Intersection Models and Nash Equilibria for Traffic Flow on Networks

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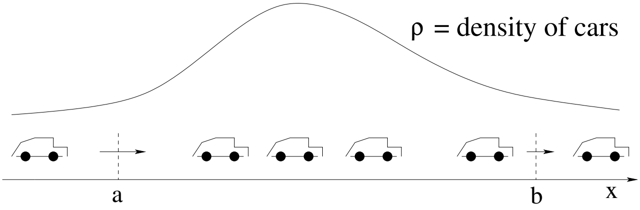
(in collaboration with Khai Nguyen)

A conservation law describing traffic flow

(Lighthill - Witham - Richards, 1955-56)



$$\rho_t + [\rho\, v_i(\rho)]_x \;=\; 0$$

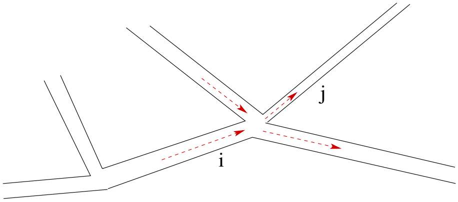
$v_i(\rho) \;=\;$ velocity of cars on road i (depends only on the density)

$f_i(\rho) \;=\; \rho\, v_i(\rho) \;=\;$ flux on the i-th road of the network

$$f_i'' \;<\; 0\,, \qquad\qquad f_i(0) \;=\; f_i(\rho_i^{jam}) \;=\; 0$$

Modeling traffic flow at a junction

incoming roads: $i \in \mathcal{I}$ outgoing roads: $j \in \mathcal{O}$



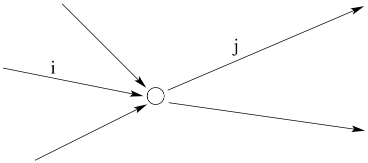
Boundary conditions account for:

- θ_{ij} = fraction of drivers from road i that turn into road j.
- c_i = relative priority of drivers from road i
  (fraction of time drivers from road i get green light, on average)

$$\sum_j \theta_{ij} = 1 \qquad \sum_i c_i = 1$$

Boundary conditions at junctions

incoming roads: $i \in \mathcal{I}$ outgoing roads: $j \in \mathcal{O}$



Boundary conditions should relate

- $\rho_i(t, 0-)$ $i \in \mathcal{I}$
- $\rho_j(t, 0+)$ $j \in \mathcal{O}$

depending on drivers' turning preferences θ_{ij}

Conservation equations: $$\sum_i f_i(\rho_i^-)\theta_{ij} \;=\; f_j(\rho_j^+) \qquad j \in \mathcal{O}$$

Boundary conditions for several incoming and outgoing roads

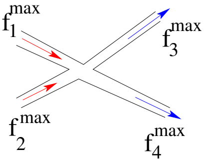
- H.Holden, N.H.Risebro, A mathematical model of traffic flow on a network of unidirectional roads, *SIAM J. Math. Anal.* **26**, 1995.
- G.M.Coclite, M.Garavello, B.Piccoli, Traffic flow on a road network, *SIAM J. Math. Anal.* **36**, 2005.
- M.Herty, S.Moutari, M.Rascle, Optimization criteria for modeling intersections of vehicular traffic flow, *Netw. Heterog. Media* **1**, 2006.
- M.Garavello, B.Piccoli, Conservation laws on complex networks, *Ann.I.H.Poincaré* **26** 2009.
- M.Garavello, B.Piccoli, *Traffic Flow on Networks*, AIMS, 2006.
- A.B., S.Canic, M.Garavello, M.Herty, and B.Piccoli, Flow on networks: recent results and perspectives, *EMS Surv. Math. Sci.* **1** (2014), 47–111.

$$\begin{cases} \rho_1, \ldots, \rho_N = \text{initial densities} \quad \text{(constant on each road)} \\ \theta_{ij} = \text{fraction of drivers from road } i \text{ that turn into road } j \end{cases}$$

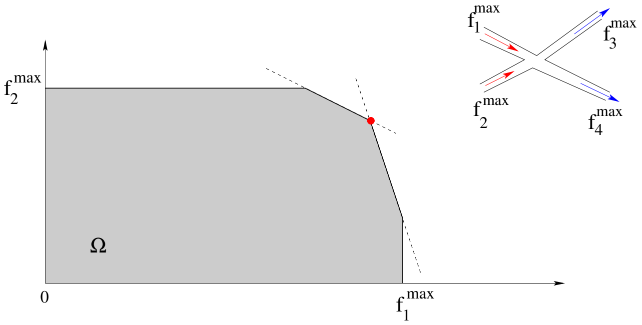
$$\begin{cases} f_i^{max} = \text{maximum flux on the incoming road } i \in \mathcal{I} \\ f_j^{max} = \text{maximum flux on the outgoing road } j \in \mathcal{O} \end{cases}$$

Feasible region $\Omega \subset \mathbb{R}^n$. Vector of incoming fluxes $(f_1, \ldots, f_n) \in \Omega$ iff

- $f_i \in [0, f_i^{max}] \qquad i \in \mathcal{I}$
- $\sum_i f_i \theta_{ij} \leq f_j^{max} \qquad j \in \mathcal{O}$



The feasible region Ω



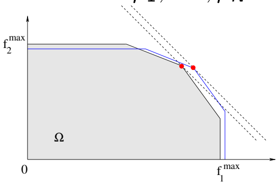
Riemann solver $\Longleftrightarrow$ rule for selecting a point in the feasible region Ω.

Natural choice: maximize the total flux through the node: $\displaystyle\sum_{i\in\mathcal{I}} f_i$

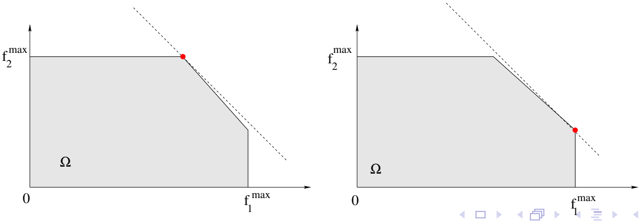
Continuity of the Riemann Solver

Selection rule: maximize the total flux $\sum_{i \in \mathcal{I}} f_i$

- If the turning preferences θ_{ij} remain constant, the fluxes f_i depend Lipschitz continuously on the Riemann data $\rho_1, \ldots, \rho_N$.

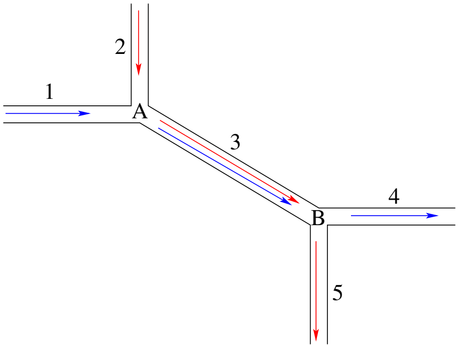


- The Riemann solver is discontinuous w.r.t. changes in the θ_{ij}



Why can the θ_{ij} vary in time?

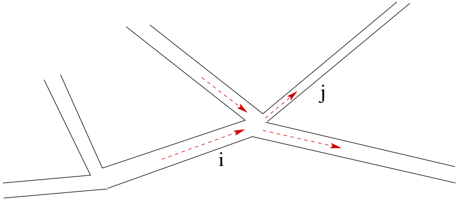
Drivers' turning preferences θ_{ij} must be determined as part of the solution



# of vehicles on road i that wish to turn into road j is conserved:

$$(\rho\theta_{ij})_t + (\rho v_i(\rho)\theta_{ij})_x \;=\; 0$$

Traffic flow on a network of roads

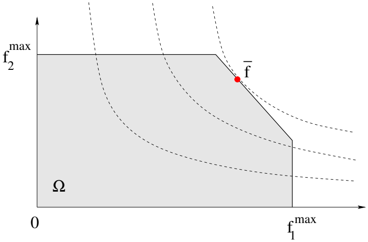


On the *i*-th incoming road, car flow is described by

$$\begin{cases} \rho_t + f_i(\rho)_x = 0 & \text{conservation law} \\ \theta_{ij,t} + v_i(\rho)\theta_{ij,x} = 0 & \text{linear transport equation} \end{cases}$$

θ_{ij} are *passive scalars,* relevant only at intersection

Continuous Riemann Solvers (A.B. - F.Yu, *Discr. Cont. Dyn. Syst.*, 2015)



- The selection rule: maximize $\prod_{i\in\mathcal{I}} f_i$ yields a Riemann solver which is **Hölder continuous** w.r.t. all variables

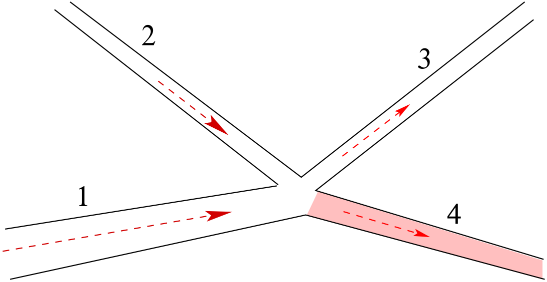
$$(\rho_i, \theta_{ij})_{i\in\mathcal{I}, j\in\mathcal{O}} \mapsto (f_i)_{i\in\mathcal{I}}$$

- One can also construct a Riemann solver which is **Lipschitz continuous** w.r.t. all variables
- Unfortunately all this is useless, because if the θ_{ij} are allowed to vary **the Cauchy problem is ill posed** anyway

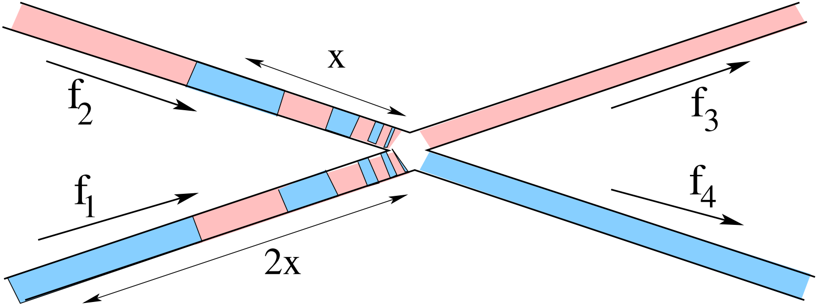
Ill-posedness of the Cauchy problem at intersections

Modeling assumptions

- If all cars arriving at the intersection can immediately move to outgoing roads, no queue is formed.
- If outgoing roads are congested, the inflow of cars from road 1 is twice as large as the inflow from road 2.



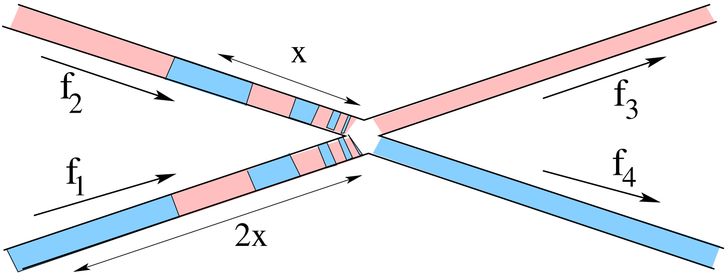
Example 1: θ_{ij} with unbounded variation, two solutions



$$f_k(\rho) \;=\; 2\rho - \rho^2 \qquad \text{maximum flux on every road:} \quad f_k^{max} = 1$$

Initial data: $\rho_k \;=\; 1, \qquad k = 1, 2, 3, 4$

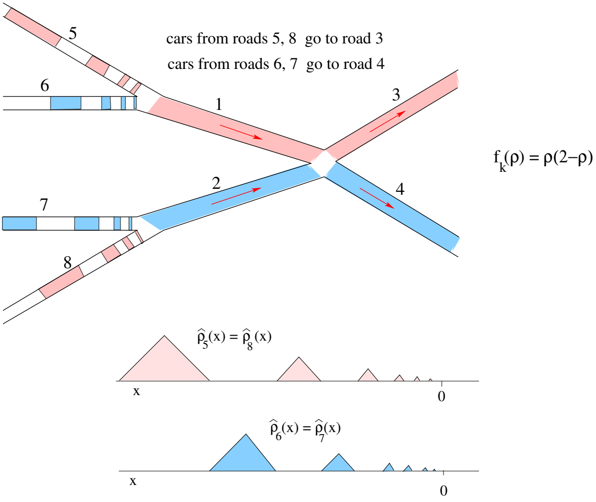
$$\hat{\theta}_{13}(x) \;=\; \hat{\theta}_{24}(x) \;=\; \begin{cases} 1 & \text{if} \quad -2^{-n} < x < -2^{-n-1}, \qquad n \text{ even} \\ 0 & \text{if} \quad -2^{-n} < x < -2^{-n-1}, \qquad n \text{ odd} \end{cases}$$



Solution 1. Incoming fluxes: $f_1(t,0) = 1, \quad f_2(t,0) = 1$

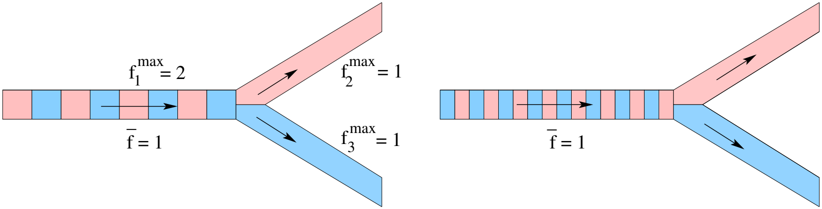
Solution 2. Incoming fluxes: $f_1(t,0) = \frac{2}{3}, \quad f_2(t,0) = \frac{1}{3}$

Example 2: θ_{ij} constant, $Tot.Var.(\rho_i)$ small, two solutions

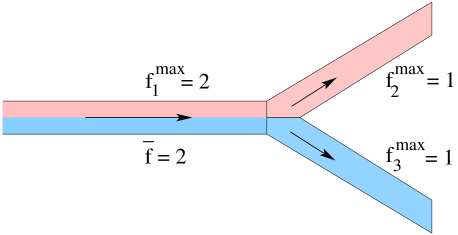


At some time $T > 0$, the same initial data as in Example 1 is created at the junction of roads 1 and 2

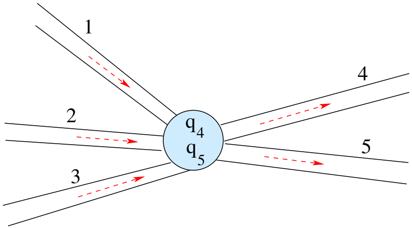
Example 3: lack of continuity w.r.t. weak convergence



As $n \to \infty$, the weak limit is $\quad \theta_{12} = \theta_{13} = \frac{1}{2}$



An intersection model with buffers



- the intersection contains a buffer with finite capacity (a traffic circle)
- $t \mapsto q_j(t) =$ queues in front of outgoing roads $j \in \mathcal{O}$, within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the fastest possible rate

- M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* 2009.
- M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* 2012.
- M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems, 2013.*

Toward the analysis of global optima and Nash equilibria, we need

- well posedness for $\mathbf{L}^\infty$ initial data ρ_k^0, θ_{ij}^0
- continuity of travel time w.r.t. weak convergence

$$\begin{cases} \rho_{k,t} + f_k(\rho_k)_x = 0 & \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} = 0 & \text{linear transport equations} \end{cases}$$

Intersection models with buffers

(A.B., K.Nguyen, *Netw. Heter. Media*, 2015)

$q_j(t)$ = size of the queue, inside the intersection, of cars waiting to enter road j

(SBJ) - Single Buffer Junction

$M > 0$ = maximum number of cars that can occupy the intersection

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

Incoming fluxes $\bar{f}_i$ satisfy $$\bar{f}_i \;\leq\; c_i\Big(M - \sum_{j\in\mathcal{O}} q_j\Big), \qquad i \in \mathcal{I}$$

Well-posedness of the Cauchy problem with buffers

Theorem A.B.- K.Nguyen, *Netw. Heter. Media*, 2015.

Assume that the flux functions satisfy

$$f_k'' < 0, \qquad f_k(0) = f_k(\rho_k^{jam}) = 0 \qquad\qquad k \in \mathcal{I} \cup \mathcal{O}$$

Consider any $\mathbf{L}^\infty$ initial data $\quad \rho(0,x) = \bar{\rho}_k(x) \in [0, \rho_k^{jam}]$,

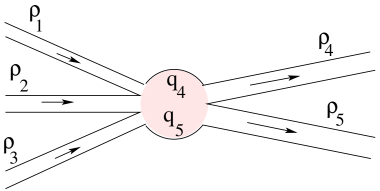
$$q_j(0) \;=\; \bar{q}_j\,, \qquad \theta_{ij}(0,x) = \bar{\theta}_{ij} \in [0,1] \qquad \text{with} \sum_{j\in\mathcal{O}} \bar{q}_j < M, \qquad \sum_{j\in\mathcal{O}} \bar{\theta}_{ij}(x) = 1$$

Then the Cauchy problem has a unique entropy admissible solution, defined for all $t \geq 0$.

Moreover, the travel times depend continuously on the initial data, in the topology of weak convergence.

$$\bar{\rho}_k^n(x) \rightharpoonup \bar{\rho}_k \qquad\qquad \bar{\rho}_i^n \bar{\theta}_{ij}^n \rightharpoonup \bar{\rho}_i \bar{\theta}_{ij}, \qquad\qquad \bar{q}_j^n \to \bar{q}_j$$

Variational formulation (A.B., K.Nguyen)



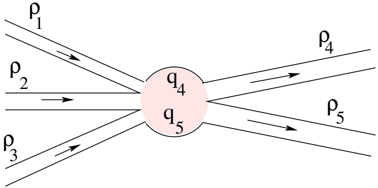
(q_4, q_5) —Lax formula→ $(V_1, V_2, V_3, V_4, V_5)$ —boundary values→ (q_4, q_5)

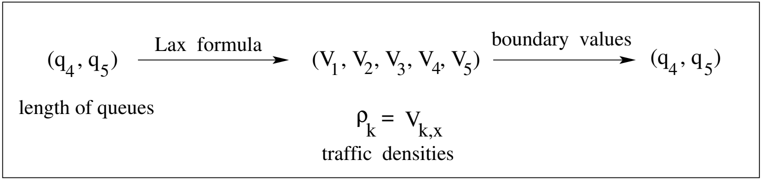
length of queues

$$V_k(t,x) = \int^x \rho_k(t,x)\,dx$$

- If the queue sizes $q_j(t)$ within the buffer are known, then the initial-boundary value problems can be independently solved along each incoming road.

  These solutions can be computed by solving suitable variational problems.
  From the value functions V_k, the traffic density $\rho_k = V_{k,x}$ along each incoming or outgoing road is recovered by a Lax type formula.

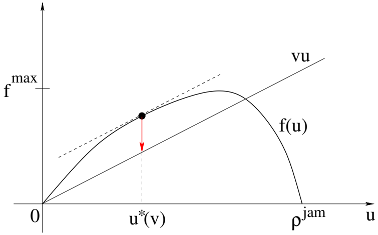


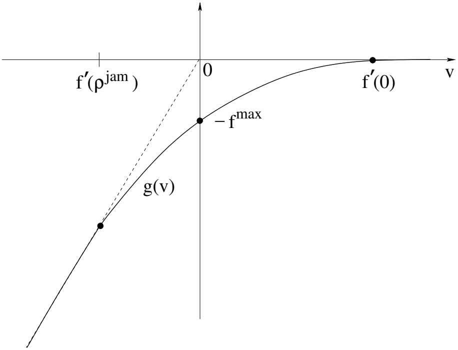


- Conversely, if these value functions V_k are known, then the queue sizes q_j can be determined by balancing the boundary fluxes of all incoming and outgoing roads
- The solution of the Cauchy problem is obtained as the unique fixed point of a contractive transformation
- The present model accounts for backward propagation of queues along roads leading to a crowded intersection, it achieves well-posedness for general $\mathbf{L}^\infty$ data, and continuity of travel time w.r.t. weak convergence

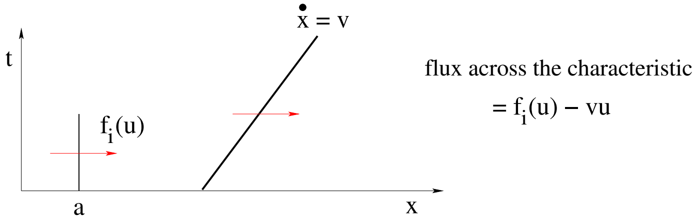
The Legendre transform of the flux function f

Legendre transform: $$g(v) \doteq \inf_{u\in[0,\,\rho^{jam}]} \{vu - f(u)\}$$





A variational problem describing traffic on road $i \in \mathcal{I}$



$$g_i(v) \;=\; -\,[\text{flux of cars from left to right, across the characteristic}]$$

For boundary conditions **(SBJ)**, define

$$h_i(\mathbf{q}) \;\doteq\; \min\left\{ f_i^{max}\,,\quad c_i \cdot \left(M - \sum_{j \in \mathcal{O}} q_j \right) \right\} \qquad\qquad i \in \mathcal{I}$$

Incoming roads with boundary condition **(SBJ)**

initial data: $\overline{V}_i(x) \doteq \int_{-\infty}^{x} \bar{\rho}_i(y)\,dy,$ queue lengths: $q_j(t)\,, \quad j \in \mathcal{O}$

To find $V_i(\bar{t}, \bar{x})$, consider the optimization problem:

$$\text{maximize:} \qquad \overline{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t))\,dt$$

$$\text{running payoff:} \qquad L_i(x(t), \dot{x}(t)) = \begin{cases} g_i(\dot{x}(t)) & \text{if} \quad x(t) < 0 \\ -h_i(\mathbf{q}(t)) & \text{if} \quad x(t) = 0 \end{cases}$$

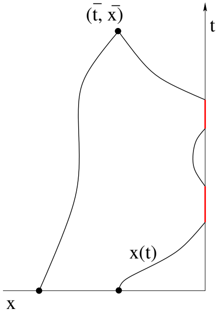
terminal condition: $x(\bar{t}) = \bar{x}$

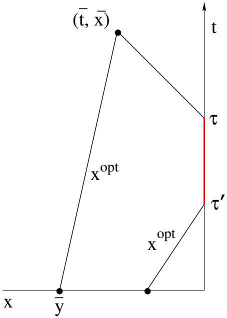
Optimization Problem 1

$$\text{maximize:} \qquad \overline{V}_i(x(0)) + \int_0^{\bar{t}} L_i(x(t), \dot{x}(t))\, dt$$

among all absolutely continuous functions $x : [0, \bar{t}] \mapsto \mathbb{R}$ such that

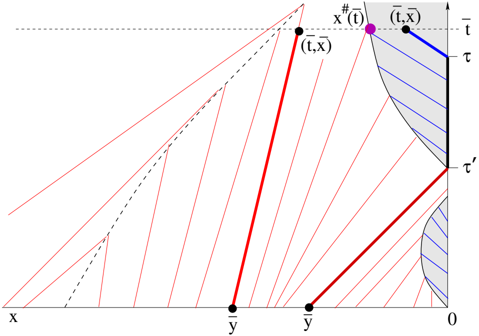
$$x(\bar{t}) \;=\; \bar{x}, \qquad\qquad x(t) \leq 0 \quad \text{for all } t \in [0, \bar{t}]$$





The Value Function V_i

- an optimal solution x^{opt} exists, and is the concatenation of at most three affine functions
- $\dot{x}^{opt} \in [f_i'(0)\,,\ f_i'(\rho_i^{jam})]$ is the speed of a characteristic
- the traffic density $\rho_i(t,x) = V_{i,x}(t,x)$ is an entropy solution of the conservation law, satisfying initial + boundary conditions



$$V_i(t,x) \doteq \max\left\{ \max_{y\le 0}\left[\overline{V}_i(y) + t\,g_i\left(\frac{x-y}{t}\right)\right] ,\right.$$

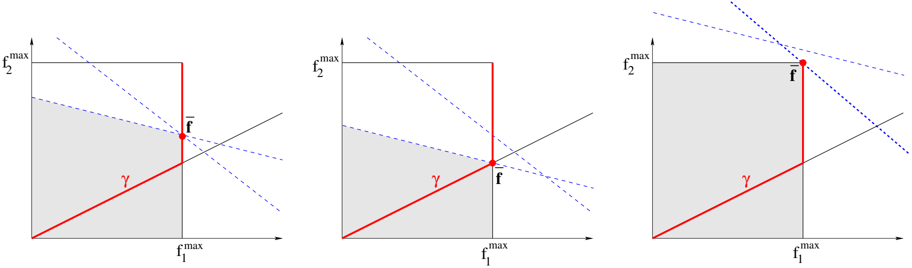
$$\left. \max_{0\le\tau'\le\tau\le t,\ y\le 0}\left[\overline{V}_i(y) + \tau'\,g_i\left(\frac{-y}{\tau'}\right) - \int_{\tau'}^{\tau} h_i(\mathbf{q}(s))\,ds + (t-\tau)\,g_i\left(\frac{x}{t-\tau}\right)\right]\right\}.$$

$V_i(t,x)$ = total amount of cars which at time t are still inside the half line $]-\infty, x]$

$\overline{V}_i(0) - V_i(t,0)$ = total amount of cars which have exited from road i during $[0,t]$

The limit Riemann Solver for buffer of vanishing size

Letting the size of the buffer $M \to 0$ one obtains a Riemann Solver which is **Lipschitz continuous** w.r.t. all variables ρ_i, θ_{ij}



$$s \mapsto \gamma(s) = (f_1(s), \ldots, f_m(s)), \qquad f_i(s) \doteq \min\{c_i s\,,\ f_i^{max}\}$$

Then the incoming fluxes are $\bar{f}_i = f_i(\bar{s})$

$$\text{where:} \qquad \bar{s} = \max\left\{ s \geq 0\,;\ \sum_{i \in \mathcal{I}} f_i(s)\,\theta_{ij} \leq f_j^{max} \quad \text{for all } j \in \mathcal{O} \right\}.$$

Optimization Problems for Traffic Flow on a Network

- Existence of a globally optimal solution
- Existence of a Nash equilibrium solution

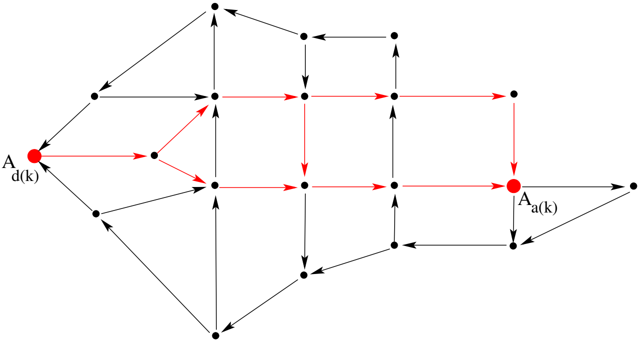
Basic setting

n groups of drivers with different origins and destinations, and different costs

Drivers in the k-th group depart from $A_{d(k)}$ and arrive to $A_{a(k)}$

can use different paths $\Gamma_1, \Gamma_2, \ldots$ to reach destination

Departure cost: $\varphi_k(t)$ arrival cost: $\psi_k(t)$



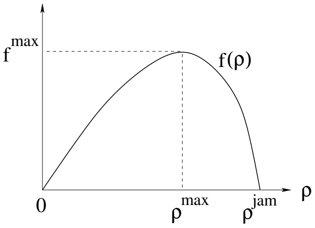
Basic assumptions

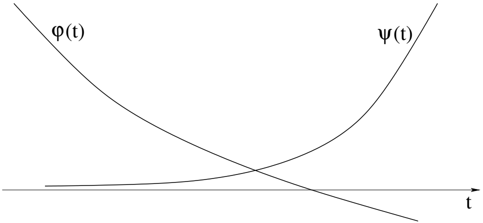
(A1) The flux functions $\rho \mapsto f_i(\rho) = \rho\, v(\rho)$ are all strictly concave down.

$$f_i(0) \;=\; f_i(\rho_i^{jam}) \;=\; 0\,, \qquad\qquad f_i'' \;<\; 0\,.$$

(A2) For each group of drivers $k = 1, \ldots, N$, the cost functions φ_k, ψ_k satisfy

$$\varphi_k' \;<\; 0, \qquad \psi_k, \psi_k' \;<\; 0, \qquad \lim_{|t|\to\infty} \Big(\varphi_k(t) + \psi_k(t)\Big) \;=\; +\infty$$





Admissible departure rates

G_k = total number of drivers in the k-th group, $\quad k = 1, \ldots, n$

Γ_p = viable path to reach destination, $\quad p = 1, \ldots, N$

$t \mapsto \bar{u}_{k,p}(t)$ = departure rate of k-drivers traveling along the path Γ_p

The set of departure rates $\{\bar{u}_{k,p}\}$ is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0\,, \qquad \sum_p \int_{-\infty}^{\infty} \bar{u}_{k,p}(t)\,dt \;=\; G_k \qquad k = 1, \ldots, n$$

$\tau_p(t)$ = arrival time for a driver starting at time t, traveling along Γ_p

(depends on the overall traffic conditions)

If this is a k-driver, his total cost is $\varphi_k(t) + \psi_k(\tau_p(t))$.

Optima and Equilibria

An admissible family $\{\bar{u}_{k,p}\}$ of departure rates is **globally optimal** *if it minimizes the sum of the total costs of all drivers*

$$J(\bar{u}) \doteq \sum_{k,p} \int \Big(\varphi_k(t) + \psi_k(\tau_p(t))\Big)\bar{u}_{k,p}(t)\,dt$$

An admissible family $\{\bar{u}_{k,p}\}$ of departure rates is a **Nash equilibrium** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

$$\varphi_k(t) + \psi_k(\tau_p(t)) = C_k \qquad \text{for all } t \in Supp(\bar{u}_{k,p})$$

$$\varphi_k(t) + \psi_k(\tau_p(t)) \geq C_k \qquad \text{for all } t \in \mathbb{R}$$

Existence results

Theorem (A.B. - Khai Nguyen, *Netw. Heter. Media*, 2015)

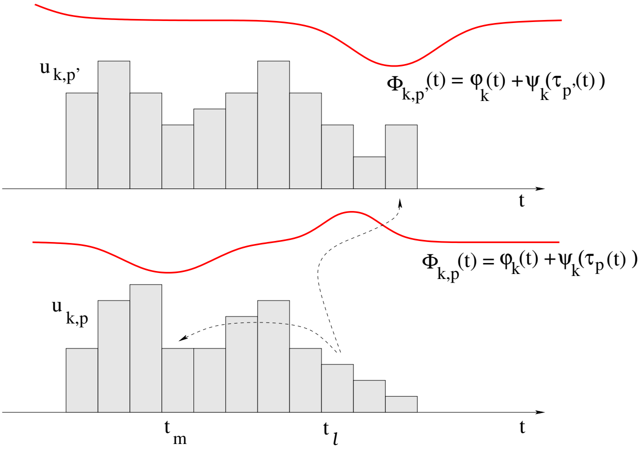
Under the assumptions (A1)-(A2), on a general network of roads, there exists at least one globally optimal solution.

If, in addition, the travel time admits a uniform upper bound, then a Nash equilibrium exists.

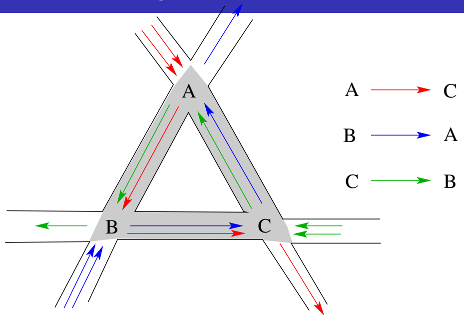
- Proof is achieved by finite dimensional approximations
  + a topological argument (relying on the continuity of the travel time w.r.t. weak convergence of the departure rates)
- For a single group of drivers on a single road, solutions are unique.
  Uniqueness is not expected to hold, on a general network.
- An earlier existence result was proved in
  A.B. - Ke Han, Netw. & Heter. Media, 2013,
  with highly simplified boundary conditions at road intersections.

Construction of Nash equilibria

By finite dimensional approximations + topological methods



Can traffic get completely stuck ?



- Assume: at each node, equal numbers of cars are allowed to enter from the two incoming roads. Then:
- for every two cars entering, only one exits the triangle of roads ABC
- at any time t,

  [# of cars that has reached destination] $\leq$ [# of cars inside the triangle ABC]
- only finitely many cars can reach destination. All the others are stuck forever.

References

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