Workshop "Mathematical Foundations of Traffic"

Macroscopic traffic flow models with non-local mean velocity

Paola Goatin

Inria Sophia Antipolis - Méditerranée
paola.goatin@inria.fr

joint works with
Sebastien Blandin (*IBM Research Collaboratory, Singapore*)
Francesco Rossi (*Université Aix-Marseille*)
Sheila Scialanga (*Università di Roma I*)

IPAM, September 28, 2015

European Research Council

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Non-local conservation laws

(Systems of) equations of the form

$$\partial_t U + \mathrm{div}_{\mathbf{x}} F(t, \mathbf{x}, U, w * U) = 0$$

with $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^d$, $U(t, \mathbf{x}) \in \mathbb{R}^N$, $w(t, \mathbf{x}) \in \mathbb{R}^{m \times N}$

Applications:

- sedimentation [Betancourt&al, Nonlinearity 2011]
- granular flows [Amadori-Shen, JHDE 2012]
- crowd dynamics [Colombo&al, ESAIM COCV 2011; AMS 2011; M3AS 2012]
- supply chains [ColomboHertyMercier, ESAIM COCV 2011]
- conveyor belts [Göttlich&al, Appl. Math. Modell., 2014]
- gradient constraint [Amorim, Bull. Braz. Math. Soc., 2012]

Non-local conservation laws

(Systems of) equations of the form

$$\partial_t U + \mathrm{div}_{\mathbf{x}} F(t, \mathbf{x}, U, w * U) = 0$$

with $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^d$, $U(t,\mathbf{x}) \in \mathbb{R}^N$, $w(t,\mathbf{x}) \in \mathbb{R}^{m \times N}$

General well posedness results:

- 1D scalar equations
  [AmorimColomboTeixeira, ESAIM M2AN 2015]
- multiD scalar equations
  [ColomboHertyMercier, ESAIM COCV 2011]
- multiD systems
  [CrippaMercier, NoDEA 2012; AggarwalColomboGoatin, SINUM 2015]

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

A model with non-local velocity[1]

LWR model with downstream non-local velocity

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V(t,x)\right) = 0$$

where

$$V(t,x) = v\left(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\, dy\right), \quad \eta > 0$$

with $w_\eta \in \mathbf{C}^1([0,\eta];\mathbb{R}^+)$ non-increasing and $\int_0^\eta w_\eta(x)dx = 1$
$v : [0,\rho_{\max}] \to \mathbb{R}^+$ s.t. $-A \leq v' \leq 0$, $v(0) = v_{\max}$, $v(\rho_{\max}) = v_{\min}$

[1] [BlandinGoatin, 2015; GoatinScialanga, submitted]

A model with non-local velocity[1]

LWR model with downstream non-local velocity

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) V(t,x)\right) = 0$$

where

$$V(t,x) = v\left(\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\, dy\right), \quad \eta > 0$$

with $w_\eta \in \mathbf{C^1}([0,\eta];\mathbb{R}^+)$ non-increasing and $\int_0^\eta w_\eta(x)dx = 1$
$v : [0,\rho_{\max}] \to \mathbb{R}^+$ s.t. $-A \leq v' \leq 0$, $v(0) = v_{\max}$, $v(\rho_{\max}) = v_{\min}$

Related works:

- sedimentation model: $F(\rho, \rho * w) = \rho(1-\rho)^\alpha V(\rho * w)$, $\alpha = 0$ or $\alpha \geq 1$
  [Betancourt&al, Nonlinearity 2011]
- Arrhenius look-ahead dynamics: $F(\rho, \rho * w) = \rho(1-\rho)e^{-(\rho * w)}$
  [SopasakisKatsoulakis, SIAM 2006]
  [KurganovPolizzi, NHM 2009]
  [LiLi, NHM 2011]

---

[1] [BlandinGoatin, 2015; GoatinScialanga, submitted]

Finite acceleration

The model avoids the infinite acceleration drawback of classical macroscopic models:

$$\begin{aligned} & \dot{x}(t) = V(t, x(t)), && t > 0 \\ \Longrightarrow \quad & \ddot{x}(t) = V_t(t, x(t)) + V(t, x(t)) V_x(t, x(t)), && t > 0 \end{aligned}$$

If $\rho(t, \cdot) \in \mathbf{L^1} \cap \mathbf{L}^\infty$, we have

$$\begin{aligned} \|V_t\|_\infty &= 2 w_\eta(0) \|v\|_\infty \|v'\|_\infty \|\rho\|_\infty \\ \|V_x\|_\infty &= 2 w_\eta(0) \|v'\|_\infty \|\rho\|_\infty \end{aligned}$$

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Well-posedness

Theorem [BlandinGoatin, 2015; GoatinScialanga, submitted]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$. Then the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho V(t,x)\right) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

admits a unique weak entropy entropy solution ($\rho \in \mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathrm{BV}$), such that

$$\min_{\mathbb{R}}\{\rho_0\} \leq \rho(t,x) \leq \max_{\mathbb{R}}\{\rho_0\} \quad \text{for a.e. } x \in \mathbb{R},\ t > 0$$

Kružkov entropy condition[2]

Definition

A function $\rho \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathrm{BV})(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ is an entropy weak solution if

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \left(|\rho - \kappa| \varphi_t + |\rho - \kappa| \, V \, \varphi_x - \mathrm{sgn}(\rho - \kappa) \, \kappa \, V_x \; \varphi\right) (t, x) dx dt + \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0, x) \; dx \geq 0$$

for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2; \mathbb{R}^+)$ and $\kappa \in \mathbb{R}$.

[2] [ColomboHertyMercier, ESAIM COCV 2011; Betancourt&al, Nonlinearity 2011]

Uniqueness[3]

Theorem

Let ρ, σ be two entropy weak solutions of CP with initial data ρ_0, σ_0 respectively. Then, for any $T > 0$ there holds

$$\|\rho(t,\cdot) - \sigma(t,\cdot)\|_{\mathbf{L}^1} \leq e^{\mathcal{K}T} \|\rho_0 - \sigma_0\|_{\mathbf{L}^1} \qquad \forall t \in (0,T].$$

where

$$\begin{aligned}\mathcal{K} = \ & w_\eta(0) \left\|v'\right\|_\infty \left(\sup_{t\in[0,T]} \|\rho(t,\cdot)\|_{\mathrm{BV}(\mathbb{R})} + 2\|\rho_0\|_\infty \right) \\ & + \|\rho_0\|_1 \left(2(w_\eta(0))^2 \left\|v''\right\|_\infty \|\rho_0\|_\infty + \left\|v'\right\|_\infty \left\|w_\eta'\right\|_{\mathbf{L}^\infty([0,\eta])} \right)\end{aligned}$$

Proof. Doubling of variables.

[3] [Betancourt&al, Nonlinearity 2011]

Existence: a Lax-Friedrichs numerical scheme

Take Δx s.t. $\eta = N\Delta x\ \exists N \in \mathbb{N}$:

$$\rho_j^{n+1} = H(\rho_{j-1}^n, \ldots, \rho_{j+N}^n) = \rho_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

with numerical flux

$$F_{j+1/2}^n = \frac{1}{2}\rho_j^n V_j^n + \frac{1}{2}\rho_{j+1}^n V_{j+1}^n + \frac{\alpha}{2}(\rho_j^n - \rho_{j+1}^n)$$

where $V_j := v\left(\Delta x \sum_{k=0}^{N-1} w_\eta^k\ \rho_{j+k}\right)$,
and we assume

$$\Delta t \leq \frac{2}{2\alpha + A\Delta x\, w_\eta(0)}\,\Delta x \qquad (CFL)$$

$$\alpha \geq v_{\max} + A\Delta x\ w_\eta(0)$$

Lax-Friedrichs numerical scheme

Given

$$H(\rho_{j-1}^n, \dots, \rho_{j+N}^n) = \rho_j^n + \frac{\lambda\alpha}{2}\left(\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n\right) + \frac{\lambda}{2}\left(\rho_{j-1}^n V_{j-1}^n - \rho_{j+1}^n V_{j+1}^n\right)$$

For $k = 2, \dots, N-2$

$$\frac{\partial H}{\partial \rho_{j+k}} = \frac{\lambda}{2}\Delta x\left(\rho_{j-1} w_\eta^{k+1} v'\left(\Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\right) - w_\eta^{k-1}\rho_{j+1} v'\left(\Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\right)\right)$$

Lax-Friedrichs numerical scheme

Given

$$H(\rho_{j-1}^n, \ldots, \rho_{j+N}^n) = \rho_j^n + \frac{\lambda\alpha}{2}\left(\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n\right) + \frac{\lambda}{2}\left(\rho_{j-1}^n V_{j-1}^n - \rho_{j+1}^n V_{j+1}^n\right)$$

For $k = 2, \ldots, N-2$

$$\frac{\partial H}{\partial \rho_{j+k}} = \frac{\lambda}{2}\Delta x \left(\rho_{j-1} w_\eta^{k+1} v'\left(\Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j-1+k}\right) - w_\eta^{k-1} \rho_{j+1} v'\left(\Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+1+k}\right)\right)$$

$\Longrightarrow$ The scheme is not monotone!

Estimates

L^∞ estimates

Let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$. Then

$$\rho_m \leq \rho_j^n \leq \rho_M \qquad \forall j, n$$

Estimates

L^∞ estimates

Let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$. Then

$$\rho_m \leq \rho_j^n \leq \rho_M \qquad \forall j, n$$

BV estimates

Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$. Then

$$\sum_j \left|\rho_{j+1}^n - \rho_j^n\right| \leq e^{w_\eta(0)\left(5A+7\left\|v''\right\|_\infty\right)\frac{n}{2}\Delta t} \sum_j \left|\rho_{j+1}^0 - \rho_j^0\right|$$

Estimates

$\mathbf{L}^\infty$ estimates

Let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0,\rho_{\max}]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0,\rho_{\max}]$. Then

$$\rho_m \leq \rho_j^n \leq \rho_M \qquad \forall j, n$$

BV estimates

Let $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$. Then

$$\sum_j \left|\rho_{j+1}^n - \rho_j^n\right| \leq e^{w_\eta(0)\left(5A+7\left\|v''\right\|_\infty\right)\frac{n}{2}\Delta t} \sum_j \left|\rho_{j+1}^0 - \rho_j^0\right|$$

$\mathbf{L}^1$ stability estimates

Let $\rho_0, \bar\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$. Then

$$\sum_j \Delta x\left|\rho_j^n - \bar\rho_j^n\right| \leq K(w_\eta,\rho_0,\bar\rho_0,n\Delta t)\sum_j \Delta x\left|\rho_j^0 - \bar\rho_j^0\right|$$

$\mathbf{L}^\infty$ estimates

$\mathbf{L}^\infty$ estimates

Let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$. Then

$$\rho_m \leq \rho_j^n \leq \rho_M \qquad \forall j, n$$

$\mathbf{L}^\infty$ estimates

$\mathbf{L}^\infty$ estimates

Let $\rho_m = \min_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$ and $\rho_M = \max_{j\in\mathbb{Z}}\{\rho_j^0\} \in [0, \rho_{\max}]$. Then

$$\rho_m \leq \rho_j^n \leq \rho_M \qquad \forall j, n$$

$$\Uparrow$$

Lemma

Let $0 \leq \rho_m \leq \rho_j^n \leq \rho_M \leq \rho_{\max}$ for all $j \in \mathbb{Z}$. Then

$$H(\rho_m, \rho_m, \rho_m, \rho_{j+2}, \ldots, \rho_{j+N-2}, \rho_m, \rho_m) \geq \rho_m$$

$$H(\rho_M, \rho_M, \rho_M, \rho_{j+2}, \ldots, \rho_{j+N-2}, \rho_M, \rho_M) \leq \rho_M$$

BV estimates

$$
\begin{aligned}
\Delta^{n+1}_{j+\frac12} = {} & \frac{\lambda}{2}\left[\alpha + V_j + \rho_{j-1}\Delta x\ v'(\xi) w^0_\eta - \Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta \Delta_{j+k+\frac12}\right]\Delta_{j-\frac12} \\
& + \left[1 - \lambda\alpha + \frac{\lambda}{2}\rho_{j-1}\Delta x\ v'(\xi) w^1_\eta - \frac{\lambda}{2}\Delta x\ v'(\xi')\sum_{k=2}^{N-2} w^{k-1}_\eta \Delta_{j+k+\frac12}\right]\Delta_{j+\frac12} \\
& + \frac{\lambda}{2}\left[\alpha - V_{j+2} + \rho_{j-1}\Delta x\ v'(\xi) w^2_\eta - \rho_{j+1}\Delta x\ v'(\xi') w^0_\eta\right]\Delta_{j+\frac32} \\
& + \frac{\lambda}{2}\Delta x \sum_{k=2}^{N-2}\Delta_{j+k+\frac12}\left[\rho_{j-1}\ v'(\xi)(w^{k+1}_\eta - w^{k-1}_\eta) + w^{k-1}_\eta \rho_{j-1}\left(v'(\xi) - v'(\xi')\right)\right] \\
& - \frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-2}_\eta \Delta_{j+N-\frac12} \\
& - \frac{\lambda}{2}\rho_{j+1}\ \Delta x\ v'(\xi')\ w^{N-1}_\eta \Delta_{j+N+\frac12}
\end{aligned}
$$

where $\Delta^n_{j+k-1/2} = \rho^n_{j+k} - \rho^n_{j+k-1}$ for $k = 0, \ldots, N+1$

$$v'' = 0 \quad \Longrightarrow \quad \text{monotonicity preserving}$$

Discrete entropy inequalities

Proposition

If $\alpha \geq 1$ and the CFL condition holds, for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$ we have

$$\left|\rho_j^{n+1} - \kappa\right| - \left|\rho_j^n - \kappa\right| + \lambda \left(F^\kappa_{j+1/2}(\rho_j^n, \rho_{j+1}^n) - F^\kappa_{j-1/2}(\rho_{j-1}^n, \rho_j^n)\right) + \frac{\lambda}{2}\mathrm{sgn}(\rho_j^{n+1} - \kappa)\ \kappa \left(V_{j+1}^n - V_{j-1}^n\right) \leq 0$$

where

$$F^\kappa_{j+1/2}(u,v) = G_{j+1/2}(u \wedge \kappa, v \wedge \kappa) - G_{j+1/2}(u \vee \kappa, v \vee \kappa)$$

$$G_{j+1/2}(u,v) = \frac{1}{2} u\, V_j^n + \frac{1}{2} v\, V_{j+1}^n + \frac{\alpha}{2}(u - v)$$

Lax-Wendroff type argument $\Longrightarrow$ convergence

Regularity of solutions[4]

Proposition

If the initial datum $\rho_0 \in W^{1,\infty}(\mathbb{R})$, then the solution $\rho \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R})$

Indeed,

$$\left| \frac{\rho^n_{j+1} - \rho^n_j}{\Delta x} \right| \leq e^{7 w_\eta(0) \left(A + \left\| v'' \right\|_\infty \right) \frac{n}{2} \Delta t} \sup_j \left| \frac{\rho^0_{j+1} - \rho^0_j}{\Delta x} \right|$$

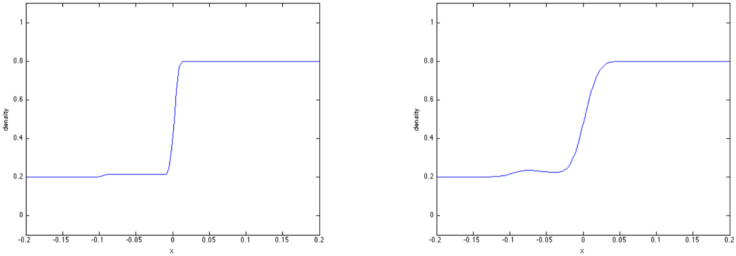
$$\left| \frac{\rho^{n+1}_j - \rho^n_j}{\Delta t} \right| \leq \left[\alpha + v_{\max} + A \left(1 + w_\eta(0) \Delta x \right) \right] \sup_j \left| \frac{\rho^n_{j+1} - \rho^n_j}{\Delta x} \right|$$

[4] [Betancourt&al, Nonlinearity 2011]

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Monotonicity preservation



(a) $v(\rho) = 1 - \rho$ (b) $v(\rho) = \ln\left(\frac{1}{\rho}\right)$

Figure: Density profiles at time $t = 0.01$ corresponding to $\rho_L = 0.2$, $\rho_R = 0.8$ and kernel $w_\eta(x) = 1/\eta$, $\eta = 0.1$.

Dependence on the location of the kernel support

We set $v(\rho) = 1 - \rho$ and

downstream: $$V_d(t,x) = 1 - \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\, dy$$

center: $$V_c(t,x) = 1 - \int_{x-\eta/2}^{x+\eta/2} \rho(t,y) w_\eta(y-x)\, dy$$

upstream : $$V_u(t,x) = 1 - \int_{x-\eta}^{x} \rho(t,y) w_\eta(y-x)\, dy$$

Dependence on the location of the kernel support

Rarefaction

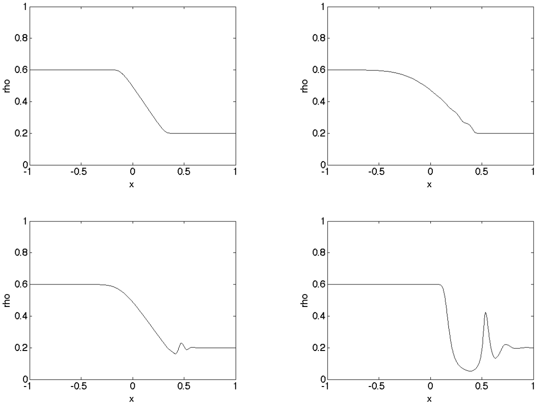


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively and initial data $\rho_L = 0.6$, $\rho_R = 0.2$

Dependence on the kernel support

Shock

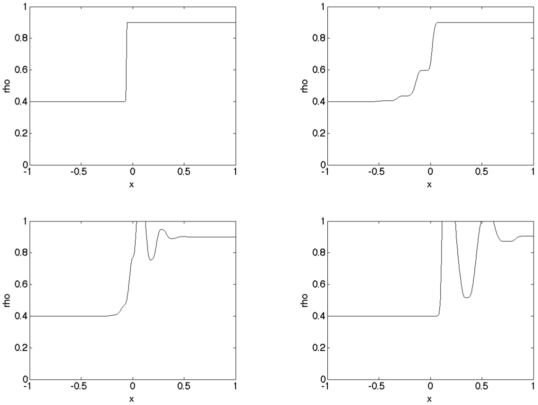


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively and initial data $\rho_L = 0.4$, $\rho_R = 0.9$

Dependence on the kernel support

Oscillating initial datum

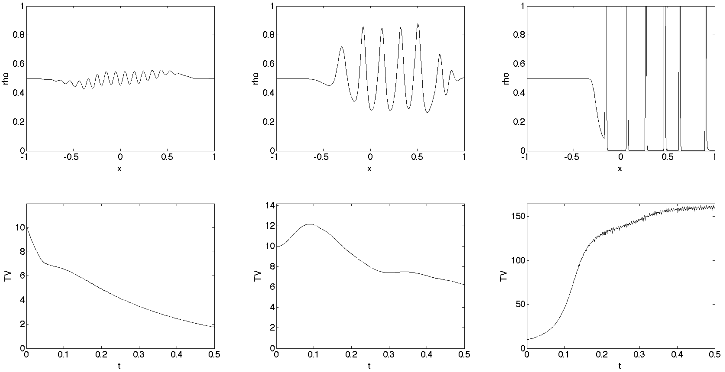
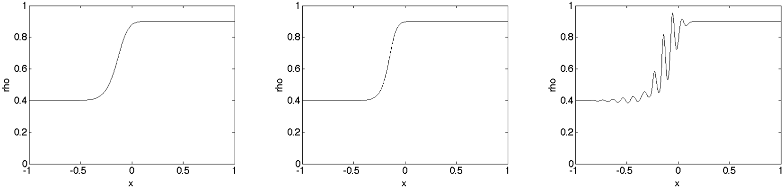


Figure: $w_\eta(x) = 1/\eta$ with downstream, central and upstream supports respectively

Kernel monotonicity



(a) $w_\eta(x) = 1/\eta$ (b) $w_\eta(x) = 2(\eta - x)/\eta^2$ (c) $w_\eta(x) = 2x/\eta^2$

Figure: $\rho(t = 0.5, \cdot)$ corresponding to $\rho_L = 0.4$, $\rho_R = 0.9$

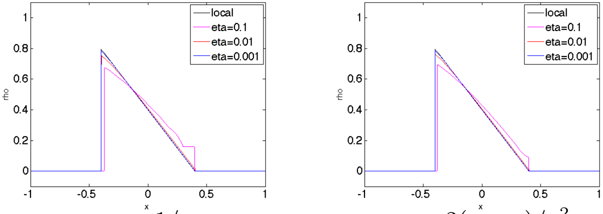
	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$		$w_\eta(x) = 2x/\eta^2$	
Δx	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error	$\gamma(\Delta x)$	$\mathbf{L^1}$-error
0.01	0.98021	3.013e-03	1.06427	3.315e-02	-0.42189	1.241e-01
0.005	0.93000	1.709e-03	1.06119	1.590e-02	-0.88509	1.287e-01
0.0025	0.61590	1.044e-03	0.87964	7.650e-03	-0.13054	1.303e-01
0.00125	0.44360	6.344e-04	1.05856	3.696e-03	0.15360	1.069e-01
0.000625	0.57113	3.632e-04	0.99995	1.547e-03	0.27699	7.093e-02

Table: Convergence orders and $\mathbf{L^1}$-errors to the reference solutions corresponding to $\Delta x = 0.00015625$

Limit $\eta \searrow 0$

$$\partial_t \rho + \partial_x (\rho v(\rho * w_\eta)) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x (\rho v(\rho)) = 0 \quad ??$$

We consider $v(\rho) = 1 - \rho$ and $\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.5 < x < -0.1 \\ 0 & \text{otherwise} \end{cases}$



$w_\eta = 1/\eta$ $\qquad$ $w_\eta = 2(\eta - x)/\eta^2$

	$w_\eta(x) = 1/\eta$		$w_\eta(x) = 2(\eta - x)/\eta^2$	
η	$\gamma(\eta)$	$\mathbf{L^1}$-error	$\gamma(\eta)$	$\mathbf{L^1}-$error
0.1	0.747605	6.417287 e-02	0.764526	4.814767 e-02
0.01	0.877130	1.147483 e-02	0.920991	8.280359 e-03
0.001	-	1.522703 e-03	-	9.932484 e-04

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Microscopic model[5]

N-dimensional dynamical system with metric interaction

$$\begin{aligned} \dot{x}_N &= v(0) \\ \dot{x}_i &= v\left(\frac{M}{N}\sum_{j=1}^{N} w_\eta^N (x_{i+j} - x_i)\right) \qquad \text{for } i = N-1, \ldots, 1 \\ x_i(0) &= x_i^0 \end{aligned}$$

where w_η^N is a regularized (continuous) kernel

$$M = \int \rho_0(x)\, dx$$

$$x_1^0 := \sup\left\{x \in \mathbb{R}\colon\ \int_{-\infty}^{x} \rho_0(y)\, dy < \frac{M}{N}\right\}$$

$$x_i^0 := \sup\left\{x \in \mathbb{R}\colon\ \int_{x_{i-1}^0}^{x} \rho_0(y)\, dy < \frac{M}{N}\right\}, \qquad i = 2, \ldots, N$$

[5] [GoatinRossi, in preparation]

Microscopic model[5]

N-dimensional dynamical system with metric interaction

$$
\begin{aligned}
\dot{x}_N &= v(0) \\
\dot{x}_i &= v\left(\frac{M}{N}\sum_{j=1}^{N} w_\eta^N (x_{i+j} - x_i)\right) \qquad \text{for } i = N-1, \dots, 1 \\
x_i(0) &= x_i^0
\end{aligned}
$$

where w_η^N is a regularized (continuous) kernel

$$
\begin{aligned}
M &= \int \rho_0(x)\, dx \\
x_1^0 &:= \sup\left\{x \in \mathbb{R} \colon \int_{-\infty}^{x} \rho_0(y)\, dy < \frac{M}{N}\right\} \\
x_i^0 &:= \sup\left\{x \in \mathbb{R} \colon \int_{x_{i-1}^0}^{x} \rho_0(y)\, dy < \frac{M}{N}\right\}, \quad i = 2, \dots, N
\end{aligned}
$$

Related works:

- FTL $\to$ LWR:
  [ColomboRossi, Rend. Sem. Mat. Univ. Padova 2014]
  [DiFrancescoRosini, ARMA 2015]

[5] [GoatinRossi, in preparation]

Discrete maximum principle

Due to the non-increasing monotonicity of w_η, there holds

- $x_i^0 - x_{i-1}^0 \geq \ell \quad \forall i \qquad \Longrightarrow \qquad x_i(t) - x_{i-1}(t) \geq \ell \quad \forall i \quad \forall t > 0$
- $x_i^0 - x_{i-1}^0 \leq L \quad \forall i \qquad \Longrightarrow \qquad x_i(t) - x_{i-1}(t) \leq L \quad \forall i \quad \forall t > 0$

Convergence

Define the empirical measure

$$\rho^N(t,\cdot) := \frac{M}{N}\sum_{i=1}^{N} \delta_{x_i(t)}$$

weak solution of

$$\partial_t \rho^N + \partial_x \left(\rho^N \, v \left(\int w_\eta^N(y-x)\, d\rho^N(t,y) \right) \right) = 0$$

Convergence

Define the empirical measure

$$\rho^N(t,\cdot) := \frac{M}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$$

weak solution of

$$\partial_t \rho^N + \partial_x \left(\rho^N v\left(\int w_\eta^N(y-x)\, d\rho^N(t,y)\right)\right) = 0$$

Theorem [GoatinRossi, in preparation]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ with compact support. Then for any $T>0$ we have

$$\rho^N \rightharpoonup \rho$$

weakly in the sense of measures.

Convergence

Define the empirical measure

$$\rho^N(t,\cdot) := \frac{M}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$$

weak solution of

$$\partial_t\rho^N + \partial_x\left(\rho^N v\left(\int w_\eta^N(y-x)\,d\rho^N(t,y)\right)\right) = 0$$

Theorem [GoatinRossi, in preparation]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ with compact support. Then for any $T>0$ we have

$$\rho^N \rightharpoonup \rho$$

weakly in the sense of measures.

Proof. Relies on ∞-Wasserstein distance

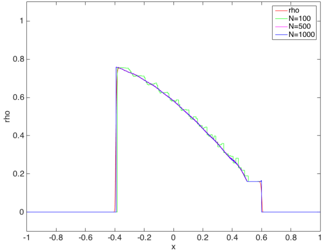
$$W_\infty(\mu,\nu) := \inf\left\{\lambda - \operatorname{ess\,sup}|y-x| : \lambda \in \Pi(\mu,\nu)\right\}$$

[ChampionDePascaleJuutinen, SIAM J. Math. Anal. 2008]

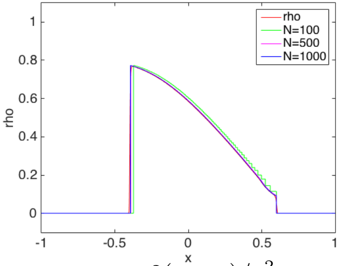
Micro-macro limit

Numerically, we consider

$$\tilde{\rho}_N(t,\cdot) := \sum_{i=1}^{N-1} \frac{M/N}{x_{i+1}(t) - x_i(t)} \chi_{[x_i(t),x_{i+1}(t)[}$$



$w_\eta = 1/\eta$



$w_\eta = 2(\eta - x)/\eta^2$

Outline of the talk

1. Non-local conservation laws
2. A traffic flow model with non-local velocity
3. Well-posedness
4. Numerical tests
5. Micro-macro limit
6. Perspectives

Perspectives

Some open problems:

- boundary conditions
- topological VS metric interactions
- finite acceleration models for pollution estimation

Perspectives

Some open problems:

- boundary conditions
- topological VS metric interactions
- finite acceleration models for pollution estimation

Thank you!