The Spherical Harmonics Discrete Ordinate Method for Atmospheric Radiative Transfer

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SHDOM Capabilities

- Unpolarized radiative transfer with a Cartesian grid (1D, 2D, or 3D)
- Solar (external collimated) or thermal (internal isotropic) sources
- Arbitrary input fields of extinction, phase function, etc.
- Monochromatic or spectrally integrated (with k-distribution for molecular absorption lines)
- Spatially variable surface reflectance; various types of bidirectional reflectance functions
- Periodic or open boundary conditions
- Flexible output: intensity in many directions, hemispheric irradiance, absorbed energy, etc. at all or some grid points

Representation of the Radiation Field

Source function $S(\mu, \phi; x, y, z)$ represented:

- **Spatially – discrete Cartesian grid points**
- **Angularly – spherical harmonic series:**

$$S(\mu, \phi) = \sum_{m=0}^{M} \sum_{l=m}^{L} Y_{lm}(\mu, \phi) S_{lm}$$

[$\mu = \cos\theta$, θ is zenith angle, ϕ is azimuth angle.]

Intensity field $I(\mu, \phi; x, y, z)$ is derived from source function by integrating the radiative transfer equation (RTE).

SHDOM Solution Iterations

Iterate 1 to 4:

1. **Transform source function to discrete ordinates,** $S(\mu_j, \phi_k)$**.**
2. **Integrate source function in RTE to get intensity,** $I(\mu_j, \phi_k)$**.**
3. **Transform intensity back to spherical harmonics,** I_{lm}**.**
4. **Convert intensity to source function in spherical harmonics,** S_{lm}**.**

Why Spherical Harmonics?

1. **The scattering integral is particularly simple and fast:**

$$S_{lm} = \varpi \frac{\chi_l}{2l+1} I_{lm} + \mathcal{S}_{lm}$$

[$\mathcal{S}$ is the internal source, ϖ is the single scattering albedo, and χ_l are the phase function moments, $P(\cos\Theta_s) = \sum_{l=0}^{N} \chi_l \mathcal{P}_l(\cos\Theta_s)$].
The scattering integral takes of order N operations, compared to the Discrete Ordinate Method (S_N method) with N^2 operations (N is number of ordinates or SH terms).

2. **Spherical harmonics can represent the radiation field with less computer memory: few SH terms in the source function are needed if a grid point has no scattering, the scattering is smooth, or the intensity field is smooth.**

What About the Cost of the SH-DO Transforms?

Spherical harmonics source function transform to discrete ordinates:

$$S(\mu_j, \phi_k) = \sum_{m=0}^{M} \cos(m\phi_k) \sum_{l=m}^{L} \Lambda_{lm}(\mu_j) S_{lm}$$

[Λ_{lm} is the normalized associated Legendre function.]

Discrete ordinate intensity transform to spherical harmonics:

$$I_{lm} = \sum_{j=1}^{N_\mu} w_j \Lambda_{lm}(\mu_j) \sum_{k=1}^{N_\phi} \hat{w}_{jk} \cos(m\phi_k) I(\mu_j, \phi_k)$$

The azimuthal transform partially separates from the zenith angle transform.

Operations count: both SH-DO transforms = $9N^{3/2}$ for N discrete ordinates; S_N method source function = $2N^2$.

Why Not a Pure Spherical Harmonics Approach?

- **A purely spherical harmonics method is very poor at streaming radiation in clear sky:**
  Unphysical oscillations occur at cloud boundaries;
  Ill-conditioned matrices give very slow convergence.
- **Discrete ordinates physically model the streaming of radiation.**
- **Spherical harmonics work well in the diffusive regime.**

SHDOM tries to marry the advantages of spherical harmonics and discrete ordinates.

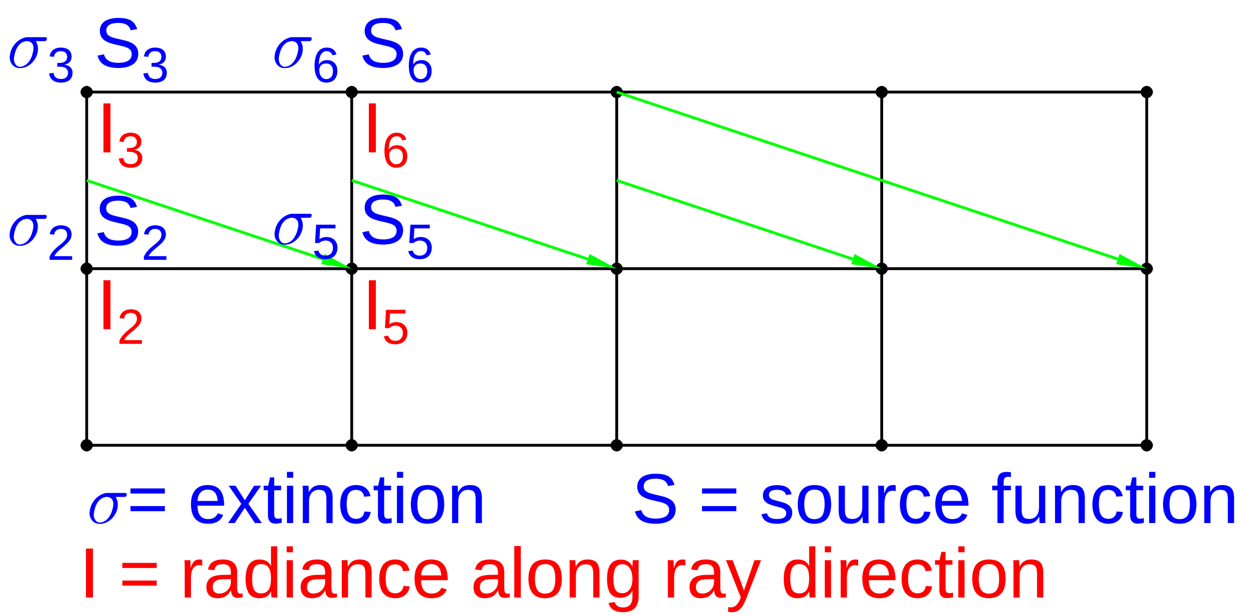
Discrete ordinate set: μ_j from Gaussian quadrature; ϕ_k equally spaced, but fewer ϕ_k for $|\mu_j|$ near 1. Choose $L = N_\mu - 1$ and $M = N_\phi/2 - 1$ for which SH functions are orthogonal.

Integrating the RTE along Discrete Ordinates

Integrate source function in RTE backwards along ordinate to get intensity $I(\mu_j, \phi_k)$ at each grid point.

$$I(s) = \exp\left[-\int_0^s \sigma(s')ds'\right] I(0) \; + \int_0^s \exp\left[-\int_{s'}^s \sigma(t)dt\right] S(s')\sigma(s')ds'$$

σ is extinction, s is distance along path, $I(0)$ is cell entering intensity.



Integrating the RTE along Discrete Ordinates (2)

- **Source function (S) and extinction (σ) at cell entering point are bilinearly interpolated between four grid points of cell face.**
- **Extinction (σ) and product (σS) are linearly interpolated between cell entering point and exitting (grid) point.**
- **No exact formula for RTE integral with linear σ and σS: use first order expression in s for small optical path, formula exact for either constant S or σ for $\tau > 2$.**
- **Integration along ray goes back to face with known intensities.**
- **Sweeping order through grid is designed to minimize number of cells traced.**

Adaptive SHDOM

Adaptive spherical harmonics truncation:

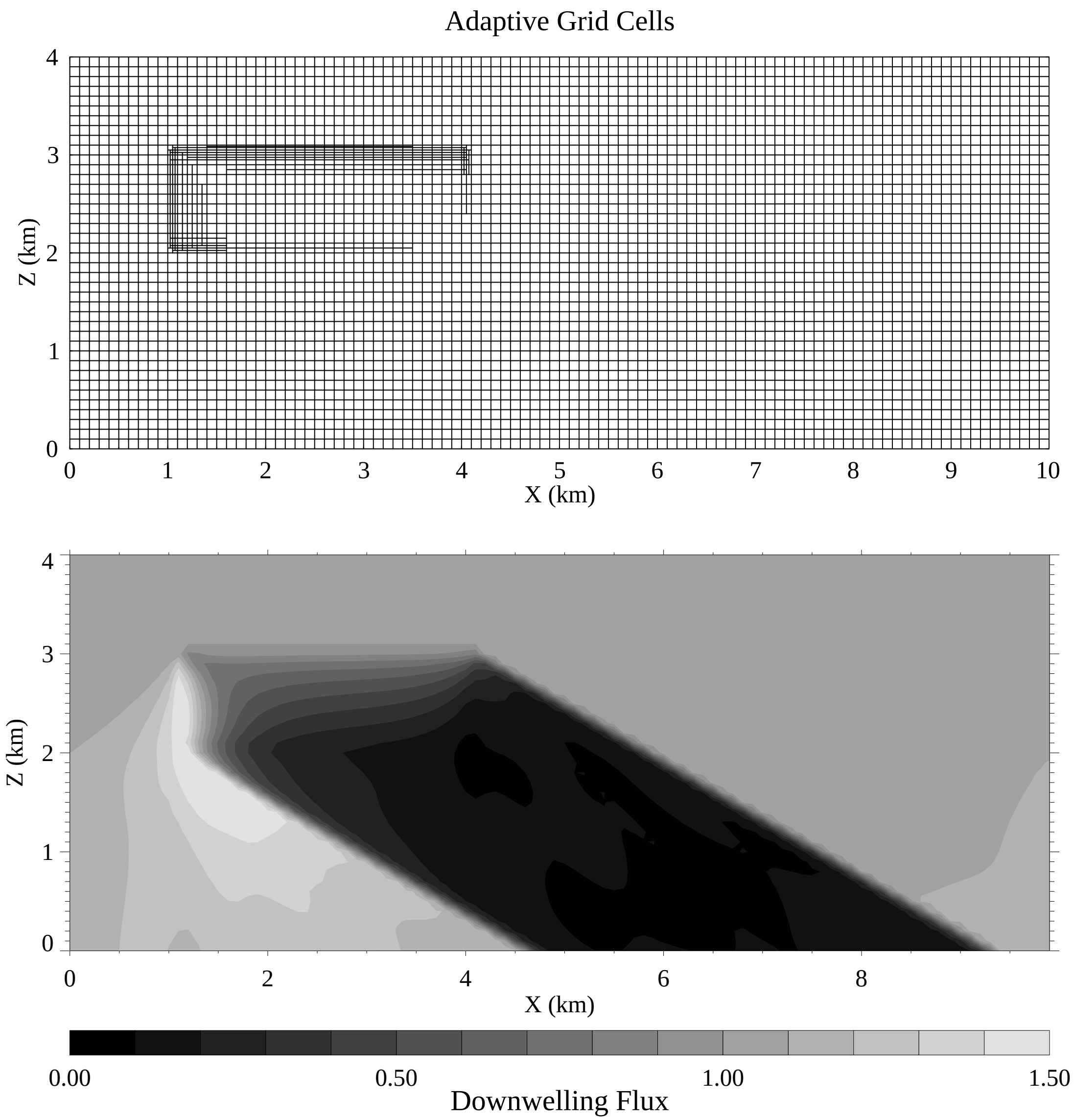
- L **varies with grid point.**
- **Only store SH terms above a threshold.**

Adaptive grid:

- **Starting with regular base grid, cells are split in half (in X, Y, or Z).**
- **Splitting criterion related to change in source function across cell**

$$C = (1 - e^{-\tau}) \; \frac{1}{\bar{\sigma}} \left\{ \frac{1}{4\pi} \sum_{lm} \left[\sigma_2 S_{lm}^{(2)} - \sigma_1 S_{lm}^{(1)} \right]^2 \right\}^{1/2}$$

- **Cells with largest criterion are split first; cell splitting accuracy is gradually lowered during iterations.**
- **Adaptive grid implemented with tree data structure.**



SHDOM example for box cloud with optical depth 20 and Mie phase function.
Solar zenith angle $= 60^\circ$, $N_\mu = 8$, $N_\phi = 16$, splitacc$= 0.03$.

Solution Method Details

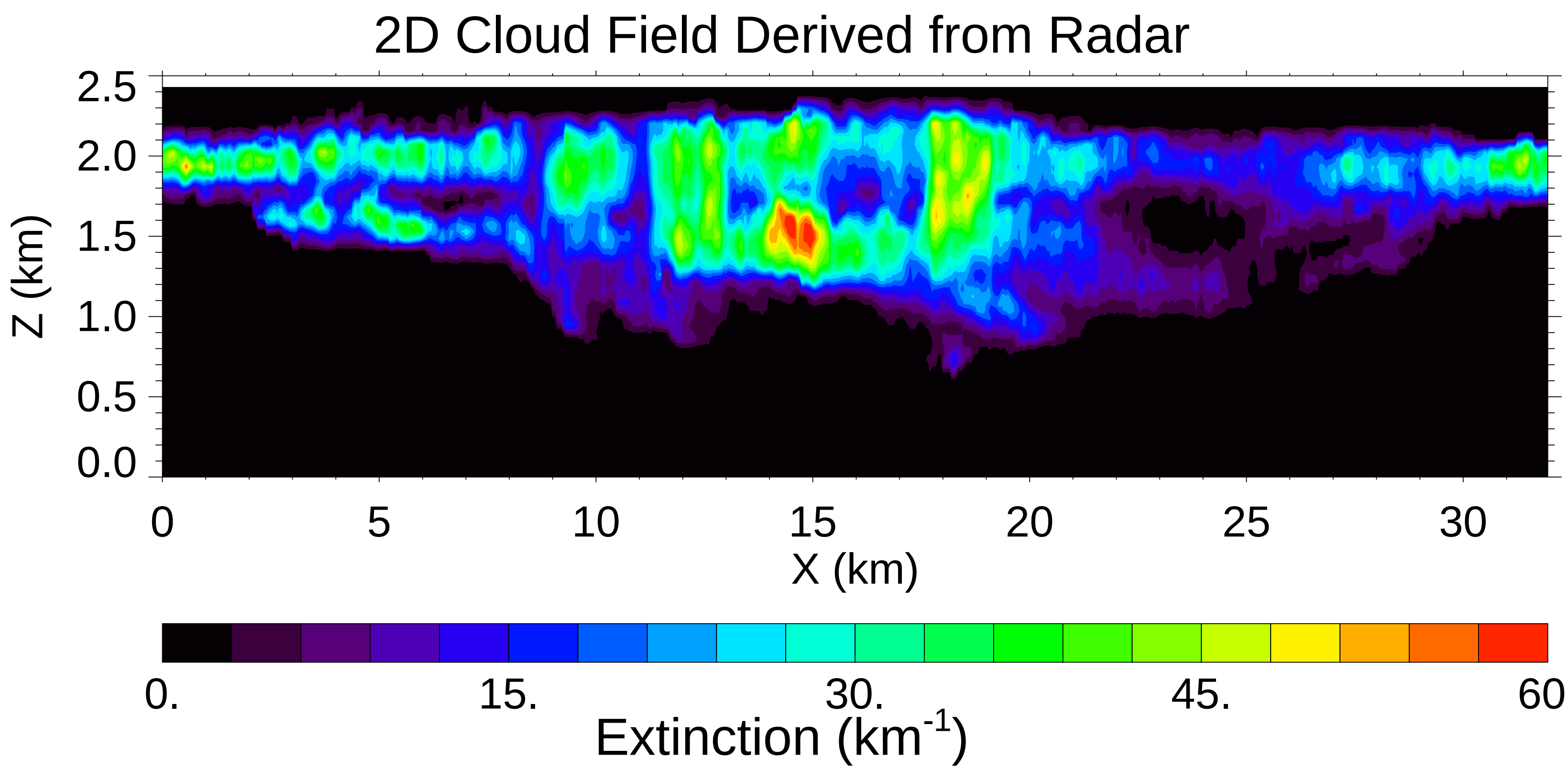
- **δ scale RTE for forward peaked phase functions ($\delta - M$ of Wiscombe, 1977).**
- **Boundary conditions: implemented easily with discrete ordinates.**
- **Phases 1 to 3 of iteration are done in loop over μ_j: full discrete ordinate source function/intensity field never in memory.**
- **Convergence acceleration speeds solution in optically thick scattering media: $\mathbf{S}'^{(n)} = \mathbf{S}^{(n)} + a(\mathbf{S}^{(n)} - \mathbf{S}^{(n-1)})$**
  2D geometric acceleration: a from 3 iterations of $\mathbf{S}$.
- **Memory use dominated by 3 arrays for SH terms at all grid points; $274N_{pts}$ words for $N_\mu = 8$, $N_\phi = 16$ with N_{pts} grid points.**

Computing Radiometric Output

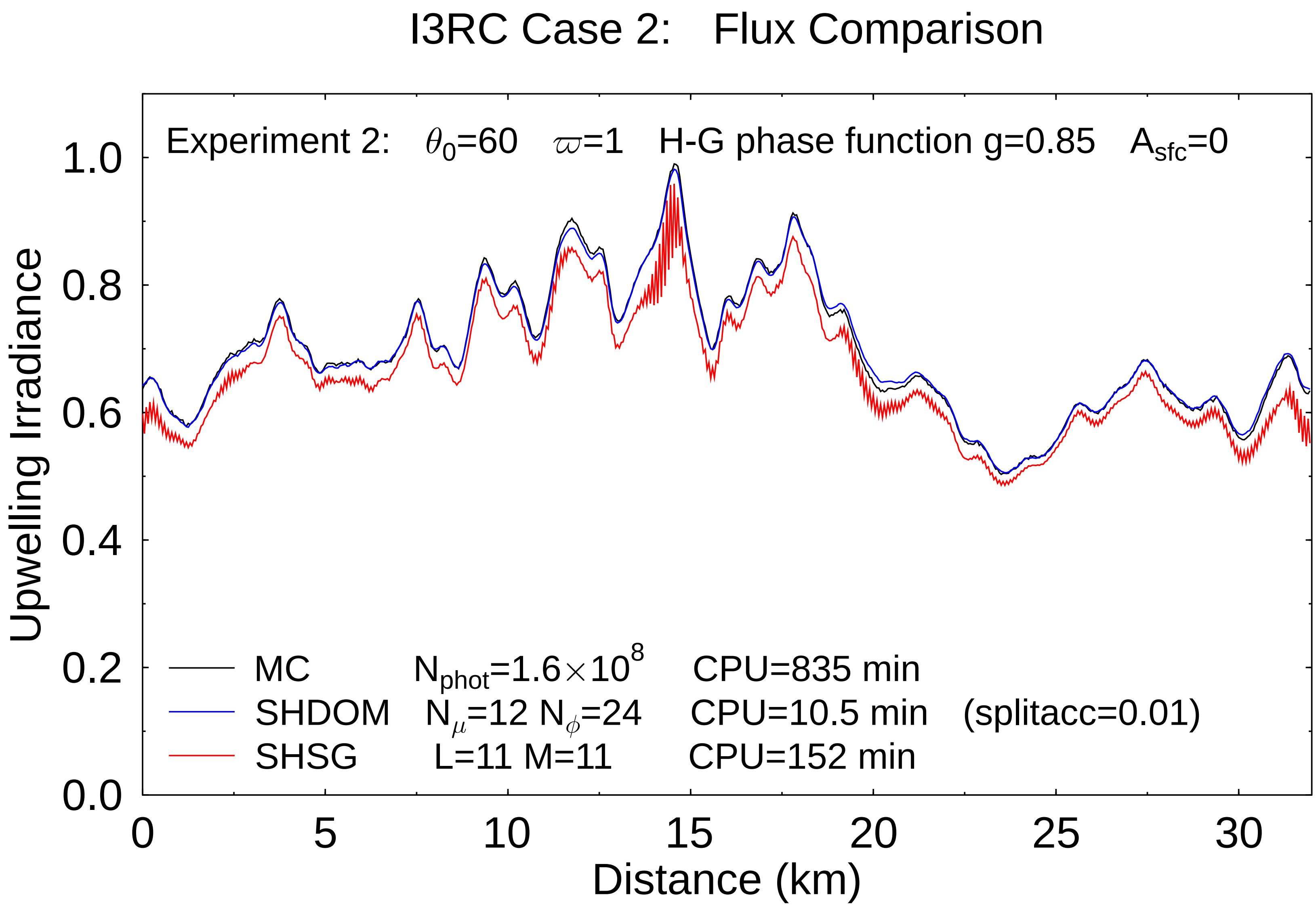
- **Hemispheric irradiance from quadrature integration of $I(\mu_j, \phi_k)$.**
- **Mean intensity and net fluxes from first four terms of I_{lm}.**
- **Net flux convergence (heating) from absorption coefficient times mean intensity.**
- **Intensity at specified angles/locations from integrating the source function. Use correct phase function for first order scattering and smoothed phase function solution (from truncated SH series) for higher scattering orders (Nakajima and Tanaka, 1988).**

SHDOM Compared with Other Methods

Intercomparison of 3D Radiation Codes (I3RC) case 2



$N_x = 640$, $N_z = 55$; optical depth (τ) ranges from 7 to 52.



MC = Monte Carlo with maximal cross section method. SHSG = 2D pure spherical harmonic method, conjugate-gradient solution (Evans, 1993).

Explicit vs Monte Carlo Radiative Transfer Methods

Explicit methods solve for the whole discrete radiation field.

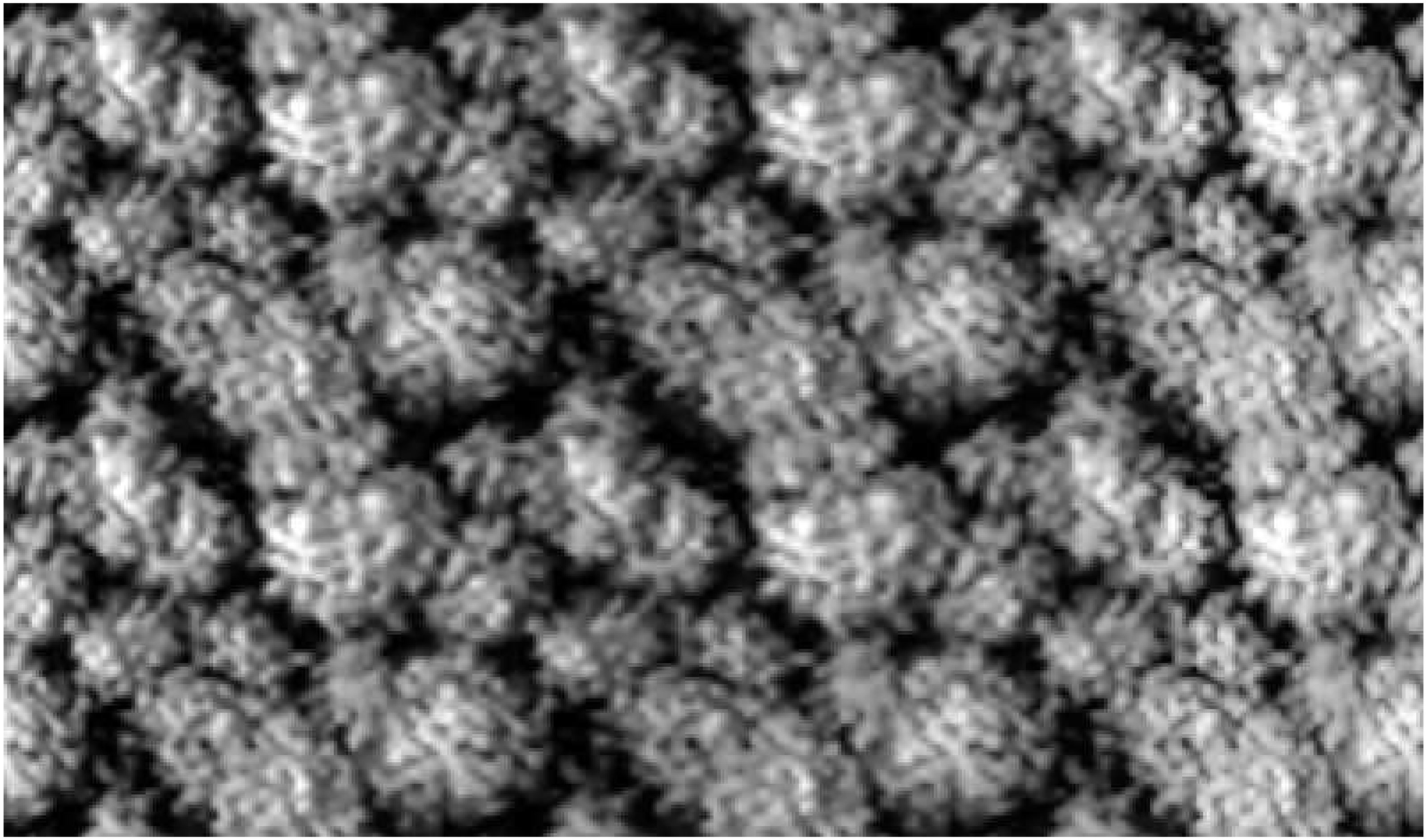
Advantages relative to Monte Carlo models:

- **Once radiation field is solved, get many outputs quickly.**
- **Faster for low dimensional problems.**

Disadvantages relative to Monte Carlo:

- **Slower for small number of outputs in 3D.**
- **Usually requires optically thin cells (small domains).**
- **Much harder to characterize accuracy.**
- **Accuracy probably improves more slowly with CPU.**

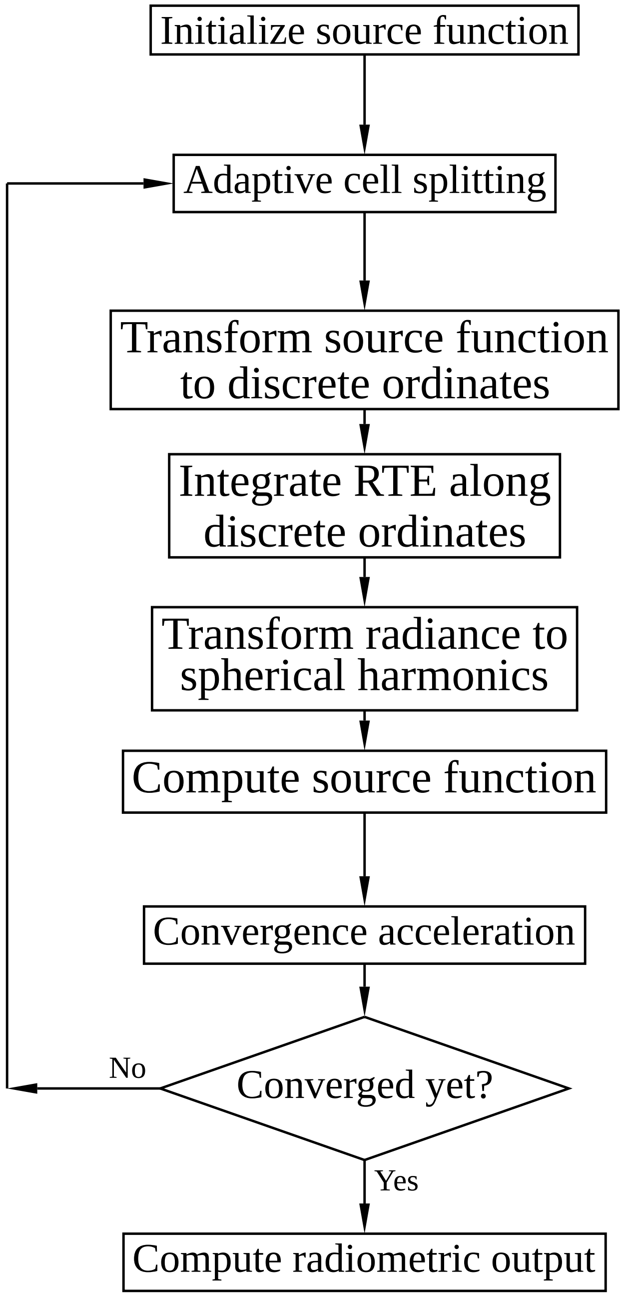
SHDOM Visualization Output: Cross-Track Scanning



Numerical cloud simulation: $N_x = 64$, $N_y = 64$, $N_z = 20$, $\langle\tau\rangle = 7.1$, $\tau_{max} = 27$.
SHDOM parameters: $\theta_0 = 60^\circ$, $N_\mu = 8$, $N_\phi = 16$, splitacc$= 0.10$.
Timing: 20 minutes for SHDOM solution on 1.1 GHz Pentium 3; 900 pixels per second for visualization.

SHDOM Visualization Output: Camera on the Ground

Flowchart Summary of SHDOM



Evans, K. F., 1998: The spherical harmonics discrete ordinate method for three-dimensional atmospheric radiative transfer. *J. Atmos. Sci.*, 55, 429–446.