:=|\mu|(\mathbb{R}^n)$

Metric $d_{\mathcal{M}}(\mu_1, \mu_2)$

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$$\Rightarrow \quad \forall \, \varphi \in C^1_c(\mathbb{R}^n) : \, \int_{\mathbb{R}^n} \varphi \, \, d\mu_t \, = \, \int_{\mathbb{R}^n} \psi_{t,\varphi}(\mathsf{0};\mathsf{x}) \, \, d\mu_\mathsf{0}(\mathsf{x})$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Autonomous linear problem

Conclusions about existence

Mutationa inclusions

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

Basic set $E := \mathcal{M}(\mathbb{R}^n),$

"Abs. value" $\lfloor \mu
floor \ := \ |\mu|(\mathbb{R}^n)$

 $\mathsf{Metric}\ d_{\mathcal{M}}(\mu_1,\mu_2)\ :=\ \sup\left\{ \left. \int_{\mathbb{R}^n} \varphi\ d\left(\mu_1-\mu_2\right) \, \right| \varphi \in C^1_c(\mathbb{R}^n),\ \|\varphi\|_{L^\infty}, \|\nabla\varphi\|_{L^\infty} \le 1 \right\}$

Transition (?) For $\mathbf{b} \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$ and $c \in W^{1,\infty}(\mathbb{R}^n)$, let $\vartheta^{\mu}_{\mathbf{b},c}:[0,1] \times \mathcal{M}(\mathbb{R}^n) \longrightarrow \mathcal{M}(\mathbb{R}^n)$, $(t,\mu_0) \longrightarrow \mu_t$ denote the unique distributional solution to the autonomous linear equation $\partial_t \mu + \operatorname{div}_x(\mathbf{b} \mu) = c \mu$.

$$\begin{cases} \partial_t \mu + \operatorname{div}_X (\mathbf{b} \ \mu) = c(x) \ \mu \\ \rho(0) = \rho_0 \end{cases}$$

$$\iff \forall \varphi \in C_c^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi \ d\mu_t = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \ d\mu_0(x)$$

Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Autonomous linear problem

Conclusions about existence

Mutations

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

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$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mathbf{b} \ \mu) = c(x) \ \mu \\ \rho(0) = \rho_0 \end{cases}$$

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Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

 $\mathcal{M}\text{-valued}$ solutions

Autonomous linear problem

Conclusions about existence

Mutationa

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

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"Abs. value" $\lfloor \mu
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$$\text{Metric } d_{\mathcal{M}}(\mu_1, \mu_2) \ := \ \sup \Big\{ \int_{\mathbb{R}^n} \varphi \ d\big(\mu_1 - \mu_2\big) \ \Big| \ \varphi \in C^1_c(\mathbb{R}^n), \ \|\varphi\|_{L^\infty}, \|\nabla \varphi\|_{L^\infty} \le 1 \Big\}$$
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Thomas Lorenz

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Autonomous linear problem

Conclusions about existence

Mutation

Summar

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

The Autonomous Linear Problem

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The auxiliary "autonomous linear" problem

$$\begin{cases} \partial_t \mu + \operatorname{div}_{\mathsf{x}}(\mathbf{b} \ \mu) = c(\mathsf{x}) \ \mu \\ \rho(0) = \rho_0 \end{cases}$$

 $\iff \forall \varphi \in C_c^1(\mathbb{R}^n): \int_{\mathbb{R}^n} \varphi \ d\mu_t = \int_{\mathbb{R}^n} \psi_{t,\varphi}(0;x) \ d\mu_0(x)$

Thomas Lorenz

in a metric space

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

Basic set $E := \mathcal{M}(\mathbb{R}^n)$,

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Existence Theoreom [TL, 2009/10]

 $\begin{array}{l} \mathcal{B}: [0,T] \times (\mathcal{M}(\mathbb{R}^n), \ d_{\mathcal{M}}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n), \|\cdot\|_{L^{\infty}}), \\ \mathcal{C}: [0,T] \times (\mathcal{M}(\mathbb{R}^n), \ d_{\mathcal{M}}) \longrightarrow (W^{1,\infty}(\mathbb{R}^n), \ \|\cdot\|_{L^{\infty}}), \end{array}$ Suppose

to be "bounded" Carathéodory functions.

Thomas Lorenz

in a metric space

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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Thomas Lorenz

in a metric space

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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Then for every $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$, there is at least one function $\rho : [0, T] \longrightarrow \mathcal{M}(\mathbb{R}^n)$ s.t.

Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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(2.)
$$\vartheta^{\mu}_{\left(\mathcal{B}(t,\,\mu_t),\,\,\mathcal{C}(t,\,\mu_t)\right)}(h,\,\mu_t)$$



Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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$$\mu_{t+h} \quad \vartheta^{\mu}_{\left(\mathcal{B}(t,\,\mu_t),\,\,\mathcal{C}(t,\,\mu_t)\right)}(h,\,\mu_t)$$



Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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(2.)
$$\frac{1}{h} \cdot d_{\mathcal{M}} \left(\mu_{t+h}, \ \vartheta^{\mu}_{\left(\mathcal{B}(t, \mu_{t}), \ \mathcal{C}(t, \mu_{t})\right)} \left(h, \mu_{t}\right) \right)$$



Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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$$(2.) \lim_{h \downarrow 0} \frac{1}{h} \cdot d_{\mathcal{M}} \left(\mu_{t+h}, \ \vartheta^{\mu}_{\left(\mathcal{B}(t, \mu_{t}), \ \mathcal{C}(t, \mu_{t})\right)} \left(h, \mu_{t} \right) \right) = 0$$



Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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to be "bounded" Carathéodory functions.

- (1.) μ is continuous w.r.t. $d_{\mathcal{M}}$ and bounded w.r.t. $|\cdot|(\mathbb{R}^n)$,
- $(2.) \lim_{h \downarrow 0} \frac{1}{h} \cdot d_{\mathcal{M}} \left(\mu_{t+h}, \ \vartheta^{\mu}_{\left(\mathcal{B}(t, \mu_{t}), \ \mathcal{C}(t, \mu_{t})\right)} \left(h, \mu_{t} \right) \right) = 0$
- (3.) the image set $\mu([0,T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,



Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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- (3.) the image set $\mu([0,T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,
- (4.) μ is narrowly continuous

Thomas Lorenz

linear problem

Conclusions about existence

Solutions with Values in $\mathcal{M}(\mathbb{R}^n)$

Conclusions about Existence

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- (3.) the image set $\mu([0,T]) \subset \mathcal{M}(\mathbb{R}^n)$ is tight,
- (4.) μ is narrowly continuous and a distributional solution of

$$\partial_t \mu + \operatorname{div}_{\mathcal{X}}(\mathcal{B}(t,\mu) \mu) = \mathcal{C}(t,\mu) \mu.$$

Ordin. diff. eqns. in a metric space

L²-value solutions

M-value solutions

Mutational inclusions

Conclusions about Balance Laws

Summary

Differential Inclusions Without Convexity

Classical Result by Antosiewicz & Cellina

Theoreom [Antosiewicz and Cellina, 1975]

Suppose for the set-valued map $F: [0, T] \times \mathbb{R}^n \leadsto \mathbb{R}^n$

- (1.) each value $F(t,x)\subset\mathbb{R}^n$ is nonempty and compact with $F(t,x)\subset\mathbb{B}_R(0)$,
- (2.) $\forall x \in \mathbb{R}^n : F(\cdot, x) : [0, T] \leadsto \mathbb{R}^n$ is measurable,
- (3.) $\forall t \in [0, T] : F(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (w.r.t. Hausdorff metric dl).

Then for each $x_0 \in \mathbb{R}^n$, there exists an absolutely cont. solution $x : [0, T] \longrightarrow \mathbb{R}^n$ of

$$x' \in F(\cdot, x)$$
 a.e. in $[0, T]$.

Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Conclusions about Balance Laws

Summary

Differential Inclusions Without Convexity

Classical Result by Antosiewicz & Cellina

Selection Principle [Antosiewicz and Cellina, 1975]

Under the preceding assumptions about the set-valued map $F:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, there exists a function $g:C^0([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n)$ such that

(i) for every $u \in C^0([0,T],\mathbb{R}^n)$: $[0,T] \longrightarrow \mathbb{R}^n$, $t \longmapsto g(u)(t)$ is integrable,

Theoreom [Antosiewicz and Cellina, 1975]

Suppose for the set-valued map $F: [0, T] \times \mathbb{R}^n \leadsto \mathbb{R}^n$

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Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

Mutational inclusions

Conclusions about Balance Laws

Summary

Differential Inclusions Without Convexity

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- (i) for every $u \in C^0([0,T],\mathbb{R}^n)$: $[0,T] \longrightarrow \mathbb{R}^n$, $t \longmapsto g(u)(t)$ is integrable,
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$$x' \in F(\cdot, x)$$
 a.e. in $[0, T]$.

Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Conclusions about Balance Laws

Summary

Differential Inclusions Without Convexity

Classical Result by Antosiewicz & Cellina

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Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Conclusions about Balance

Summary

Differential Inclusions Without Convexity

Extensions

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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutational inclusions

Conclusions about Balance Laws

Differential Inclusions Without Convexity

Extensions

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e.g.,
$$F:[0,T]\times (E,d) \rightsquigarrow (\Theta(E),D)$$

L²-valued solutions

M-value solutions

Mutationa

Conclusions about Balance Laws

Summary

Multivalued Nonlocal Balance Laws

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L²-valued solutions

M-value solutions

Mutationa inclusions

Conclusions about Balance Laws

Summary

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Motivation

Main recults

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-value solutions

Mutationa inclusions

Conclusions about Balance Laws

Summary

Multivalued Nonlocal Balance Laws

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Mativation

Main manulant

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-value solutions

Mutationa inclusions

Conclusions about Balance Laws

Summary

Multivalued Nonlocal Balance Laws

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L²-value solutions

M-value solutions

Mutationa inclusions

Conclusions about Balance Laws

Summary

Multivalued Nonlocal Balance Laws

Existence of Weak Solutions to Multivalued Balance Laws

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Thomas Lorenz

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Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Conclusions about Balance Laws

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Multivalued Nonlocal Balance Laws

Existence of Weak Solutions to Multivalued Balance Laws

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L²-valued solutions

Mutationa

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The focus of this talk was on a class of "multiscale" traffic flow models, i.e., two nonhomogeneous transport equations with nonlocal dependence

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho,\mu) \ \rho) = \mathcal{U}(t,\rho,\mu) \ \rho + \mathcal{W}(t,\rho,\mu) \\ \partial_t \mu + \operatorname{div}_{\mathsf{x}}(\mathcal{B}(t,\rho,\mu) \ \mu) = \mathcal{C}(t,\rho,\mu) \ \mu \end{cases}$$

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M-valued solutions

Mutationa inclusions

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The components are

- * a (strongly) continuous solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ in the weak sense,
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Motivatio

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

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Goal: Sufficient conditions for the well-posed initial value problem:

- * Existence (due to compactness)
- * Uniqueness (due to LIPSCHITZ continuity of coefficients)
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Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-value solutions

M-valued solutions

Mutationa inclusions

Summary

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Motivation

Main results

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

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Tool: Construct a solution by means of EULER method, but with focus on the metric aspect:

Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summary

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→ Mutational equations: A joint framework for CAUCHY problems in and beyond vector spaces



Motivatio

Main result

Ordin. diff. eqns. in a metric space

L²-valued solutions

M-valued solutions

Mutationa inclusions

Summary

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The focus of this talk was on a class of "multiscale" traffic flow models, i.e., two nonhomogeneous transport equations with nonlocal dependence

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathsf{x}}(\mathcal{G}(t,\rho,\mu) \ \rho) = \mathcal{U}(t,\rho,\mu) \ \rho + \mathcal{W}(t,\rho,\mu) \\ \partial_t \mu + \operatorname{div}_{\mathsf{x}}(\mathcal{B}(t,\rho,\mu) \ \mu) = \mathcal{C}(t,\rho,\mu) \ \mu \end{cases}$$

The components are

- * a (strongly) continuous solution $\rho:[0,T]\longrightarrow L^2(\mathbb{R}^n)$ in the weak sense,
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Goal: Sufficient conditions for the well-posed initial value problem:

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