Braid Group, Gauge Invariance and Topological Order

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Topological Quantum Computing
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Outline

- Motivation: Topological Matter (Phases)
- IQHE: Topological Number
- IQHE: Bulk-Edge Connection (omitted)
- FQHE: Ground State Degeneracy
- FQHE: Fractional Quantum Number (Charge)
- FQHE: Fractional Statistics
- Classification of Abelian Topological Orders
- Epilogue or Prologue:

  Non-abelian Topological Order

Topological Order: Beyond the Landau's Theory

- Novel phases **at T=0:** due to **quantum** effects
- **No** symmetry breaking
- **No** order parameter(s)
- Partly manifested by a **topological** number**,** that is **robust** against weak disorders and interactions

Examples: fractional quantum Hall effect, frustrated spin models, Mott insulators, etc.

Properties of FQHE

- Manifested by an exotic **topological** number**,** **robust** against weak disorders & interactions
- Close **relation** between **bulk and edge**
- **Topology** dependent **GSD**

  (Ground State Degeneracy)
- **Fractionalization** of quantum numbers
- **Fractional statistics** of quasiparticles

Many-body Definition of Chern Number (I)

Niu, Thouless, Y.S. Wu (1985)

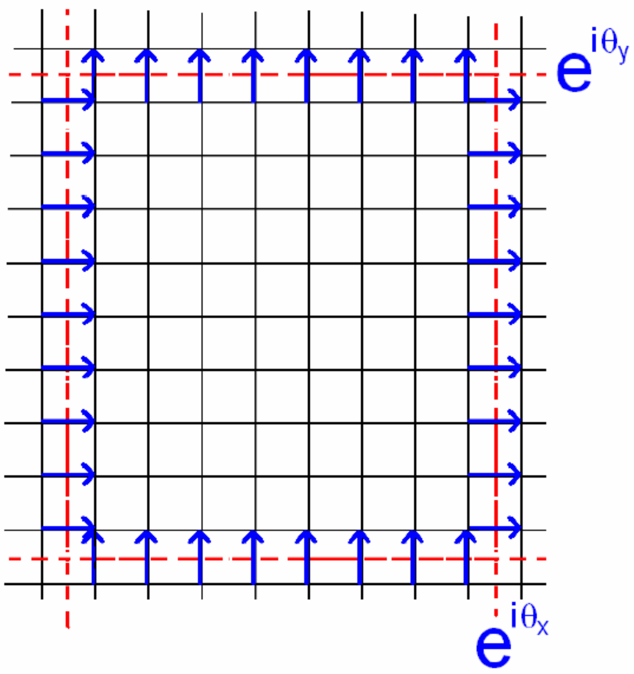
- *k*-space topology may *not be appropriate* for general many-body systems.
- Many-body definition of Chern number: Twist boundary condition
- General Many-body Hamiltonian:

$$H_0 = -\sum_{\langle ij \rangle} t_{ij,\alpha\beta} c^\dagger_{i\alpha} c_{j\beta} + H_{\text{int}}$$

$$t_{ij} \to \begin{cases} t_{ij} e^{i\theta_x}, & \langle ij \rangle \text{ cross } L_x \\ t_{ij} e^{i\theta_y} & \langle ij \rangle \text{ cross } L_y \end{cases}$$

- Parameterized Hamiltonian

$$H = H(\theta_x, \theta_y)$$



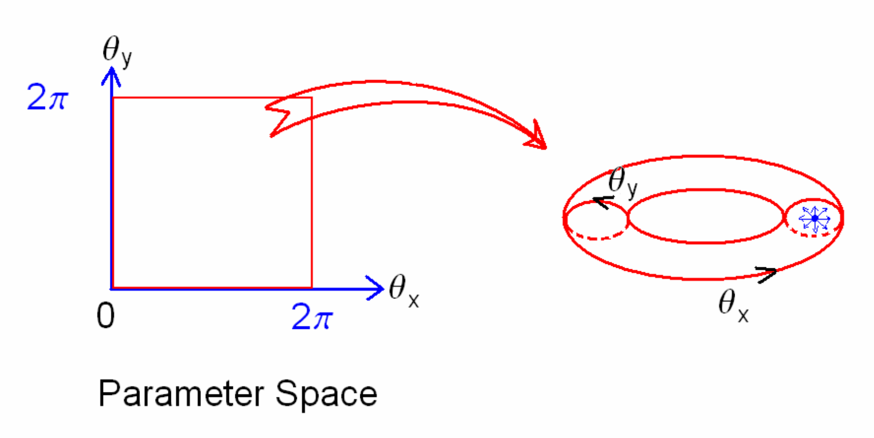
Many-body Definition of Chern Number (II)

- **Berry Phase gauge field in Parameter Space** (θ_x, θ_y)

$$A_\mu = -i \langle G(\theta_x, \theta_y) | \frac{\partial}{\partial \theta_\mu} | G(\theta_x, \theta_y) \rangle , \mu = 1, 2$$

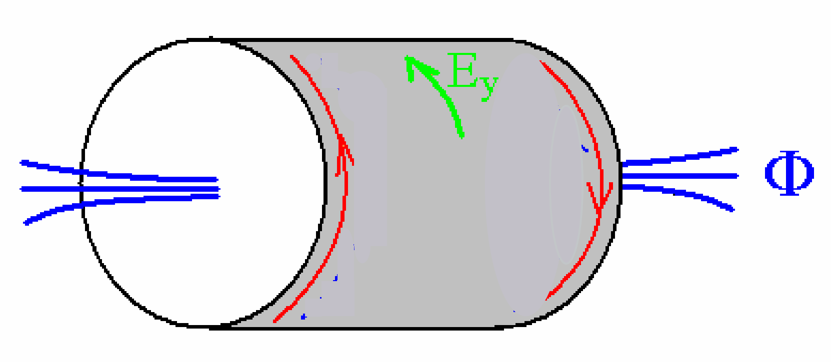
- **First Chern number: Total flux in Parameter Space.**
- **Well-defined for insulator without ground state degeneracy.**

$$\begin{aligned} N &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\theta_x d\theta_y F_{xy}(\theta_x, \theta_y) \\ &\equiv \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\theta_x d\theta_y \left(\frac{\partial A_y}{\partial \theta_x} - \frac{\partial A_x}{\partial \theta_y} \right) \end{aligned}$$



For any 2-d insulator, the many-body Chern number defines a *topological order*, which is stable against any small local perturbation

Laughlin's Gauge Argument (1981)



Adiabatically threading a unit flux $0 \rightarrow \Phi_0$

→ Gauge invariance (AB period)

→ Return to the original ground state

→ Transport an integral number of electrons

→ Integrally quantized Hall conductance

Ground State Degeneracy in FQHE

(Tao & Wu, 1984; Niu, Thouless, Wu, 1985)

Question: How is fractional topological number possible?

Answer: In Laughlin's argument on a cylinder or torus,

after adiabatically threading a unit magnetic flux,

Gauge invariance ⟹ energy spectrum unchanged

But a ground state may go to another ground state,

not necessarily back to the original state

Then need to thread an integer multiple of flux quanta,

to return to the original ground state and

to transport an integral number of electrons

⟹ Fractionally quantized Hall conductance

Topology Dependent GSD

For Fractional Quantum Hall Effect

- Sphere: unique ground state (Haldane, 1983)
- Torus: degenerate GS (Yoshioka et al., 1984)
- Proof for any number of handles: (Wen & Niu,1989)

  Use Chern-Simons effective theory

  Use Magnetic translation operators
- K-matrix Formulation of Multiple Abelian CS theory:

  (Read, 1990; Wen, Zee, Fröhlich, 1990-1991)

Fractional Charged Quasiparticles

- Laughlin wave function (1983)

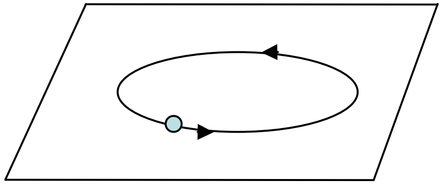
$$\Psi_0(z_{1,},\ldots,z_N) = \prod_{i<j}(z_i - z_j)^3 \exp\{-\frac{1}{2}\sum_i z_i z_i^*\}$$

$$\Psi_1(z_{1,},\ldots,z_N;Z) = \left(\prod_i (z_i - Z)\right) \Psi_0(z_{1,},\ldots,z_N)$$

- Berry's Phase Argument:

(Avoras, Schrieffer, Wilczek, 1984)

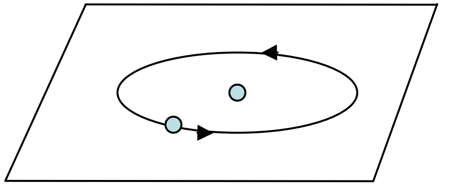
  - Move a quasiparticle along a loop
  - Quasiparticle sees electrons as unit flux
  - Electron filling factor is fractional



⇒ Quaispartilce charge is fractional

Fractional Statistics of Quasiparticles

- Hierarchy and Stability: (Halperin, 1983)
- Berry's Phase Argument:
  (Avoras, Schrieffer, Wilczek, 1984)



- Move a quasihole around the other one along a loop
  ⟺ Exchange them twice
- Quasihole sees the other as a defect
- Fractional Berry phase
  ⟹ Fractional exchange phase

Classification of Abelian Topological Orders

- Use Many-Body Wave-Functions:

  Laughlin 83; Haldane-Rezayi 85

- Early attempts for abelian FQH states:
  - ✓ Effective Chern-Simons Theory:

    Zhang, Hanson, Kivelson, 87; Niu and Wen, 88

  - ✓ Multiple Abelian Chern-Simons Theory

    (K-matrix formulation)

    Read, 90; Wen, Zee and Frohlich: 90-91

- Attempts based on Braid Group and Gauge Invariance

  Start with anyonic quasiparticles

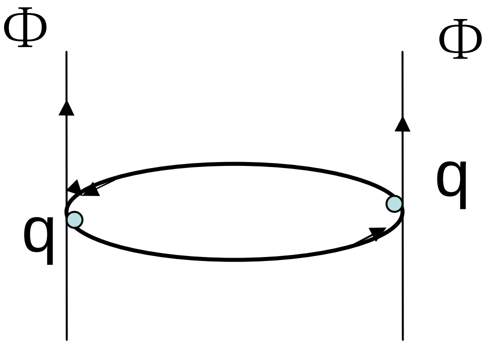
  Hatsugai, Kohmoto, Wu: 90-91

  Sato, Kohmoto, Wu, 06

Anyons and Braid Group

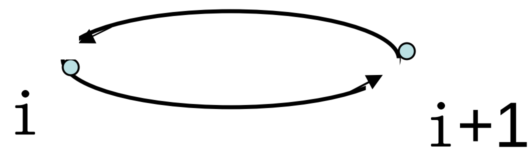
- Anyons: Wilczek (Wave function, 1982)
  Charge-Flux Composite
  Exchange phase

$$\exp\{i\theta\} \qquad (\theta/\pi = q\Phi)$$



- Braid Group on a plane: Wu (Path Intergral,1984)

Generators: σ_i
(exchange counterclockwise)



Relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad (|i-j|>1)$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

1d Repres:

$$\sigma_i = e^{i\theta}$$

(Note: If add $\sigma_i^{\ 2} = 1$, then it becomes the permutation group.)

Braiding of Three Anyons

- Path dependence of exchange phases:

  Enclosing the third particle or not

$$\longrightarrow \quad e^{i\theta} \text{ or } e^{i3\theta}$$

- Permuting wave function: not work
- Generalizing usual commutation relation of creation and annihilation operators: not work
- Key point is to keep track of path dependence:

  Path Integral or Monodromy

- Necessary to study braid group on various topology

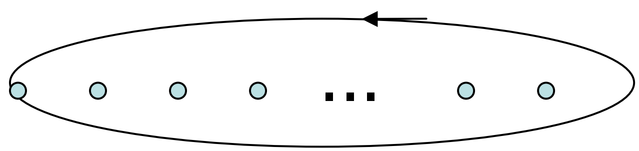
Braid Group on Sphere

(Thouless & Wu, 85)

- The same set of generators
- One more relations

$$\sigma_1\sigma_2\cdots\sigma_{N-1}^2\sigma_{N-2}\cdots\sigma_1 = 1$$

- Consequence of spherical topology



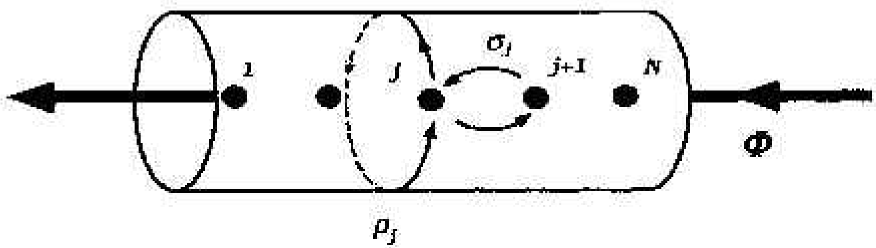
- Constraint for anyons on the sphere (Shift):

$$e^{i2\theta(N-1)} = 1$$

Relations on Cylinder (Annulus)

(Hatsugai, Kohmoto, Wu: 1990)

- More Generators:



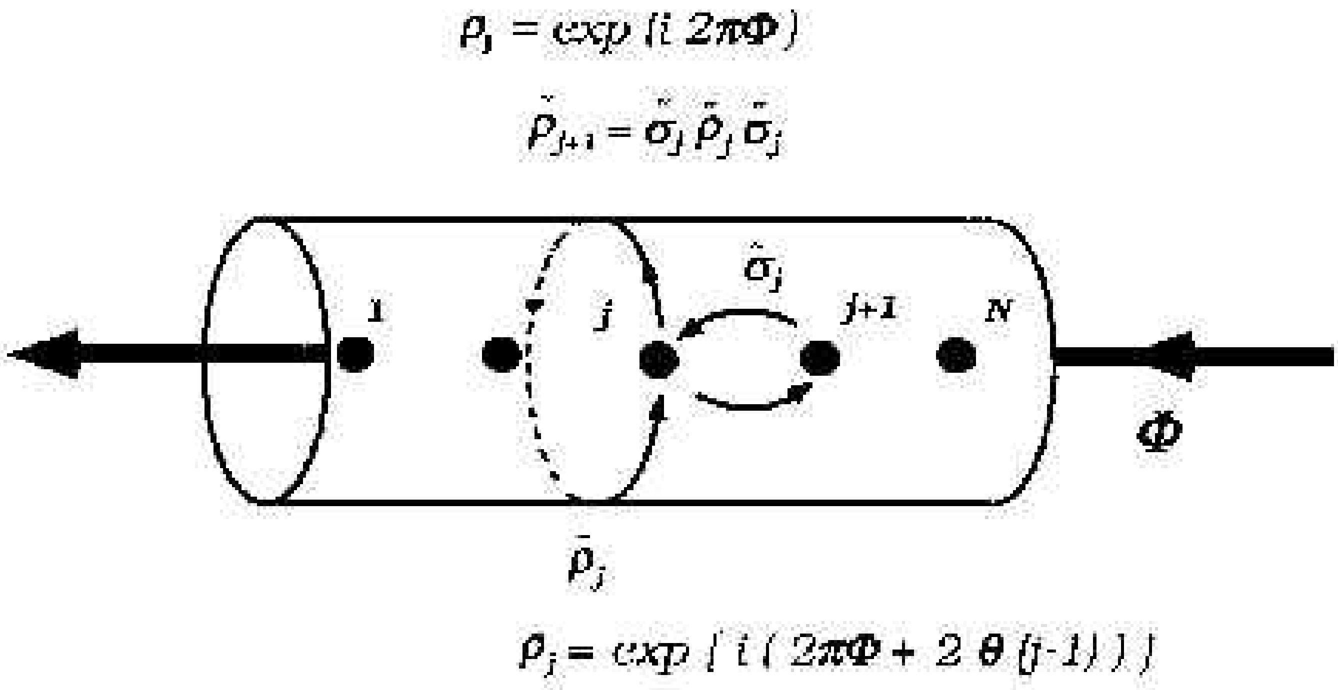
More Relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad (i \neq j \pm 1), \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

$$\rho_i\rho_j = \rho_j\rho_i, \quad \sigma_i\rho_j = \rho_j\sigma_i \quad (i \neq j, j-1),$$

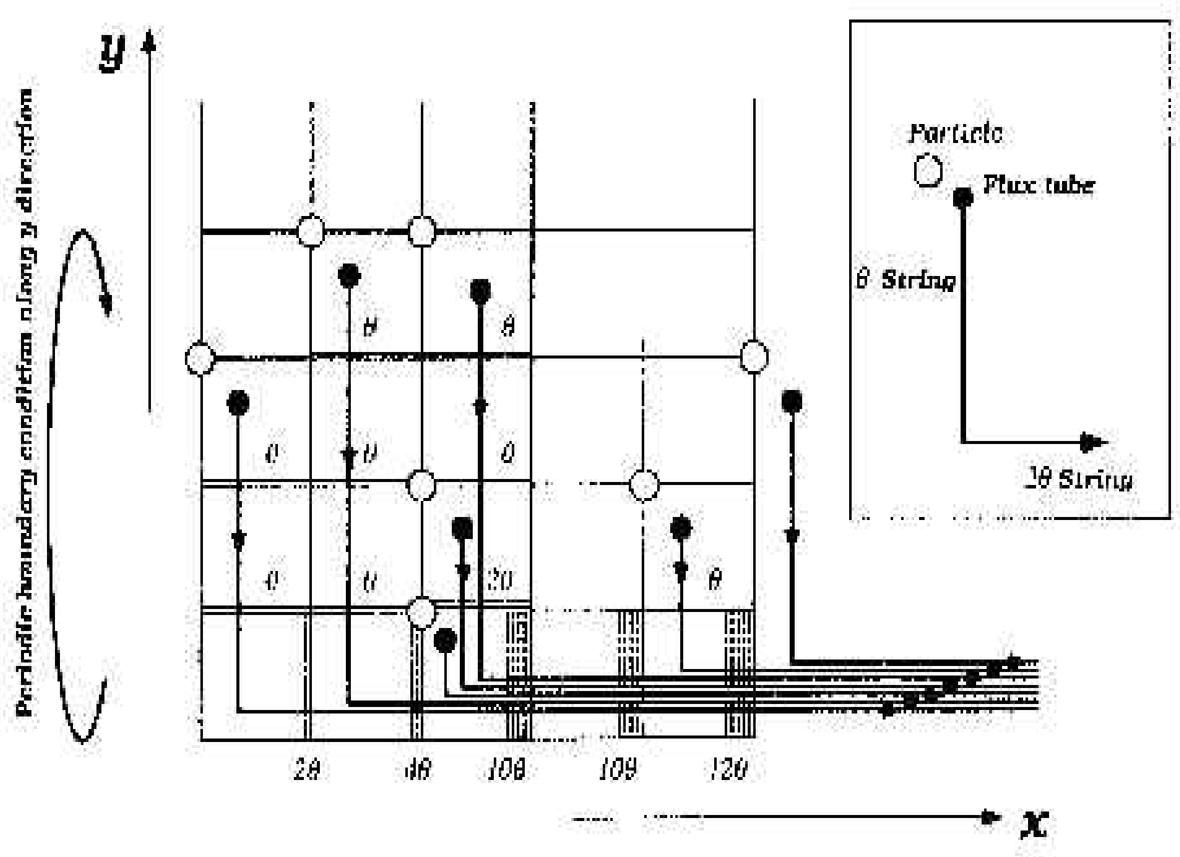
$$\rho_{j+1} = \sigma_j\rho_j\sigma_j.$$

1D Repre. on a Cylinder



AB period is determined by anyon charge.

Anyon Hopping Rule on a Cylinder



Puzzle for the AB-period of Quasiparticles (due to Gauge Invariance)

Known: The AB period of quasiparticle is hc/e^{*}.

The AB period of constituent is hc/e.

Question: How come quasiparticles can have a smaller AB period if $e^{*}=e/n$?

Answer: (Hatsugai, Kohmoto, Wu, PRL; 1991)

Degenerate states of quasiparticles is n.

Analytic Proof: Use braid group on a cylinder

Numerical Proof: Use anyon hopping rules

Gauge Invariance (A-B Period) of Fractional Charged Anyons

A higher dimensional representation of BG:

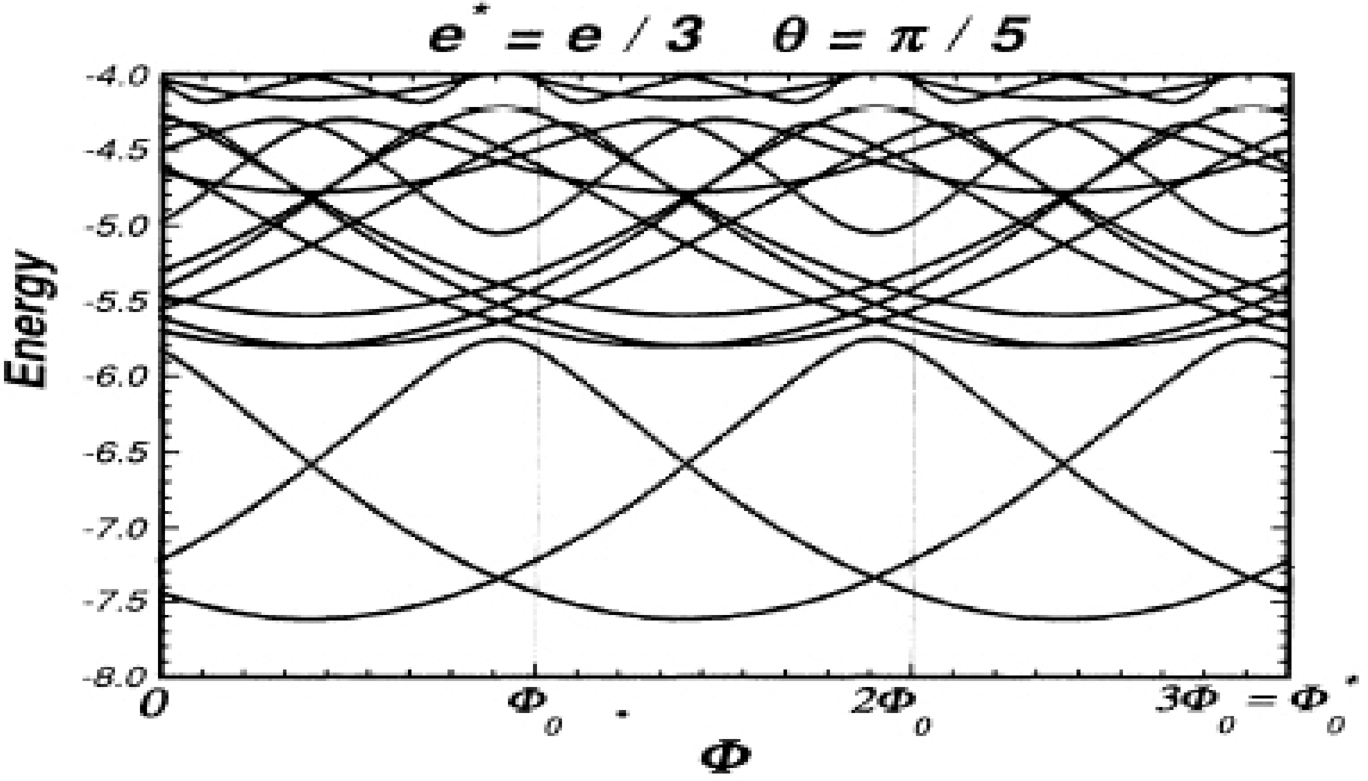
$$\rho_j(\Phi) = \exp\{i2\theta(j-1) + i2\pi e^{*}\Phi / hc\}W$$

$$W = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

On a cylinder emerges a topological Z_n symmetry in topological phases with fractionalization, requiring degenerate ground states

(related to gapless edge states?)

Numerical Evidence



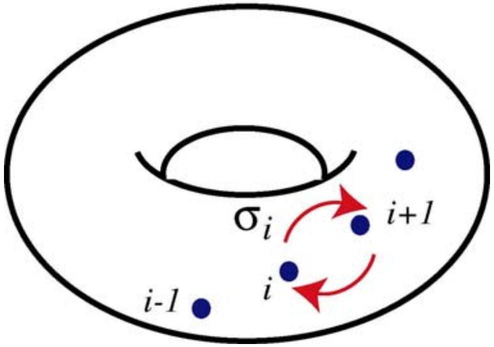
GSD is determined by fractional charge !

Braid Group on a Torus

(Enarsson, 1989)

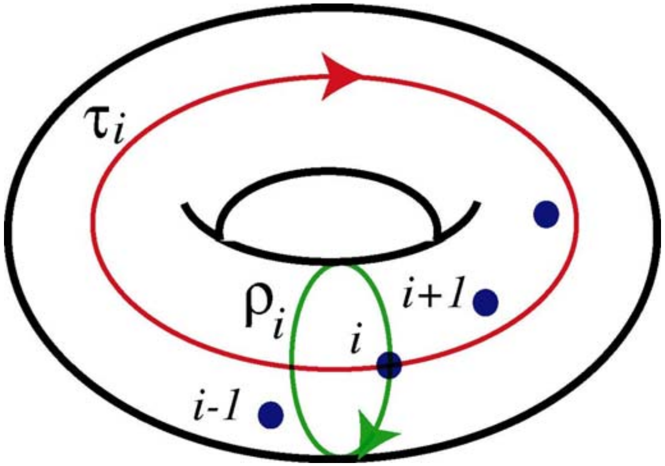
- Generators:

σ_i : Exchange operation



ρ_i : Translation operation along a nontrivial loop

τ_i :



Braid Group on a Torus (II)

- Relations: (J. Birman, 1965)

$$\sigma_k \sigma_l = \sigma_l \sigma_k, \quad 1 \le k \le N-3, \quad |l-k| \ge 2,$$
$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad 1 \le k \le N-2,$$
$$\tau_{i+1} = \sigma_i^{-1} \tau_i \sigma_i^{-1}, \quad \rho_{i+1} = \sigma_i \rho_i \sigma_i,$$
$$\tau_1 \sigma_j = \sigma_j \tau_1, \quad \rho_1 \sigma_j = \sigma_j \rho_1, \quad \sigma_i^2 = A_{i+1,i},$$

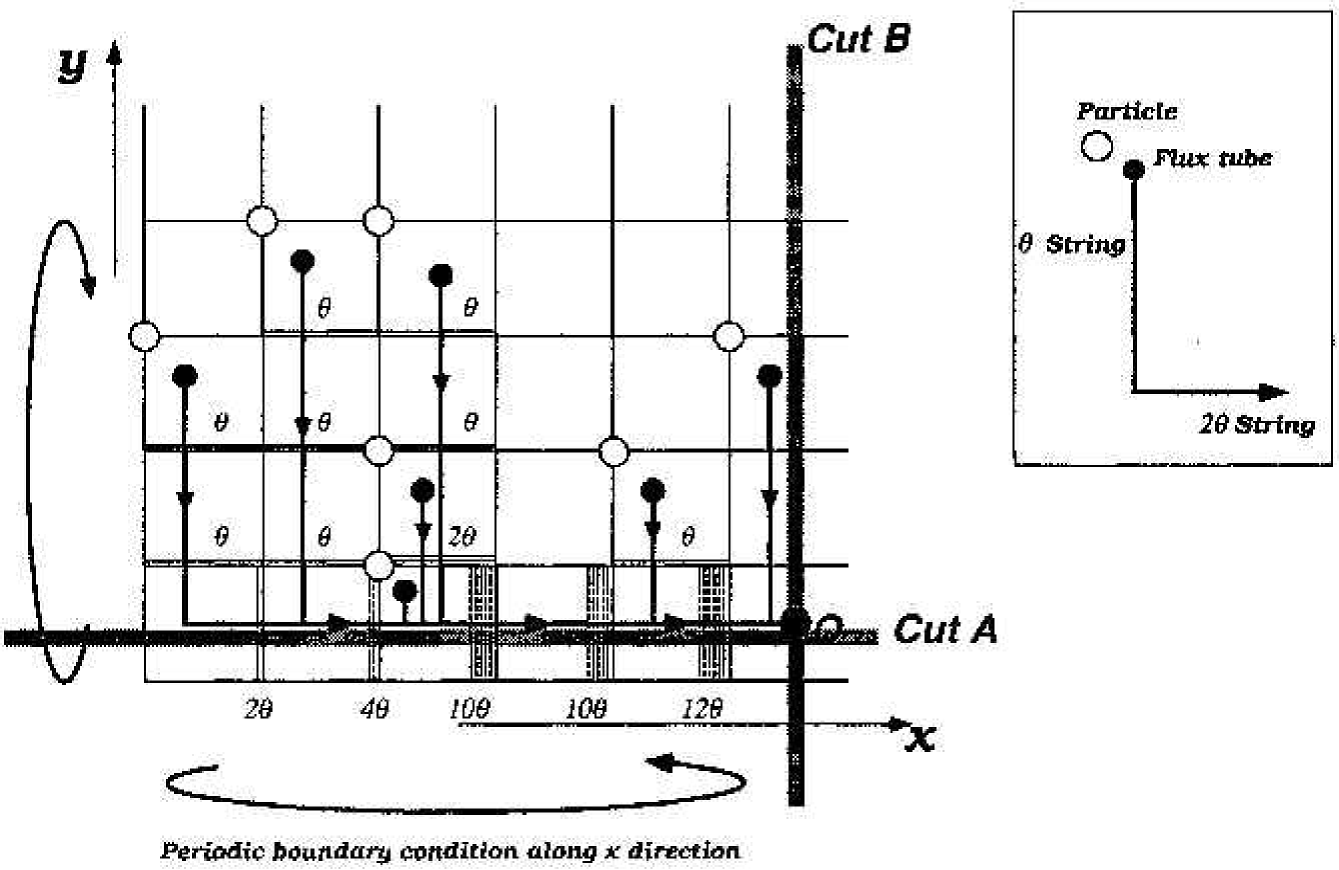
$$A_{j,i} = \tau_j^{-1} \rho_i \tau_j \rho_i^{-1}, \quad C_{j,i} = \rho_j^{-1} \tau_i \rho_j \tau_i^{-1},$$

$$A_{m,l} \tau_k = \tau_k A_{m,l}, \quad A_{m,l} \rho_k = \rho_k A_{m,l}$$
$$\tau_i \tau_j = \tau_j \tau_i, \quad \rho_i \rho_j = \rho_j \rho_i,$$
$$C_{j,i} = (\tau_i \tau_j) A_{j,i}^{-1} (\tau_j^{-1} \tau_i^{-1}), \quad A_{j,i} = (\rho_i \rho_j) C_{j,i}^{-1} (\rho_j^{-1} \rho_i^{-1}),$$
$$C_{j,i} = (A_{j,j-1}^{-1} \cdots A_{j,i+1}^{-1}) A_{j,i}^{-1} (A_{j,i+1} \cdots A_{j,j-1}),$$
$$\tau_1 \rho_1 \tau_1^{-1} \rho_1^{-1} = A_{2,1} A_{3,1} \cdots A_{N-1,1} A_{N,1},$$

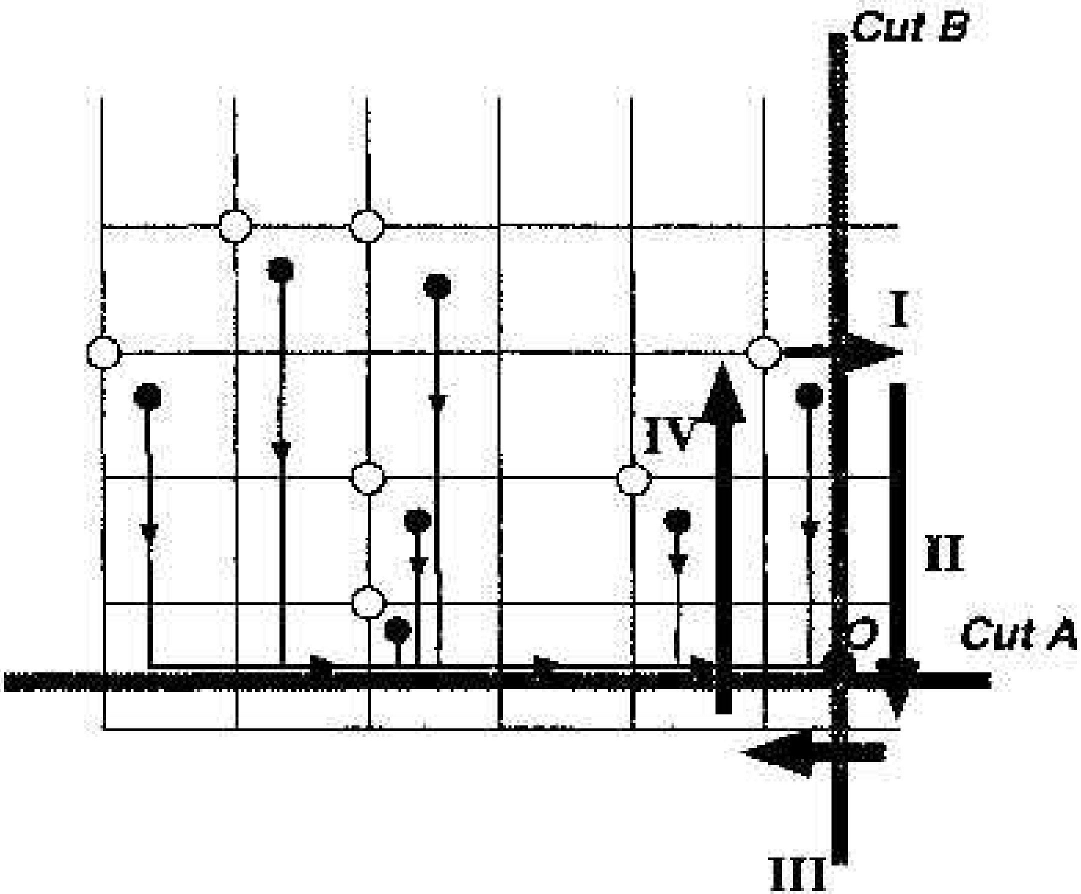
where $1 \le k < l < m \le N$ and $1 \le i < j \le N$.

Anyon Hopping on a Torus

(Wen, Dagotto, Fradkin, 90; Hatsugai, Kohmoto, Wu, 90)



End of Flux (on Torus)



Total flux ended at O is invisible: $e^{i2N\theta} = 1$

Pattern of GSD on Torus

Assume fractional particle number: $e^{*}/e=p/q$

- In abelian FQH states (anyons)

$$GSD=q^{g} \quad (g=1)$$

- In fractionalied liquids (bosons or fermions)

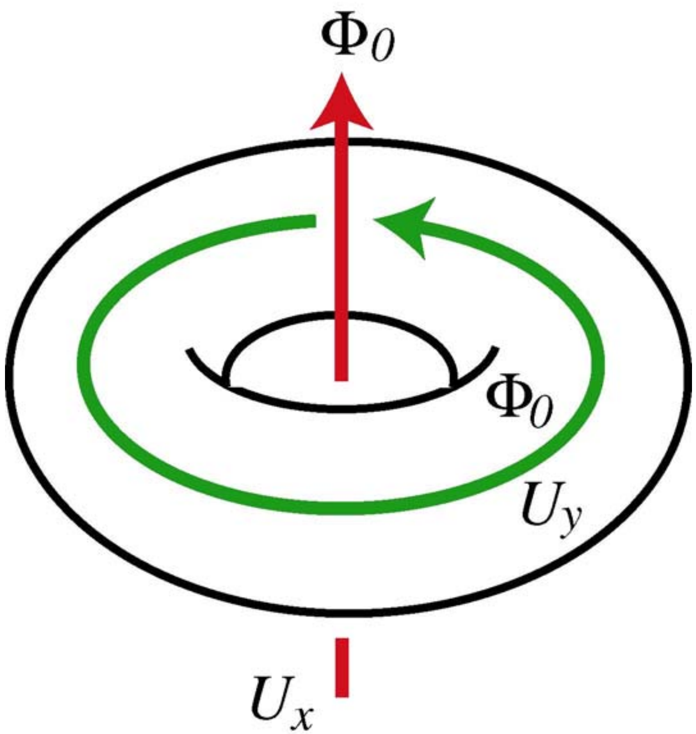
$$GSD=(q^{2})^{g} \quad (g=1)$$

Read, Sachdev 91; Senthil, Fisher, 01;
Balents, Fisher, Girvin, 02; ……

Question: q or q^2 on torus ?

Flux Insertion and Large Gauge Transformations

U_x : change A_x from 0 to $2\pi / eL$



Relations:

$$U_x \rho_i = \rho_i U_x , \quad U_x \sigma_i = \sigma_i U_x .$$

$$U_x \tau_i = e^{i2\pi e^* / e} \tau_i U_x$$

Automorphism of Braid Group:

$$\sigma_i' = U_a \sigma_i U_a^{-1} , \quad \tau_i' = U_a \tau_i U_a^{-1} , \quad \rho_i' = U_a \rho_i U_a^{-1} .$$

⟹ σ_i', ρ_i' and τ_i' satisfy same Braid Group relations

Abelian Topological Orders

Abelian quasiparticle statistics:

$$\sigma_j = e^{i\theta} 1,$$

Braid Group Relations ⟶

$$\tau_j = e^{-2i\theta(j-1)} T_x, \quad \rho_j = e^{2i\theta(j-1)} T_y,$$

$$T_x T_y = e^{-2i\theta} T_y T_x.$$

T_x, T_y, U_x and U_y act on many-body Hilbert space

Fundamental Discrete Algebra for Abelian Topological Order

(Sato, Kohmoto, Wu, PRL 06; cond-mat/0604506)

Fractional statistics $(\theta = 2\pi\, m/n)$:

$$T_x T_y = e^{-2\pi i m/n} T_y T_x.$$

Fractional charge $(e^*/e = p/q)$:

$$U_x T_x U_x^{-1} = e^{-2\pi i p/q} T_x, \quad U_y T_x U_y^{-1} = T_x,$$
$$U_x T_y U_x^{-1} = T_y, \quad U_y T_y U_y^{-1} = e^{-2\pi i p/q} T_y.$$

Non-commutativity λ: (motivated by topol. invariant)

$$U_x U_y = e^{2\pi i \lambda} U_y U_x.$$

Minimal GS Degeneracy

- Theorem: Given rational θ and e^*/e, the GSD is an integral multiple of $n\mathbf{Q}^2$.

Here $\theta = 2\pi \frac{m}{n}$, $\frac{e^*}{e} = \frac{p}{q}$, and

$$\frac{n}{q} = \frac{\mathbf{N}}{\mathbf{Q}} \quad (\mathbf{N} \text{ and } \mathbf{Q} \text{ coprime})$$

Example 1

If $\theta = \pi / n$ and $e^* = e / n$ (Laughlin states with $\nu = 1/n$)

Then, minimal GSD is just n.

Realization of Minimal GSD: (with $U_x U_y = e^{-2\pi i/n} U_y U_x,$)

$$T_x = S_{n\times n}, \quad T_y = R_{n\times n},$$
$$U_x = R^{-1}_{n\times n}, \quad U_y = S_{n\times n}.$$

Here $S_{n\times n} = \mathrm{diag}\{1, e^{i2\pi/n}, \cdots, e^{i2\pi(n-1)/n}\}$ and

$$R_{n\times n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

FQH conductance is

$$\sigma_{xy} = \frac{1}{n}\frac{e^2}{h}$$

Example 2

If $\theta = 0$ or π (quasiparticles are bosons or fermions)

Then minimal GSD is q^2 . (Oshikawa, Senthil, 2006)

Realization of Minimal GSD:

$$T_x = R_{q\times q} \otimes 1_{q\times q}, \quad T_y = 1_{q\times q} \otimes R_{q\times q},$$
$$U_x = S^p_{q\times q} \otimes 1_{q\times q}, \quad U_y = 1_{q\times q} \otimes S^p_{q\times q}.$$

In this realization,

U_x and U_y commutes with each other.

More Examples

- For q and n coprime,

$$\text{GSD} = nq^2$$

$$T_x = 1_{q\times q} \otimes R_{q\times q} \otimes S^m_{n\times n},$$
$$T_y = R_{q\times q} \otimes 1_{q\times q} \otimes R_{n\times n},$$
$$U_x = 1_{q\times q} \otimes S^p_{q\times q} \otimes 1_{n\times n},$$
$$U_y = S^p_{q\times q} \otimes 1_{q\times q} \otimes 1_{n\times n}.$$

with U_x and U_y commute.

(4) $n = \mathcal{N}q$ and $m = 1$. - Because of $Q = 1$, the minimum degeneracy is n. A representation is given by

$$T_x = S_{n\times n}, \quad T_y = R_{n\times n},$$
$$U_x = R_{n\times n}^{-\mathcal{N}p}, \quad U_y = S_{n\times n}^{\mathcal{N}p}. \tag{25}$$

U_x and U_y satisfy $U_xU_y = e^{-2\pi i(\mathcal{N}p^2/q)}U_yU_x$.

(5) $q = Qn$. - In this case, $\mathcal{N} = 1$, thus the least degeneracy is nQ^2. When Q and n are mutually prime and $p = m = 1$, we can construct the following representation:

$$T_x = S_{Q\times Q} \otimes S_{n\times n} \otimes 1_{Q\times Q},$$
$$T_y = 1_{Q\times Q} \otimes R_{n\times n} \otimes S_{Q\times Q},$$
$$U_x = R_{Q\times Q}^{-l} \otimes R_{n\times n}^{-k} \otimes 1_{Q\times Q},$$
$$U_y = 1_{Q\times Q} \otimes S_{n\times n}^{k} \otimes R_{Q\times Q}^{-l}, \tag{26}$$

where $k/n + l/Q = 1/Qn\,(\text{mod}.1)$. In this case, $U_xU_y = e^{-2\pi ik^2/n}U_yU_x$.

Physical Meaning of Parameter λ

Assume $\lambda = k/l$:

$$U_x U_y = e^{i2\pi k/l} U_y U_x \quad (U_x^l U_y = U_y U_x^l)$$

<u>Theorem</u> (Generalizing Niu-Thouless-Wu)

Hall conductance is

$$\sigma_{xy} = \frac{e^2}{hd} \sum_{r=0}^{d/l-1} I_r = \frac{e^2}{h} \frac{I}{l}.$$

Here d is GSD. All I_r have the same value I.

Summary for This work

- Characterization of the abelian topological orders by means of a discrete symmetry algebra which has a topological origin
- Reveal the interplay between anyon statistics, fractionalization and topological number
- Determine minimal ground state degeneracy without assuming relations between $\theta, e^{*} \text{and} \lambda$
- Can be generalized to higher genus surfaces
- Generalization to gapless systems?
- Generalization to non-Abelian topological order?
- Mechanism underlying relations between $\theta, e^{*} \text{and} \lambda$?

Non-Abelian Topological Order?

- Pfaffin and generalized wave functions:

  Moore and Read (1990)

  Rezayi and Read (parafermion states) (1994)

- Candidate state: $\nu = 5/2,\ 12/5\,?$
- Non-abelian anyons:

  Higher dimensional representations

  of braid groups (also Nayak and Wilczek, 1994)

  Good for Topological quantum computing

  Kitaev, 1997; Project-Q, 2003; This workshop

- Generalizations:

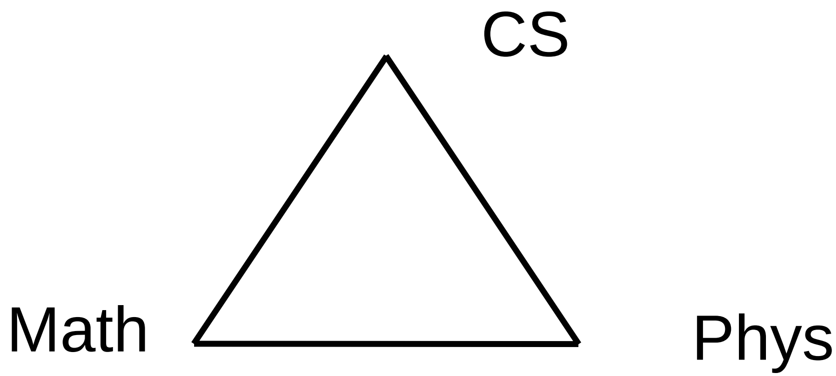
  Wen and Wu (1994); S. Simon (this workshop)

Generalization to Non-abelian Topological Order ?

- Non-abelian anyons are described by Modular tensor category (TQFT)
- Still need to introduce more (and specific) structures (like symmetries) to accommodate rich physics
- Relationship between translations ρ_j and τ_j and modular S-matrix ?
- Relationship between flux insertions U_x and U_y and the symmetry in Wang's yesterday talk ?

Perspective in Topological Quantum Computing

A New Chapter in Science!



More Excitements to Come!

Thank you!!