Finding Braids for Nonabelian Particles

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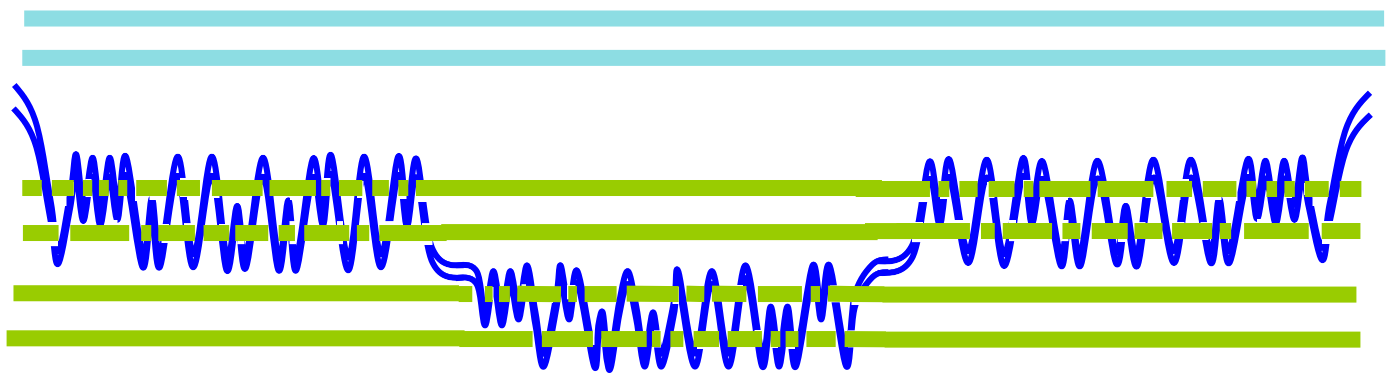
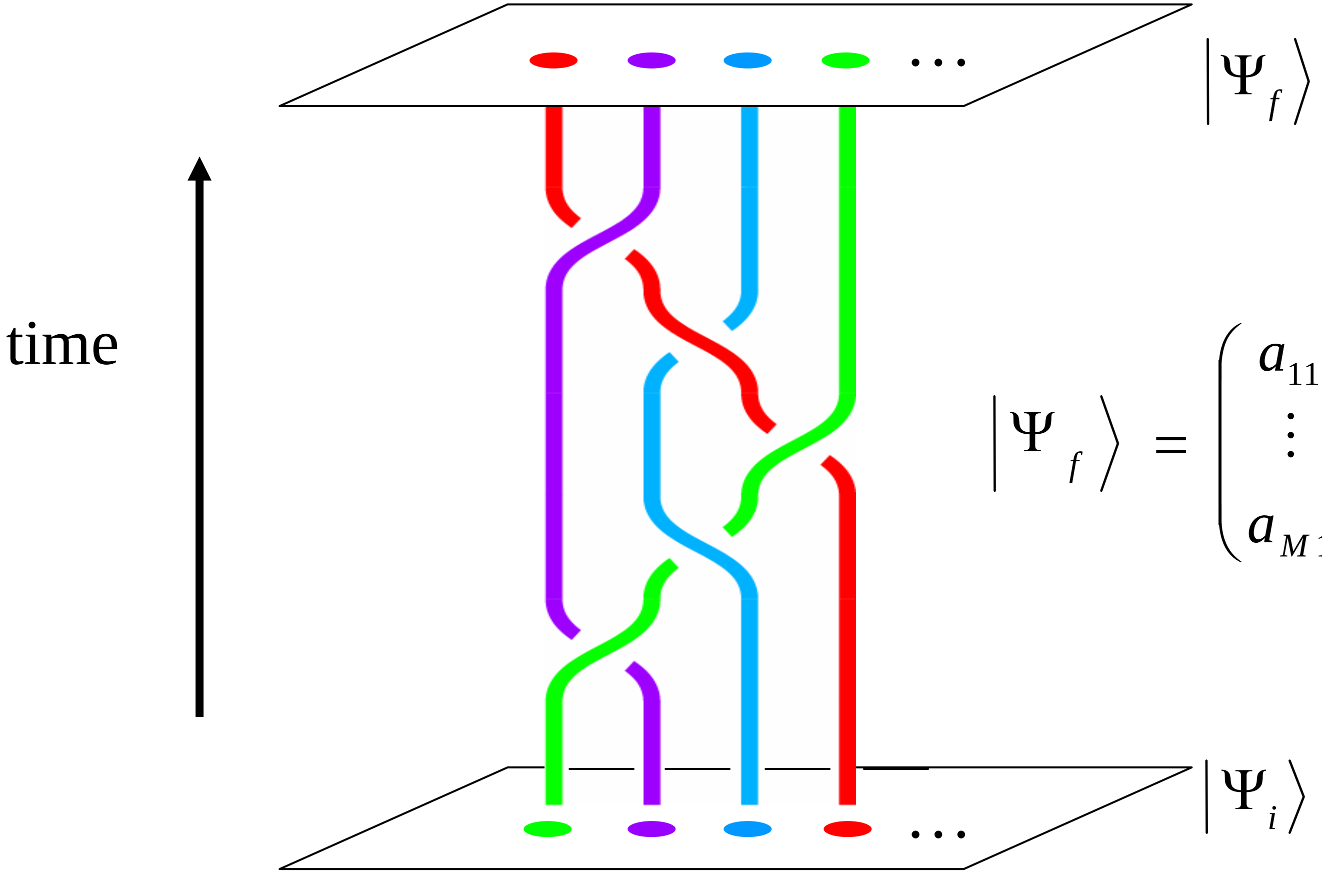
Lucent Technologies



NEB, L. Hormozi, G. Zikos, S.H. Simon, Phys. Rev. Lett. 95 140503 (2005)
L. Hormozi, G. Zikos, NEB, and S.H. Simon, Phys. Rev. B, In Press. (quant-ph/0610111)
+Some unpublished results

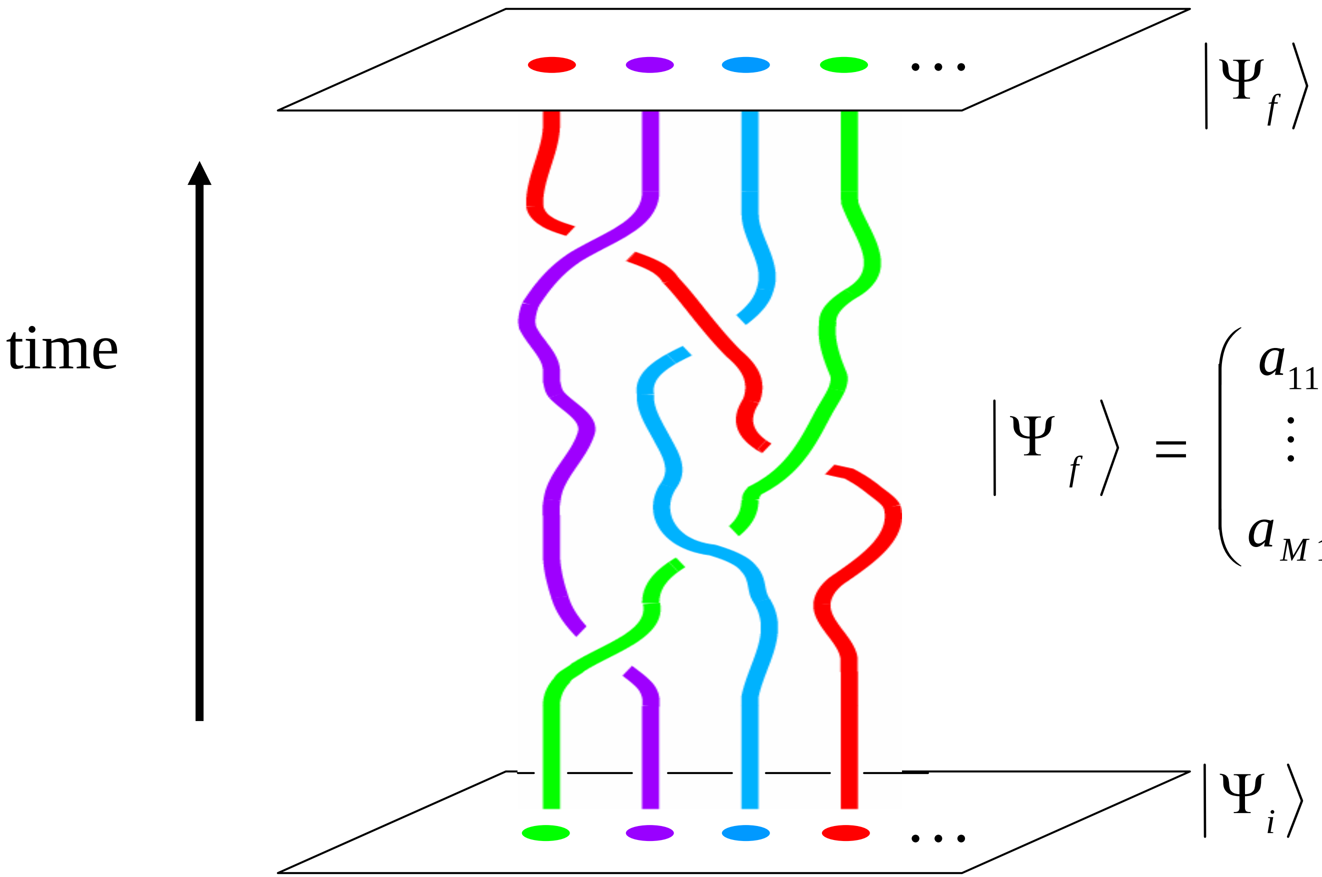
Support: US DOE

Topological Quantum Computation



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Topological Quantum Computation



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Matrix depends only on the topology of the braid swept out by quasiparticle world lines!

Robust quantum computation?

(Kitaev ‘97; Freedman, Larsen and Wang ‘01)

$SU(2)_k$ Nonabelian Particles

Describe quasiparticle excitations of the Read-Rezayi "Parafermion" states at level **k** (up to Abelian phases).

Read and Rezyai, 1999
Slingerland and Bais, 2001

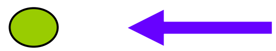
Are universal for quantum computation for **k=3** and **k > 4**.

Freedman, Larsen, and Wang, 2001

But before $SU(2)_k$, there was just plain old SU(2)….

Particles with Ordinary Spin: SU(2)

1. Particles have spin ***s*** = 0, 1/2, 1, 3/2 , …



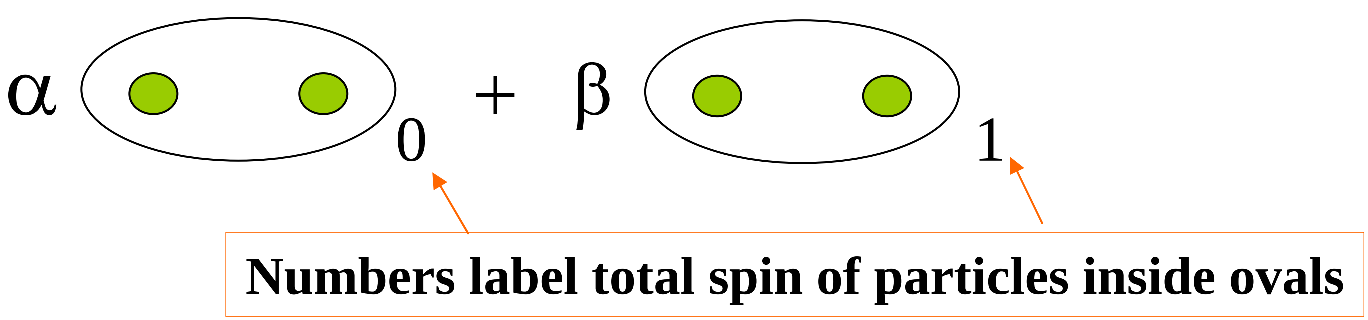
spin = ½

2. “Triangle Rule” for adding angular momentum:

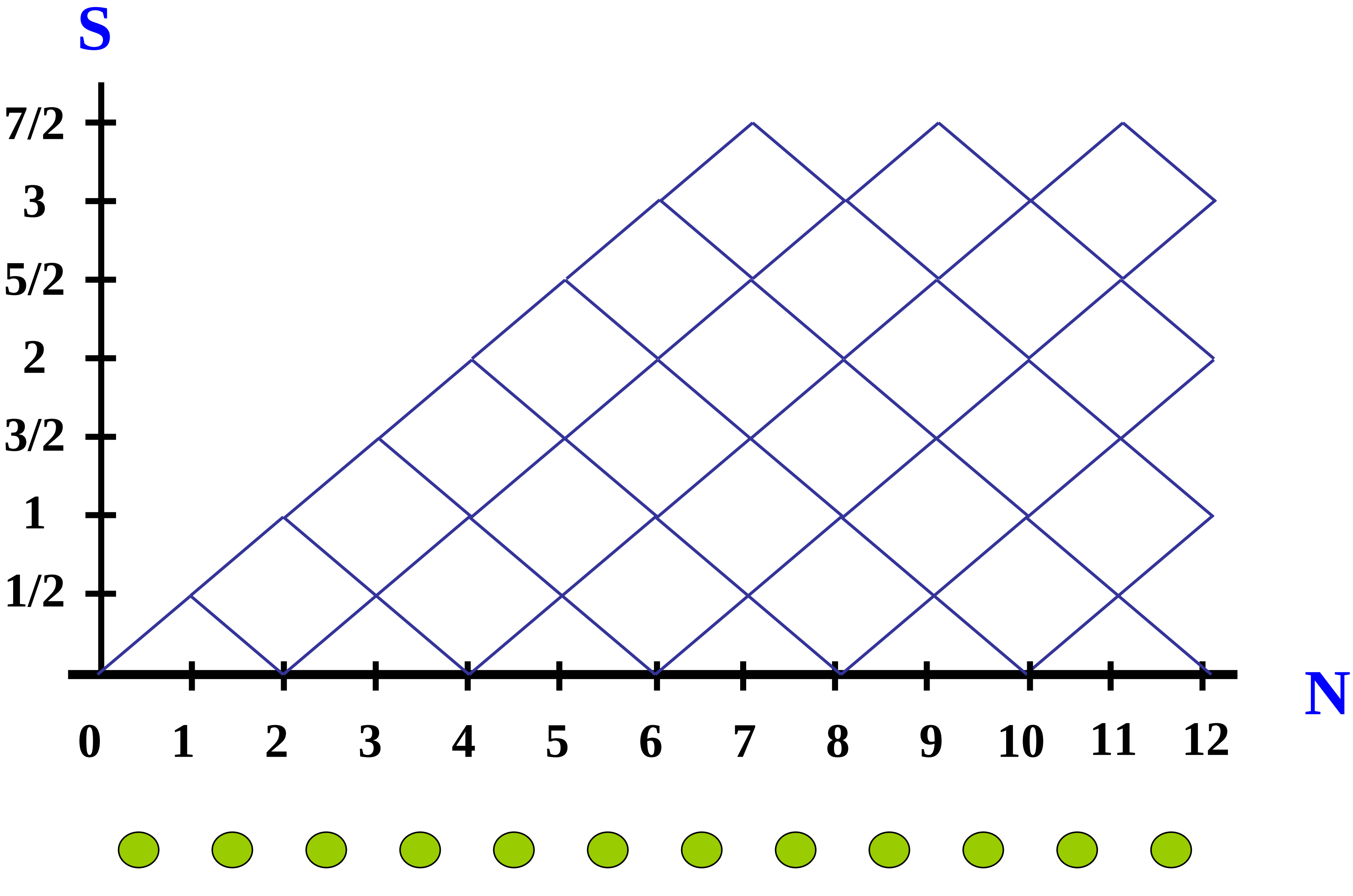
$$s_1 \otimes s_2 = \left|s_1 - s_2\right| \oplus \left(\left|s_1 - s_2\right| + 1\right) \oplus \cdots \oplus s_1 + s_2$$

For example: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

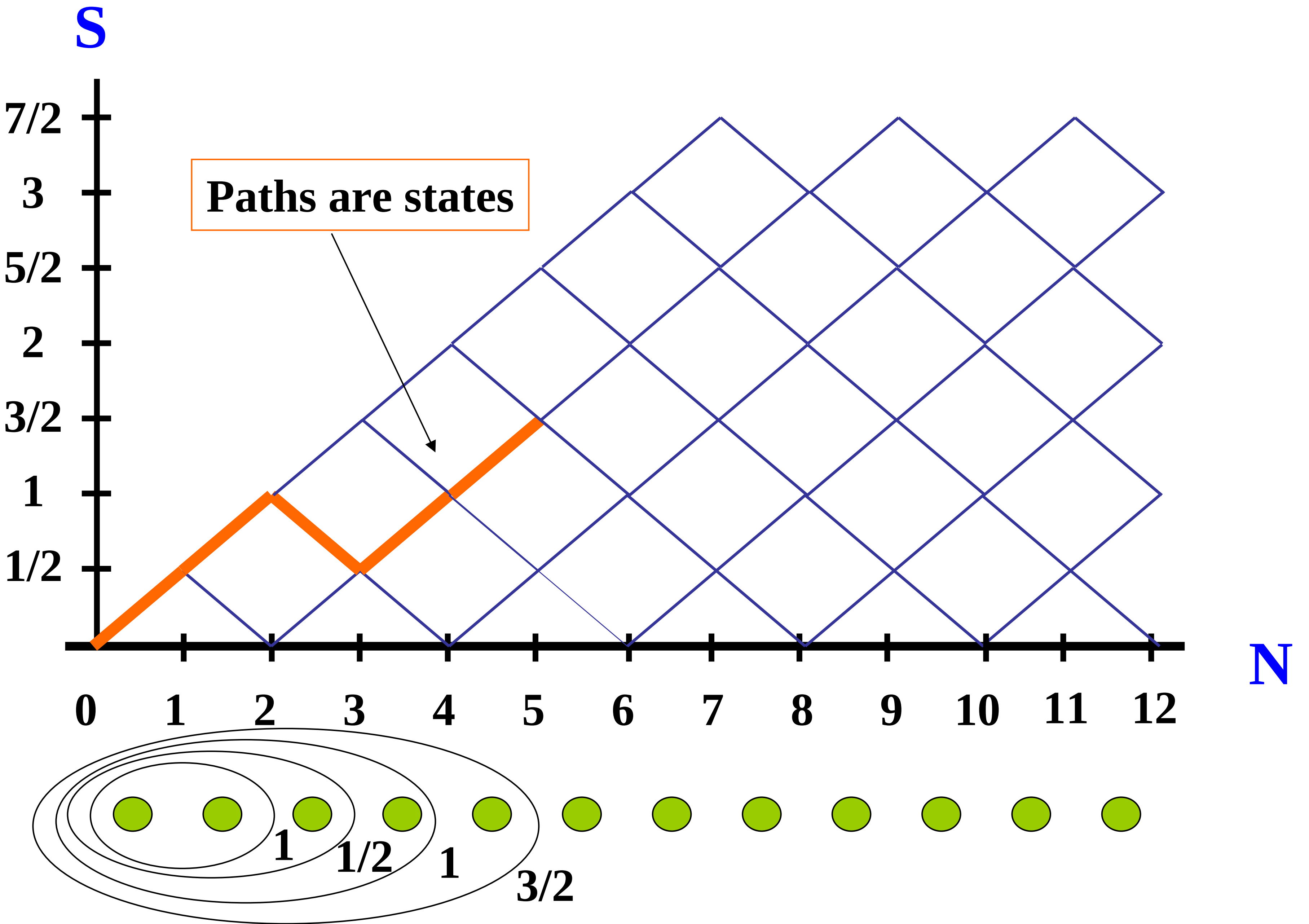
➡ Two ● particles can have total spin **0** or **1**.

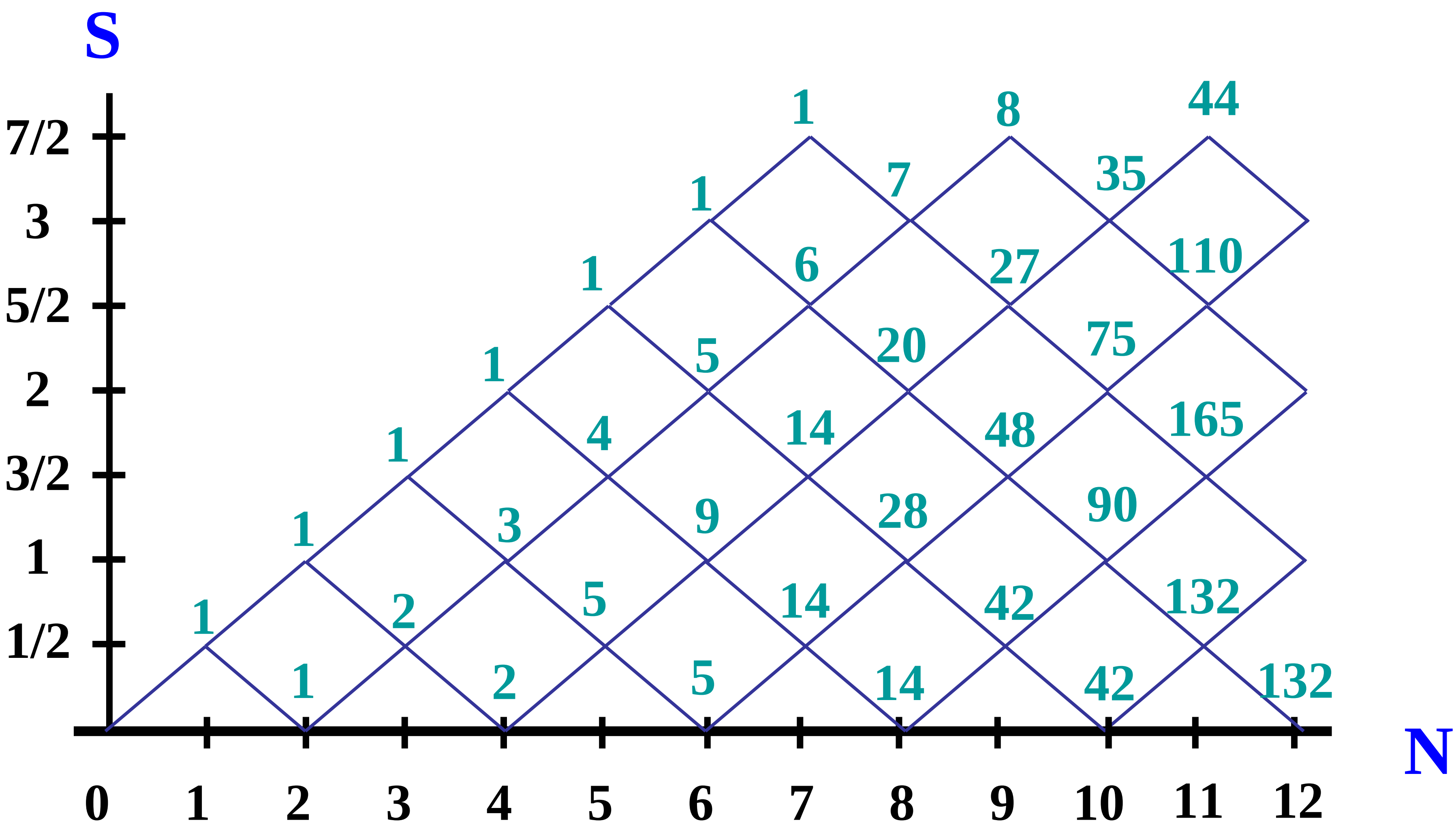


Hilbert Space



Hilbert Space





S
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14
48
165
1
3
9
28
90
1
2
5
14
42
132
1
2
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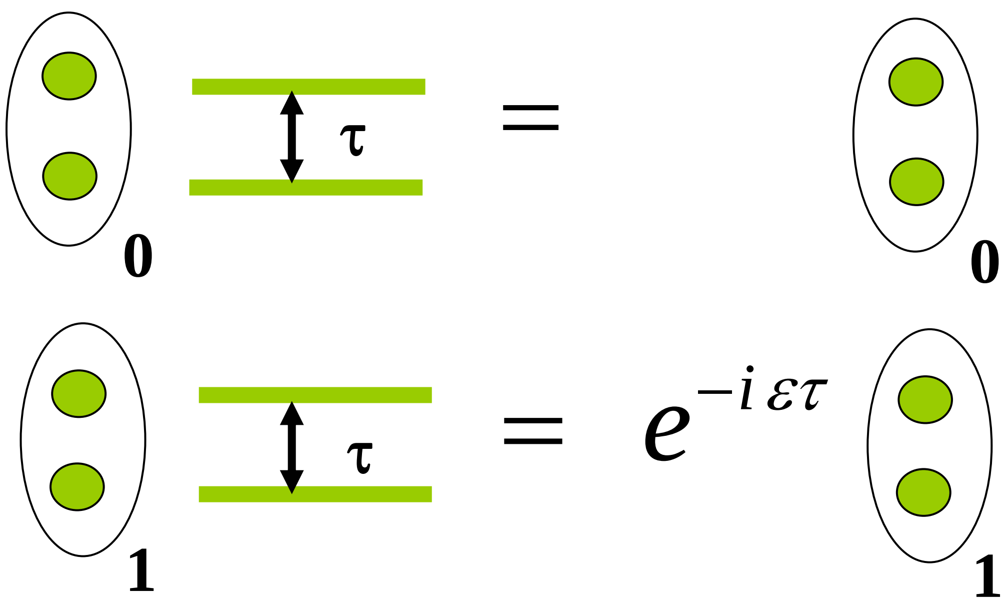
Exchange-Based Quantum Computing

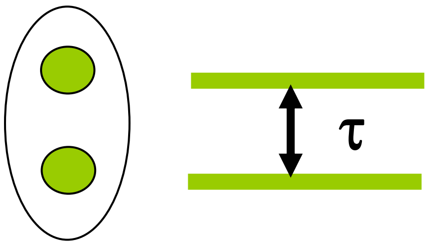
Compute by turning on and off the **exchange interaction** between neighboring spin-1/2 particles for some time **τ,** splitting the energy of the total spin 0 and total spin 1 sectors by ε.



$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varepsilon\tau} \end{pmatrix}$$

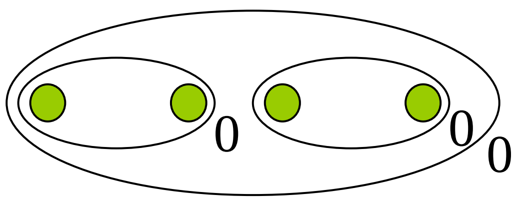
Encoded Universality

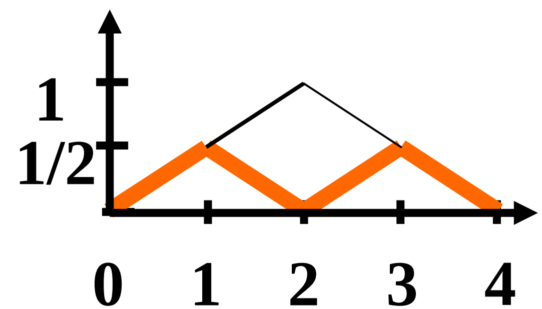
J. Kempe, D. Bacon, D.A. Lidar, and K.B. Whaley (2001)

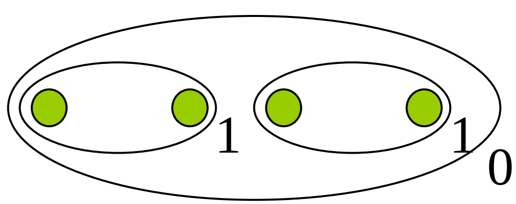


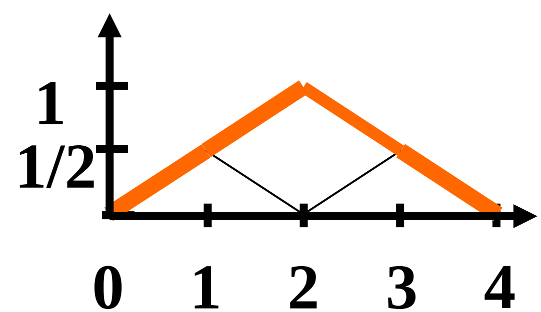
$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varepsilon\tau} \end{pmatrix}$$

4 spin encoding:

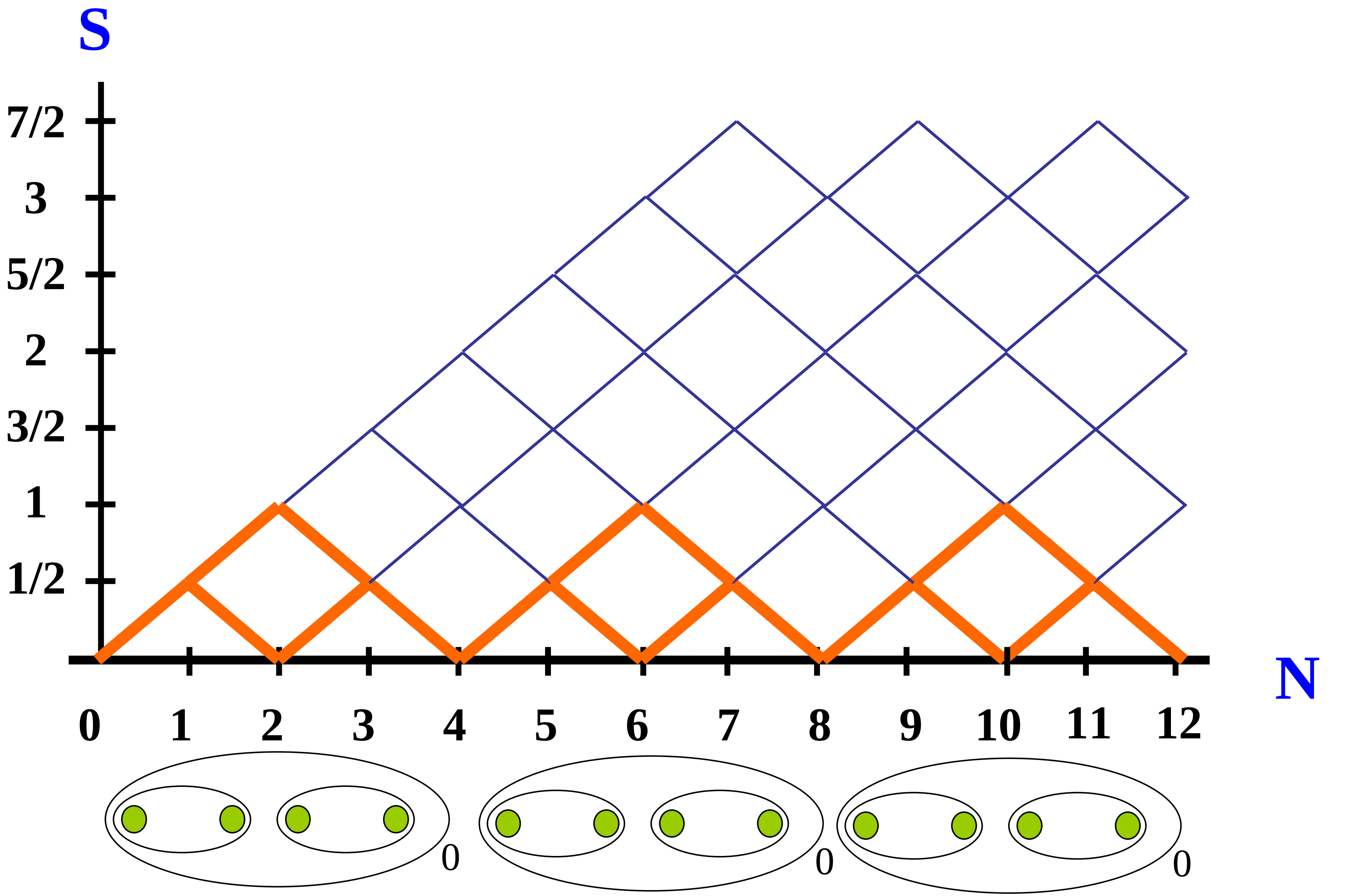
$|0\rangle =$ 



$|1\rangle =$ 



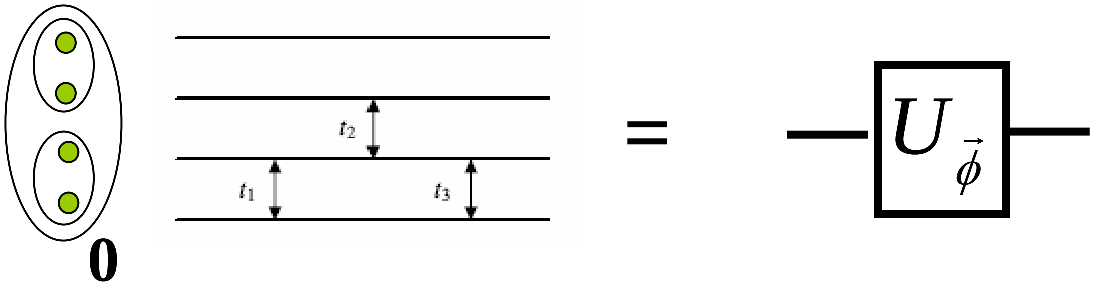
Encoded Qubit Space



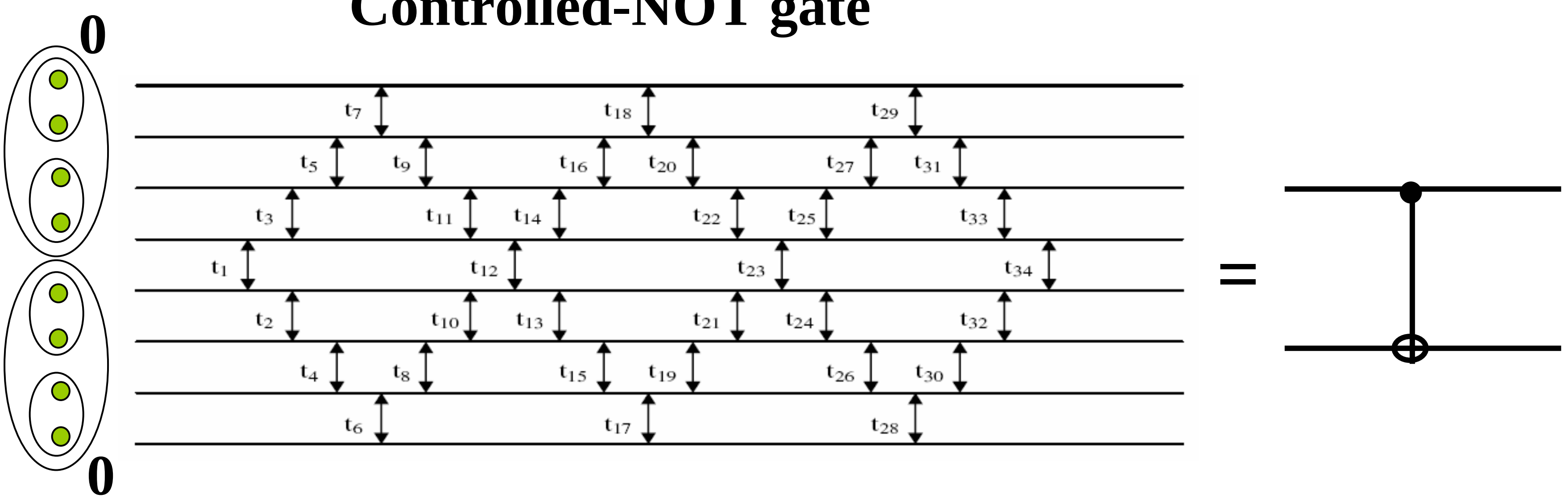
Universal Set of Gates

M.Hsieh, J. Kempe, S. Myrgren and K.B. Whaley (2003)

Single Qubit Gates

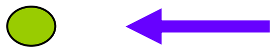


Controlled-NOT gate



Particles with Ordinary Spin: SU(2)

1. Particles have spin s = 0, 1/2, 1, 3/2 , …



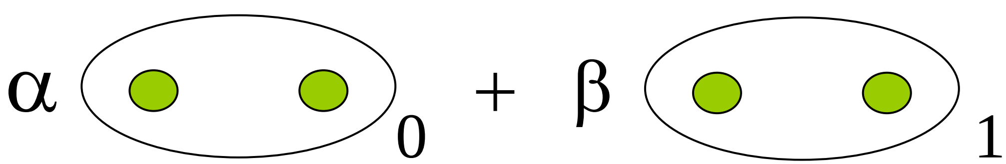
spin = ½

2. "Triangle Rule" for adding angular momentum:

$$s_1 \otimes s_2 = |s_1 - s_2| \oplus \left(|s_1 - s_2| + 1\right) \oplus \cdots \oplus s_1 + s_2$$

For example: $$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

→ Two ● particles can have total spin **0** or **1**.



Nonabelian Particles: $SU(2)_k$

1. Particles have topological charge **s** = 0, 1/2, 1, 3/2 , …, k/2

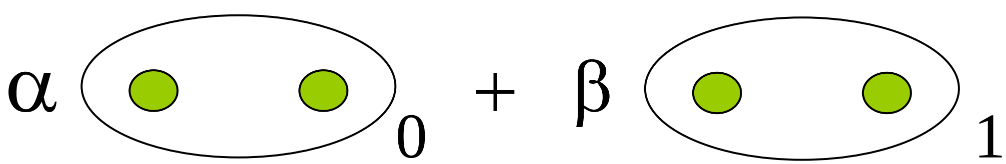
● ← topological charge = ½

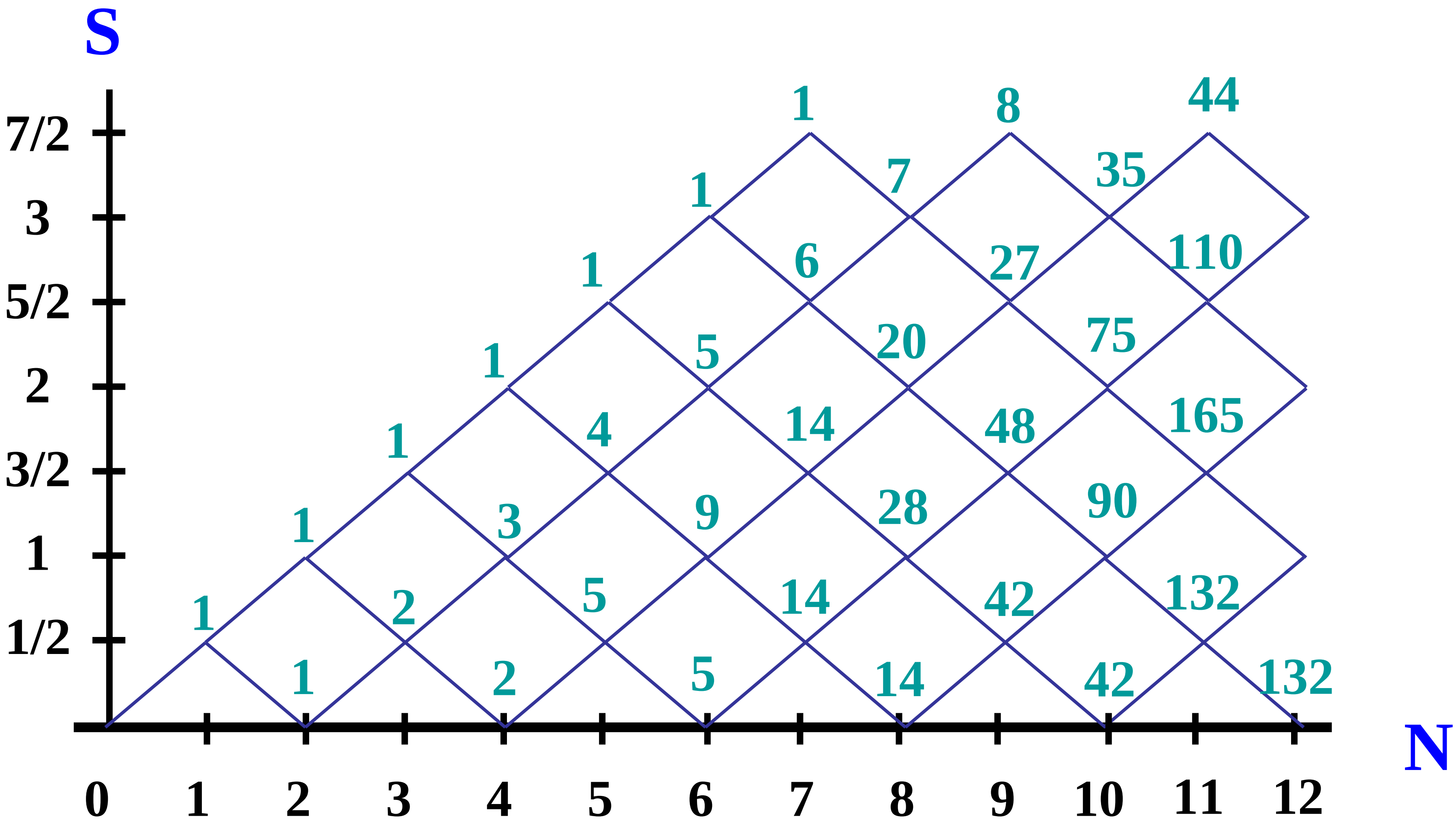
2. "Fusion Rule" for adding topological charge:

$$s_1 \otimes s_2 = |s_1 - s_2| \oplus \left(|s_1 - s_2| + 1\right) \oplus \cdots \oplus \min\left[s_1 + s_2, s_1 + s_2 - k/2\right]$$

For example: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

→ Two ● particles can have total topological charge **0** or **1**.

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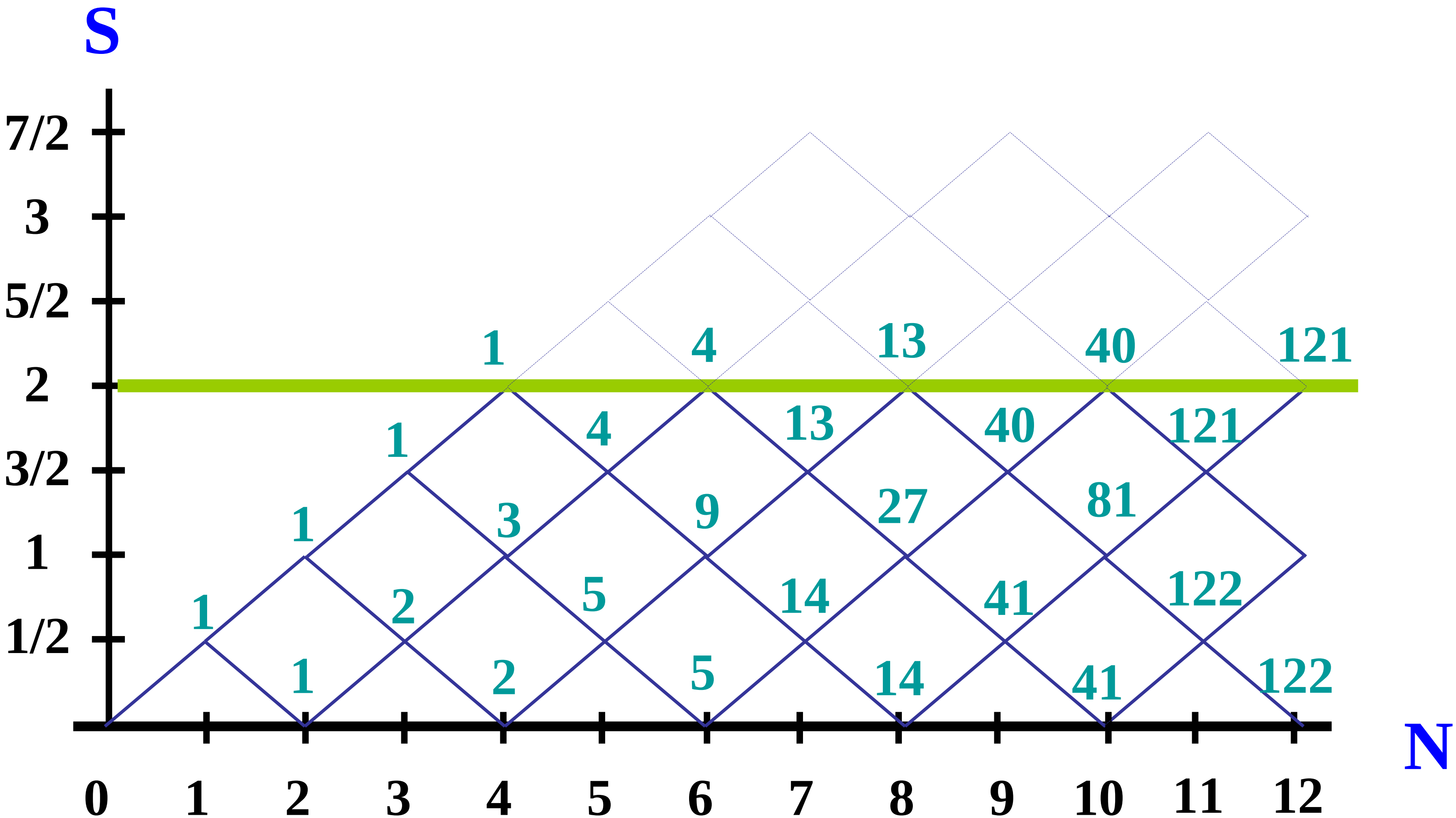
k = 4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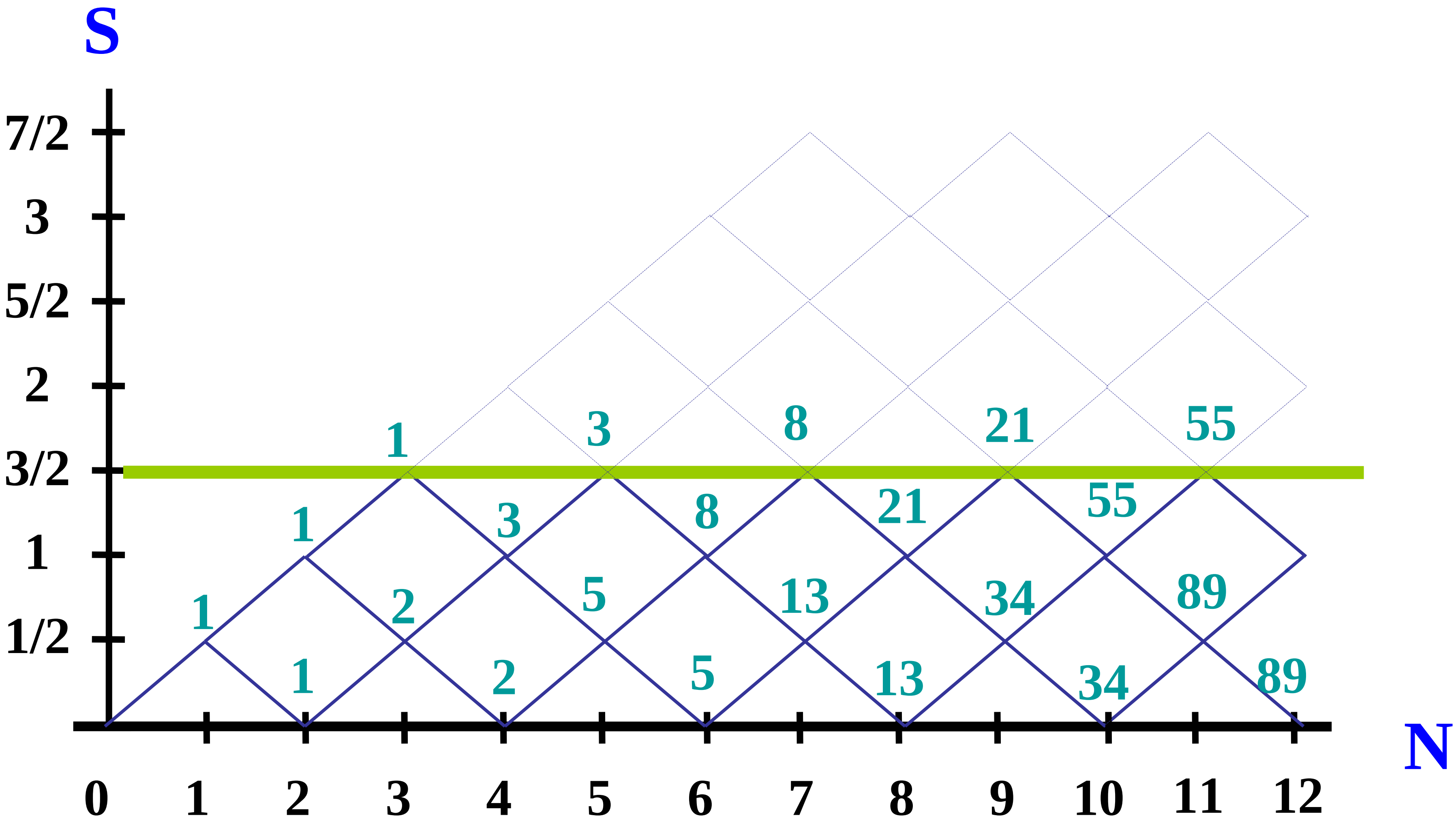
122

k = 3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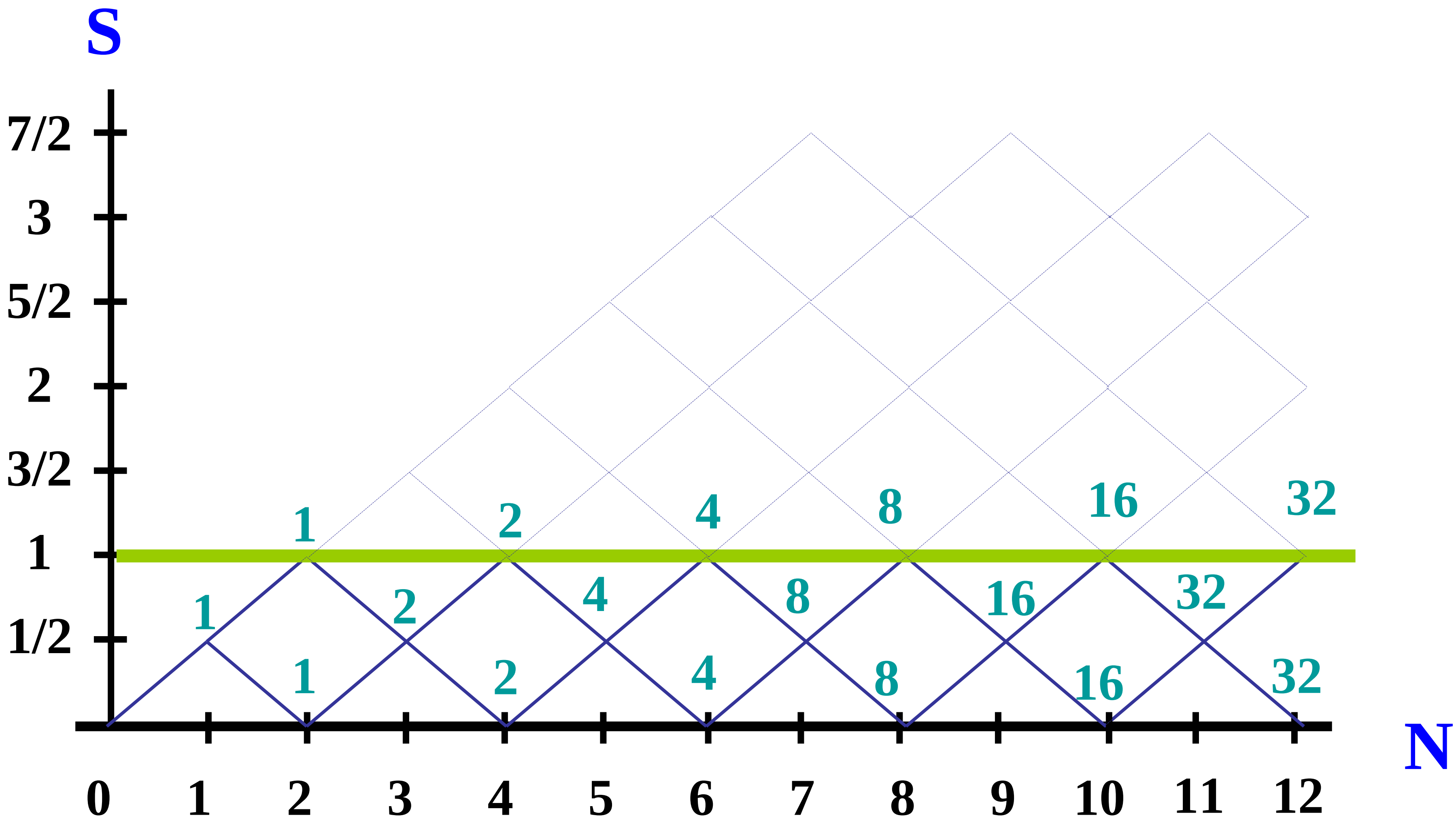
1 3 8 21 55
1 2 5 13 34 89
1 2 5 13 34 89

Fib(N+1)

k = 2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4
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16
32
N
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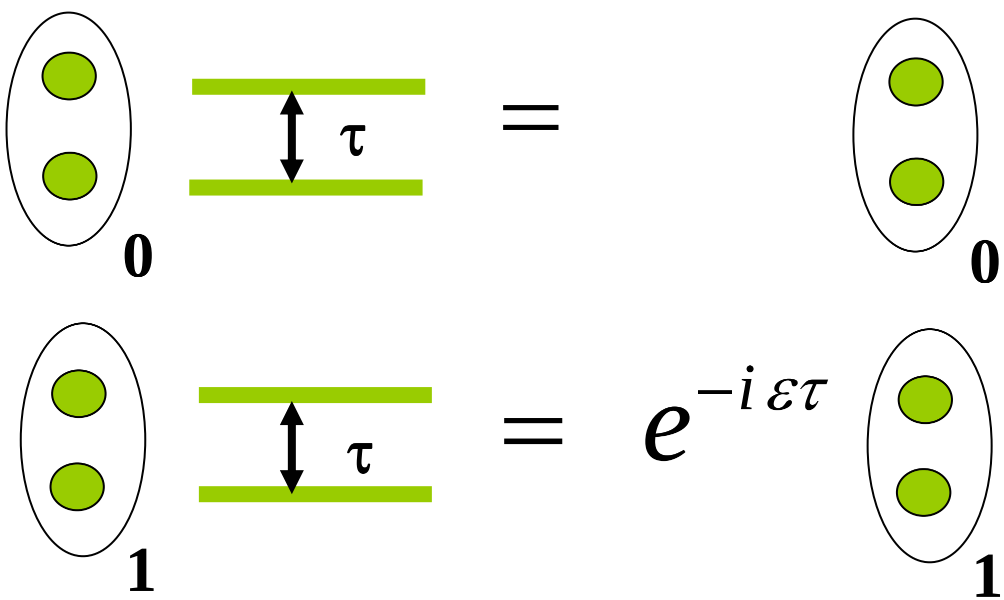
12

$2^{N/2-1}$

Exchange-Based Quantum Computing

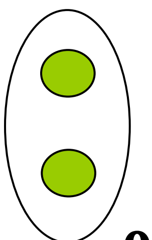
Compute by turning on and off the **exchange interaction** between neighboring spin-1/2 particles for some time **τ,** splitting the energy of the total spin 0 and total spin 1 sectors by ***ε***.



$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varepsilon\tau} \end{pmatrix}$$

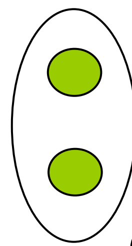
Topological Quantum Computing

Compute by braiding quasiparticles around one another.

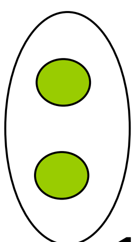


0

$$= e^{i\frac{2k+1}{2k+4}\pi}$$

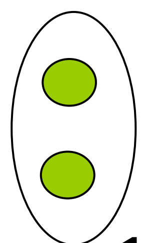


0



1

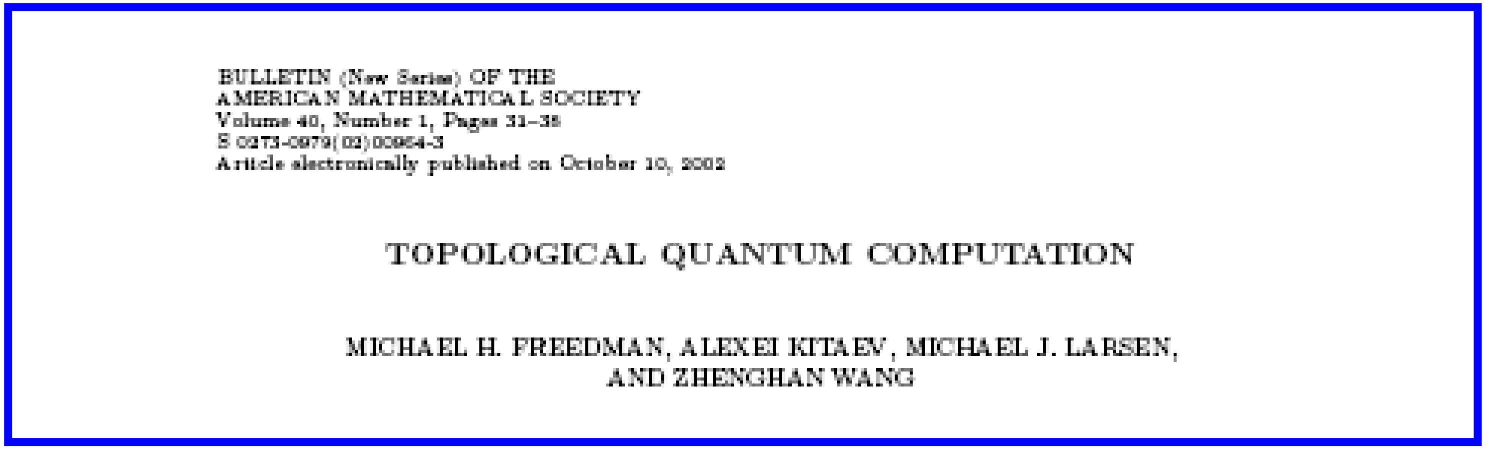
$$= e^{i\frac{1}{2k+4}\pi}$$



1

R matrix

$$R = \begin{pmatrix} e^{i\frac{2k+1}{2k+4}\pi} & 0 \\ 0 & e^{i\frac{1}{2k+4}\pi} \end{pmatrix}$$



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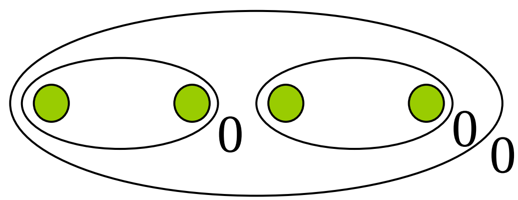
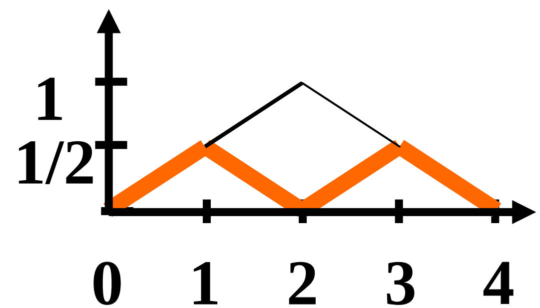
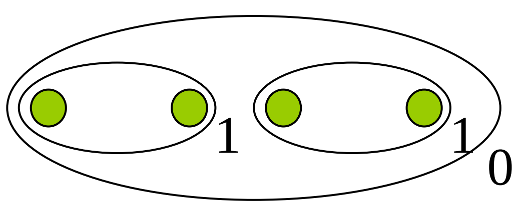
TOPOLOGICAL QUANTUM COMPUTATION

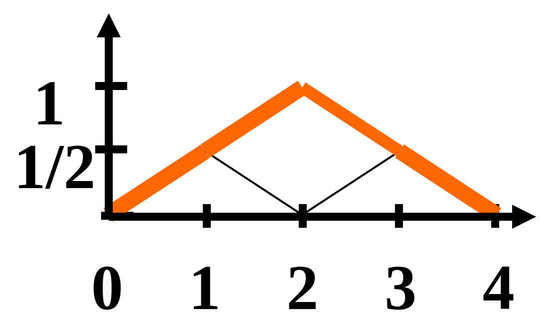
MICHAEL H. FREEDMAN, ALEXEI KITAEV, MICHAEL J. LARSEN,
AND ZHENGHAN WANG

4 particle encoding:

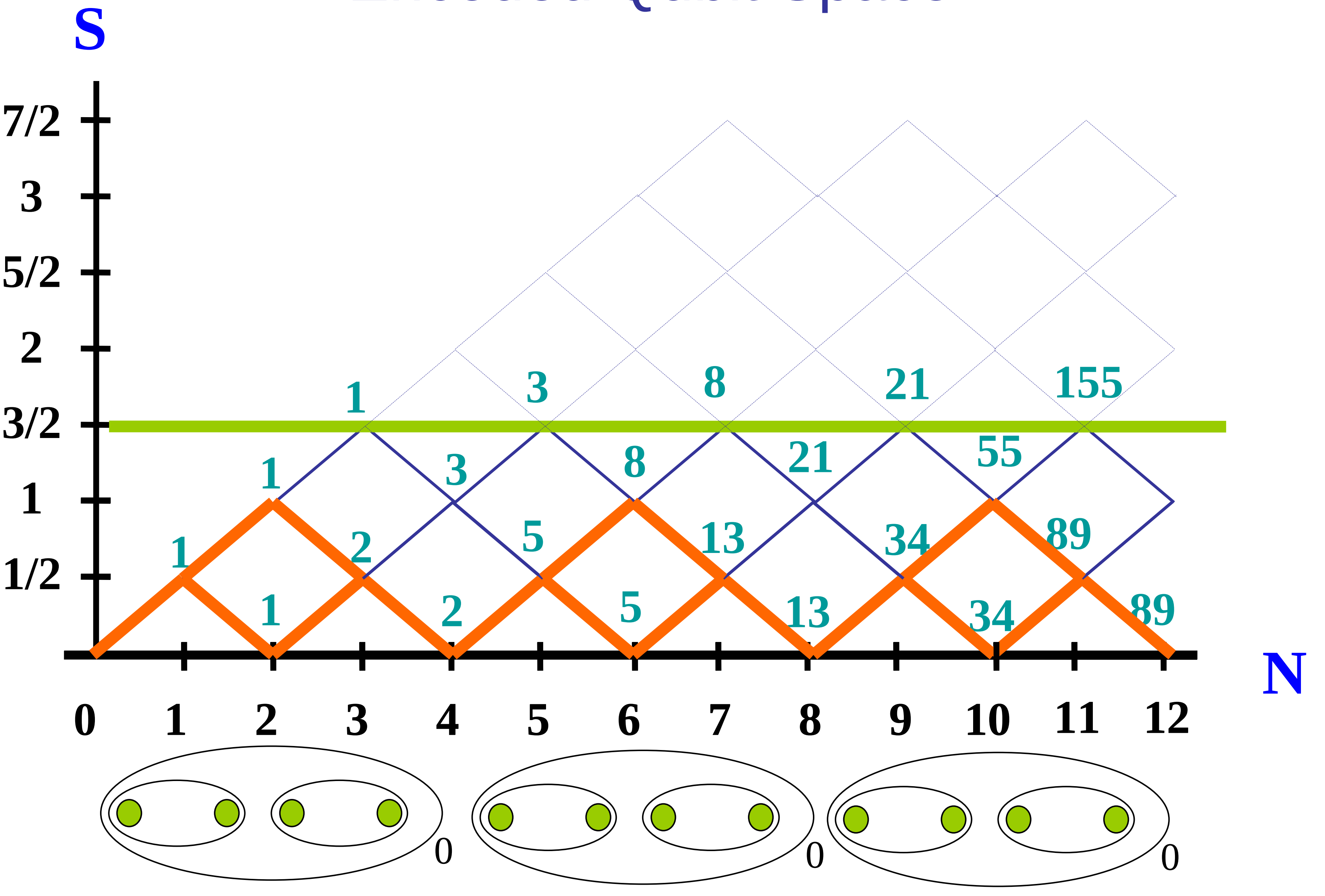
$|0\rangle =$ 



$|1\rangle =$ 



Encoded Qubit Space



Let's focus on k=3 first.

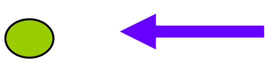
For k=3, if we only have particles with topological charge **1**, the fusion rules are simply:

$$0 \otimes 0 = 0; \quad 0 \otimes 1 = 1 \otimes 0 = 1; \quad 1 \otimes 1 = 0 \oplus 1$$

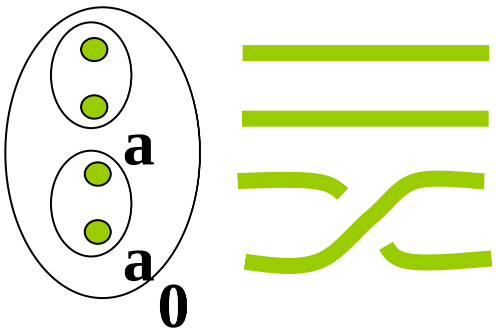
The remaining k=3 fusion rules, such as

$$0 \otimes \frac{3}{2} = \frac{3}{2}; \qquad \frac{1}{2} \otimes \frac{3}{2} = 1; \qquad 1 \otimes \frac{3}{2} = \frac{1}{2}; \qquad \frac{3}{2} \otimes \frac{3}{2} = 0$$

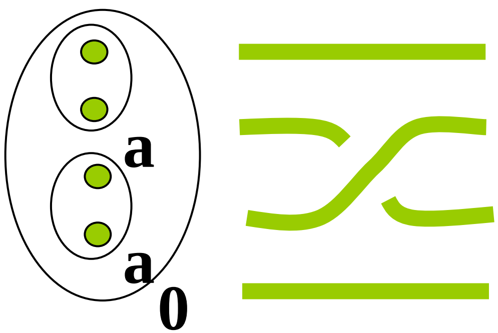
imply that the **3/2** particle is effectively abelian (it has non-branching fusion rules) and the braiding properties of particles with topological charges **1/2** and **1** are equivalent (up to a phase).

Fibonacci anyon:  **topological charge 1 in $SU(2)_3$**

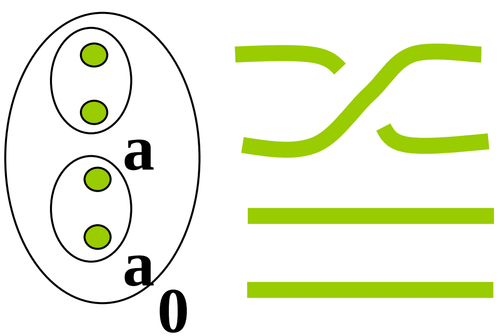
Braiding Matrices for 4 Fibonacci anyons



$$\sigma_1 = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

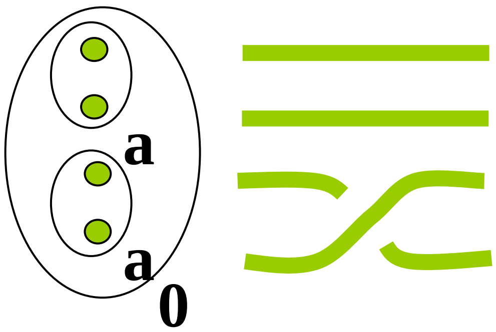


$$\sigma_2 = F\,\sigma_1 F = \begin{pmatrix} -\tau\, e^{-i\pi/5} & \sqrt{\tau}\, e^{-i3\pi/5} \\ \sqrt{\tau} e^{-i3\pi/5} & -\tau \end{pmatrix}$$

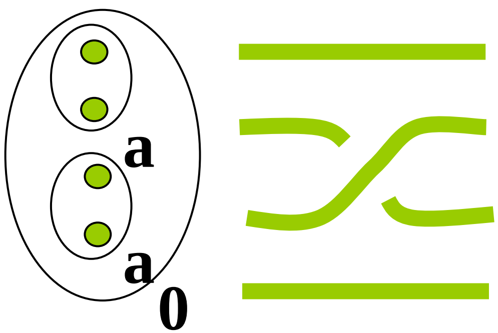


$$\sigma_3 = \sigma_1 = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

Braiding Matrices for 4 Fibonacci anyons

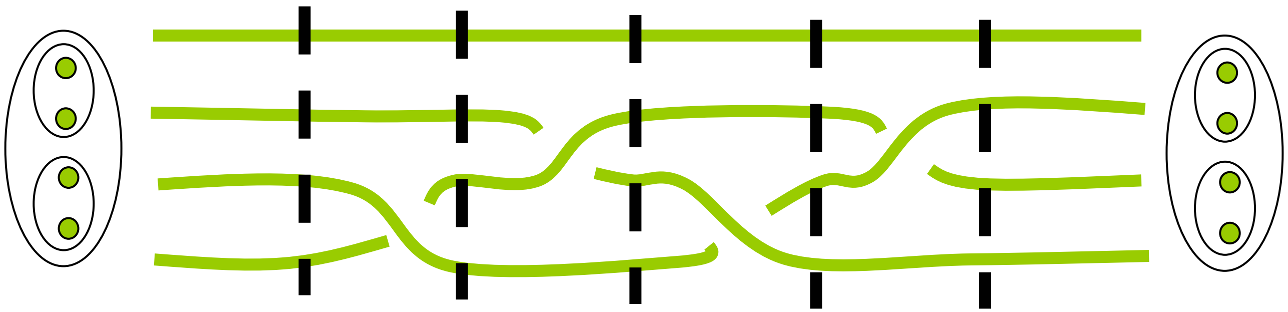


$$\sigma_1 = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$



$$\sigma_2 = F\,\sigma_1 F = \begin{pmatrix} -\tau\, e^{-i\pi/5} & \sqrt{\tau}\, e^{-i3\pi/5} \\ \sqrt{\tau}\, e^{-i3\pi/5} & -\tau \end{pmatrix}$$

$|\Psi_i\rangle$



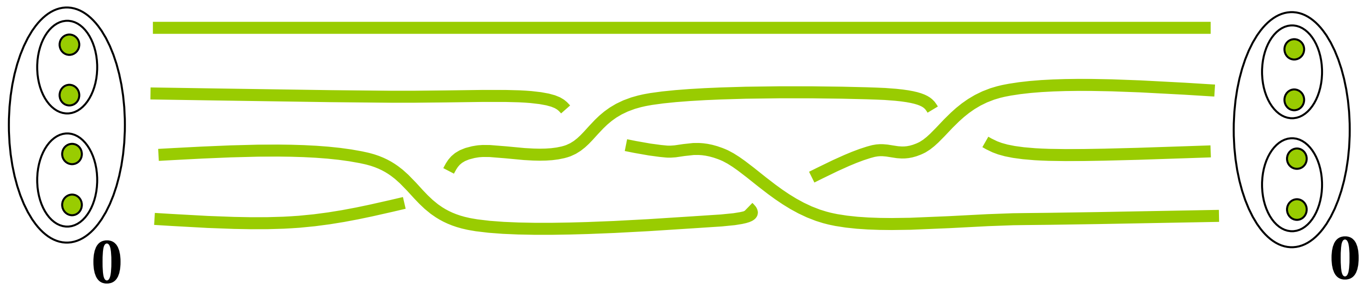
$$\sigma_1^{-1} \quad \sigma_2 \quad \sigma_1^{-1} \quad \sigma_2 \quad = \quad \boldsymbol{M}$$

$$|\Psi_f\rangle = \boldsymbol{M}^T |\Psi_i\rangle$$

Single Qubit Operations

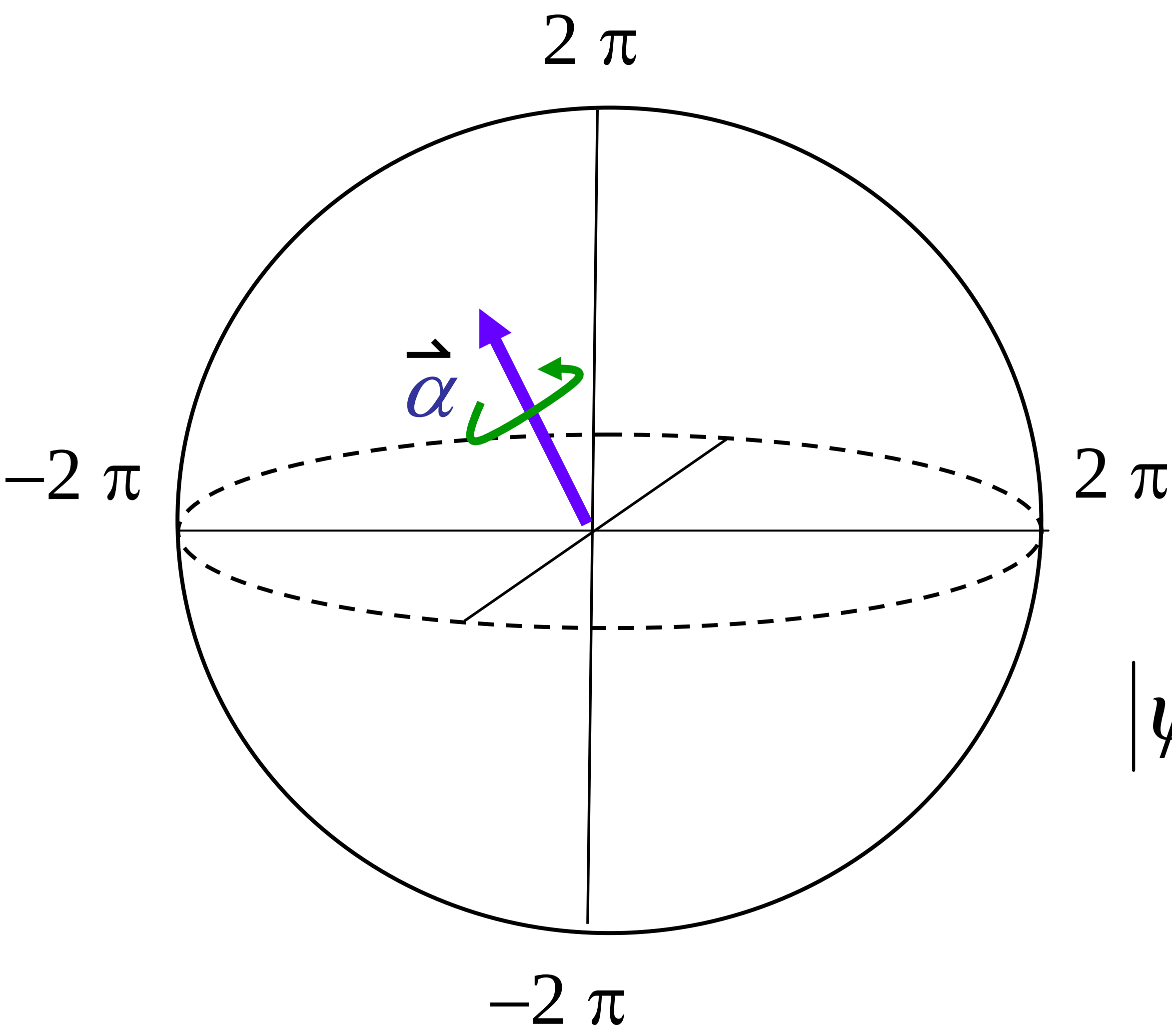
General rule: Braiding inside an oval does not change the total topological charge of the enclosed particles.

Important consequence: As long as we braid *within* a qubit, there is **no leakage error.**

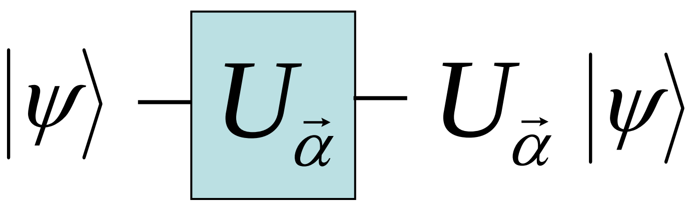


Can we do arbitrary single qubit rotations this way?

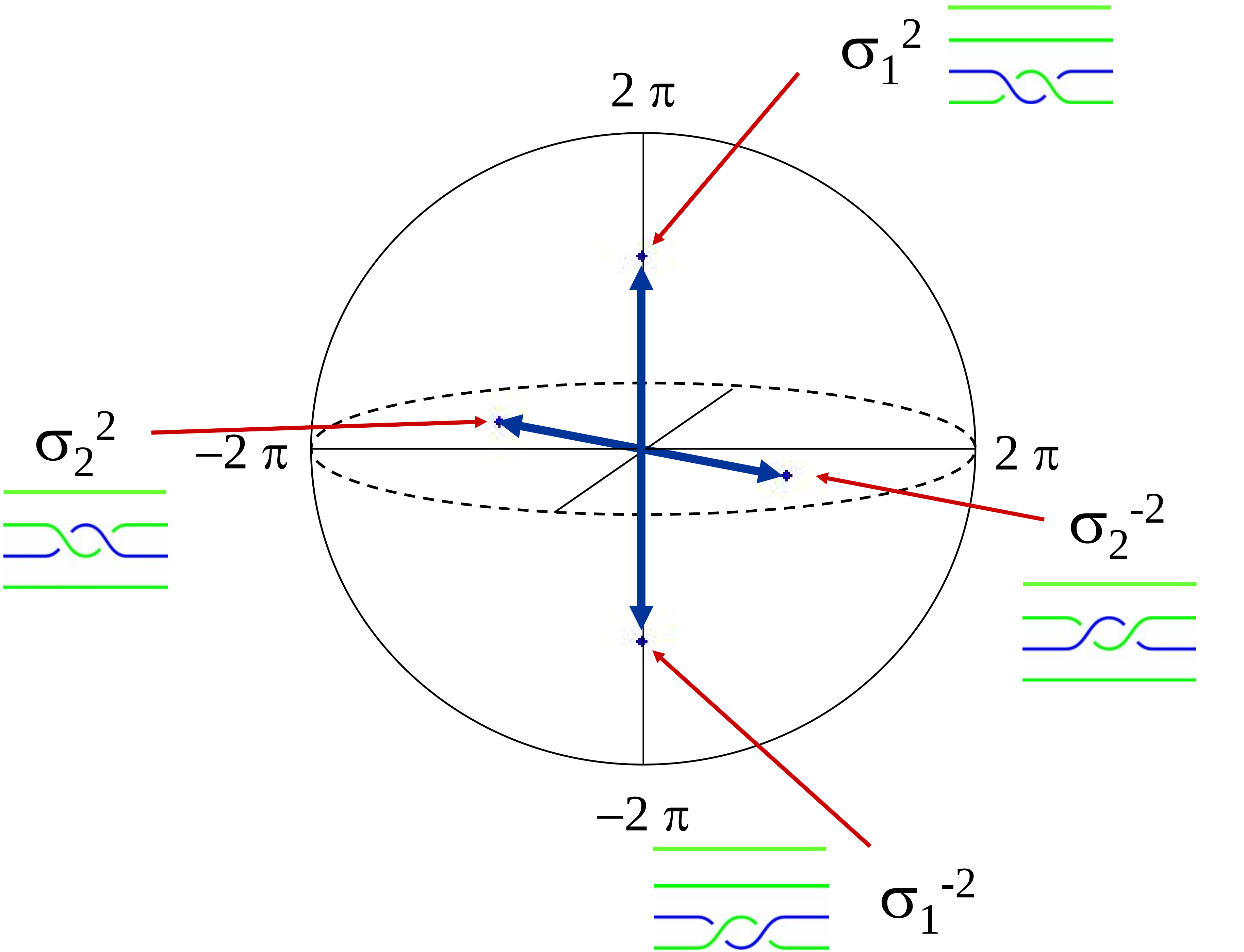
Single Qubit Operations are Rotations



The set of all single qubit rotations lives in a solid sphere of radius 2π.

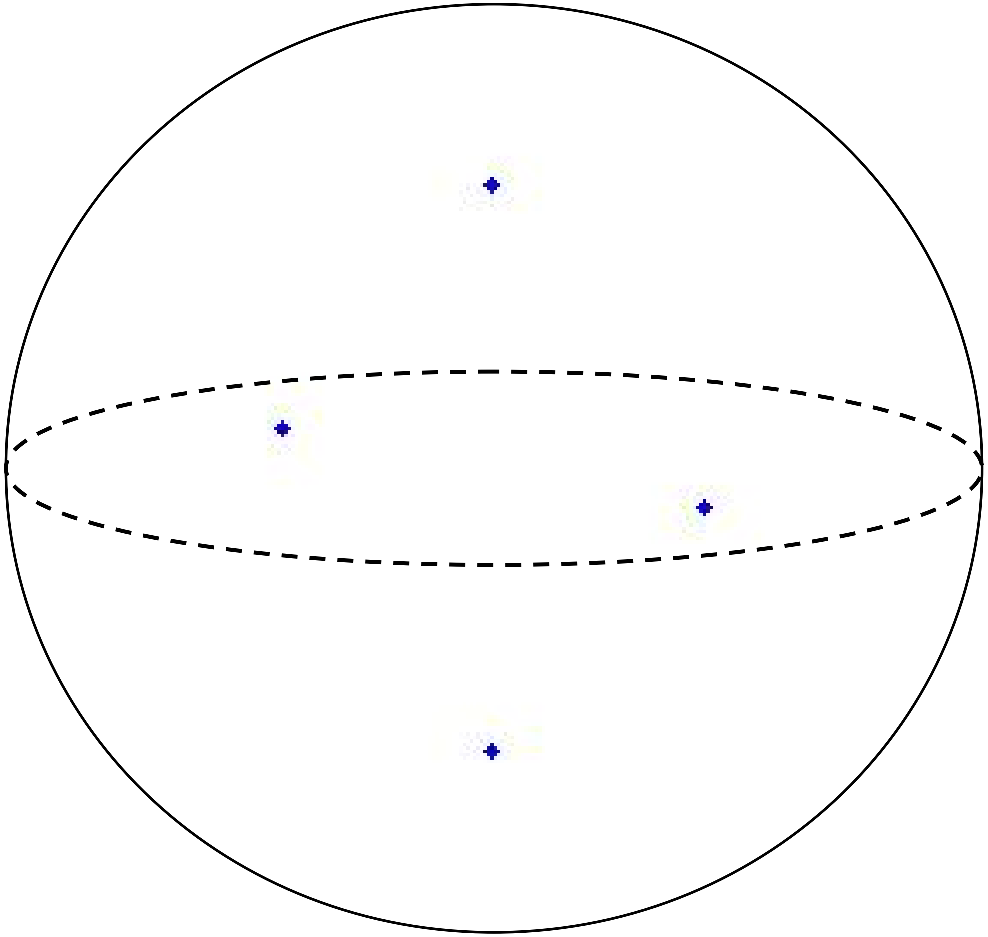


$$U_{\vec{\alpha}} = \exp \frac{i\,\vec{\alpha}\cdot\vec{\sigma}}{2}$$

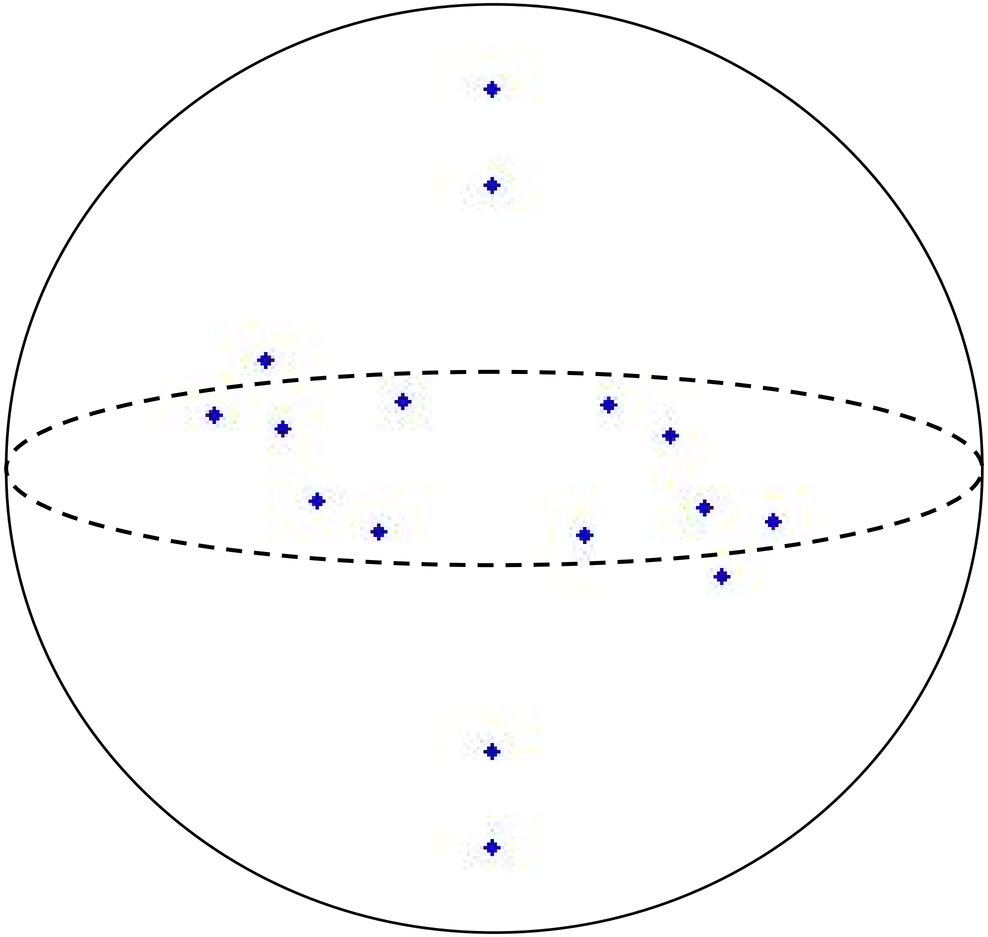


σ_1^{2}
2π
σ_2^{2}
-2π
2π
σ_2^{-2}
-2π
σ_1^{-2}

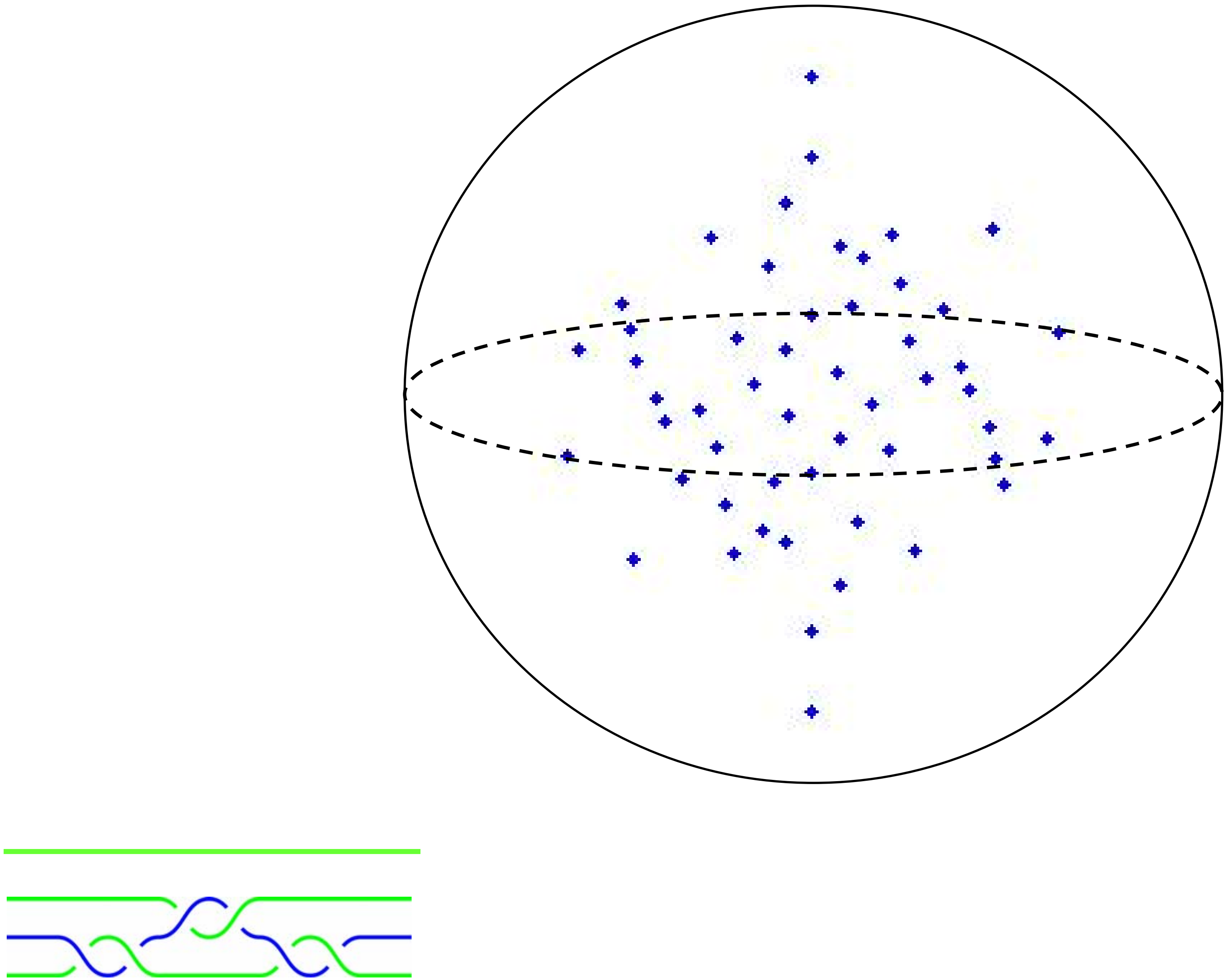
N = 1



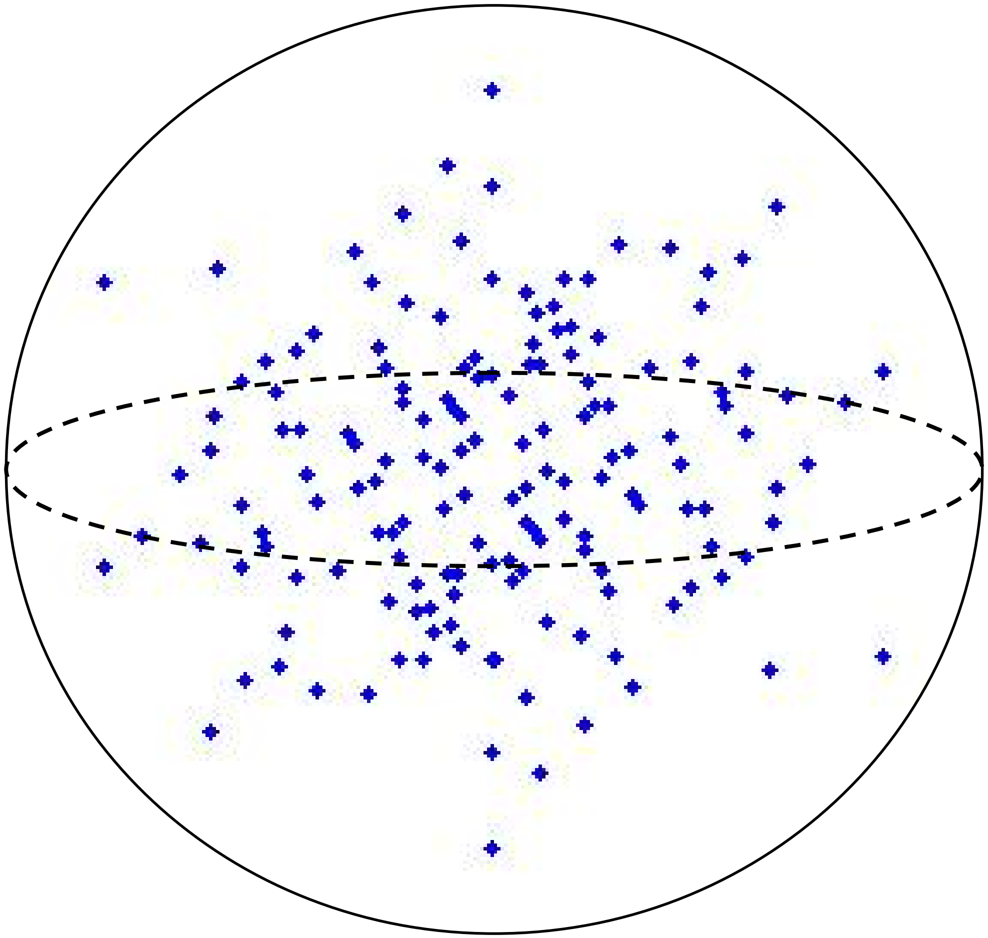
N = 2



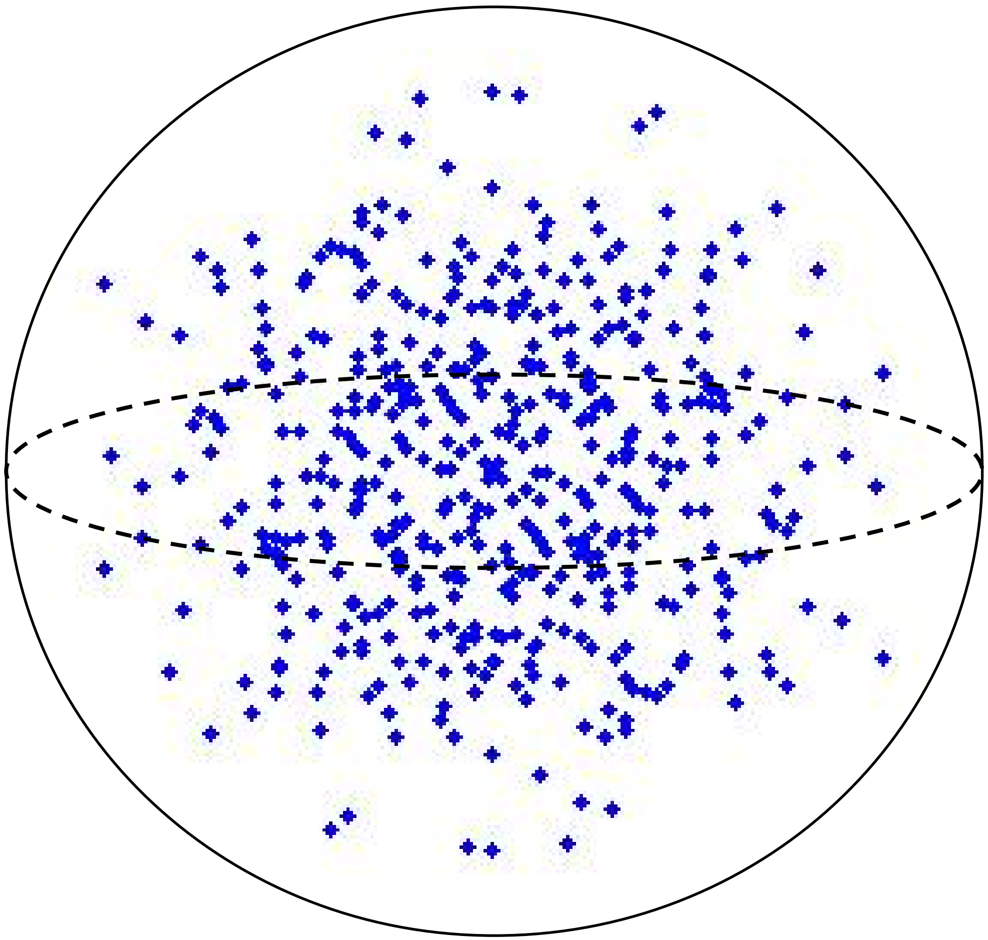
N = 3



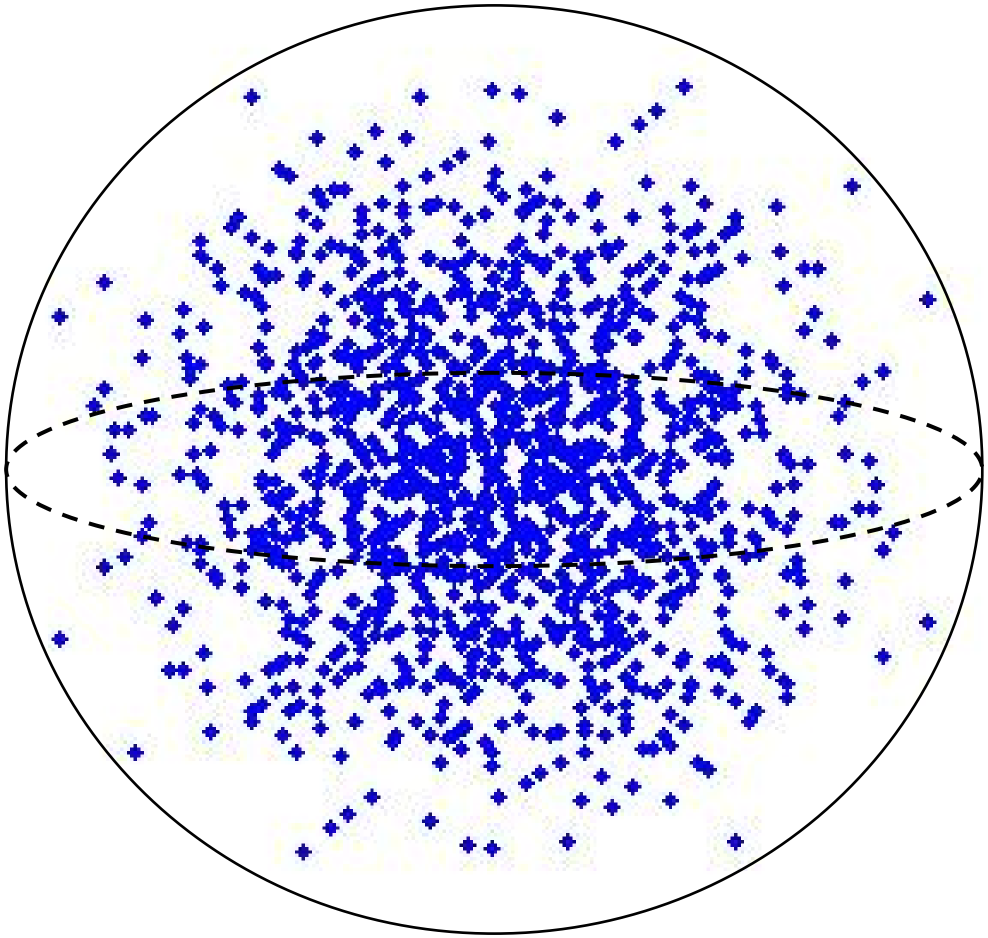
$N = 4$

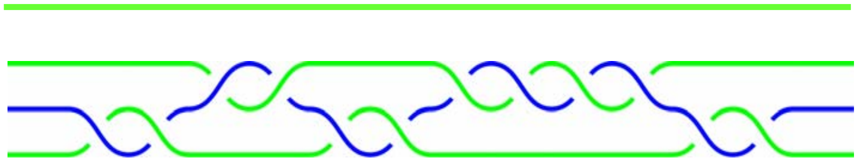


N = 5

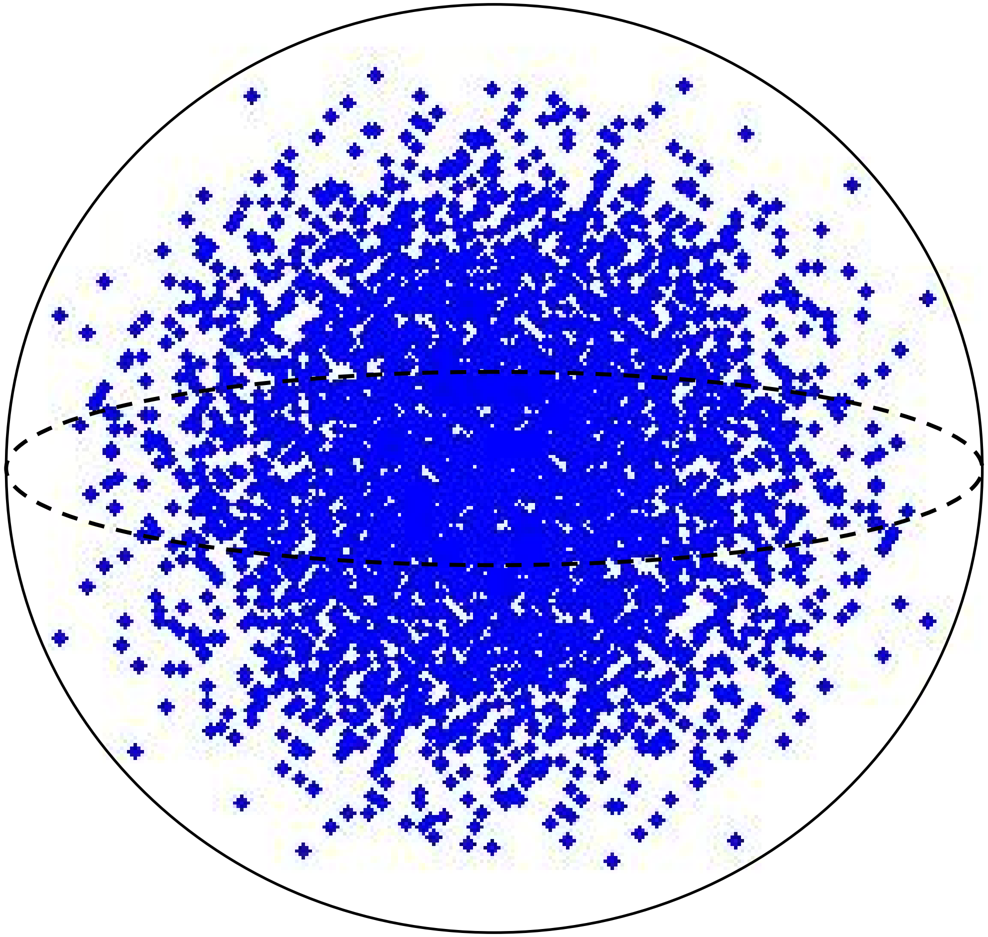


N = 6

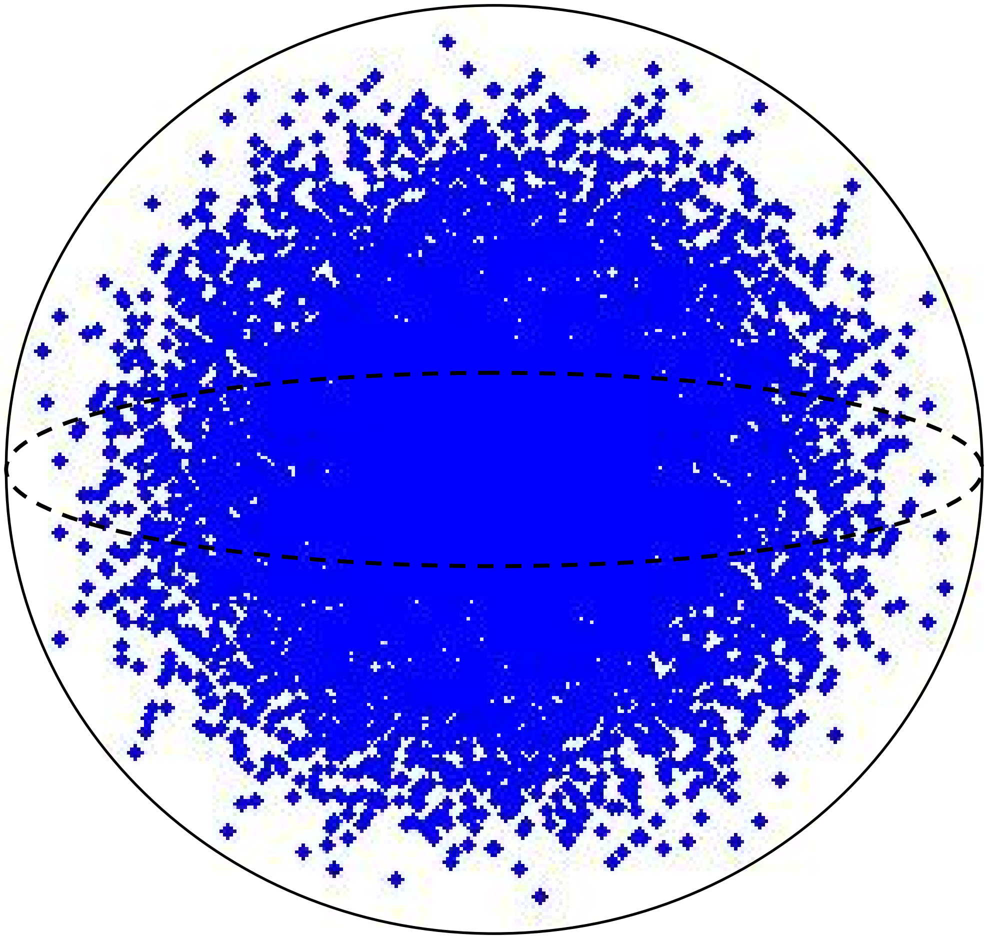




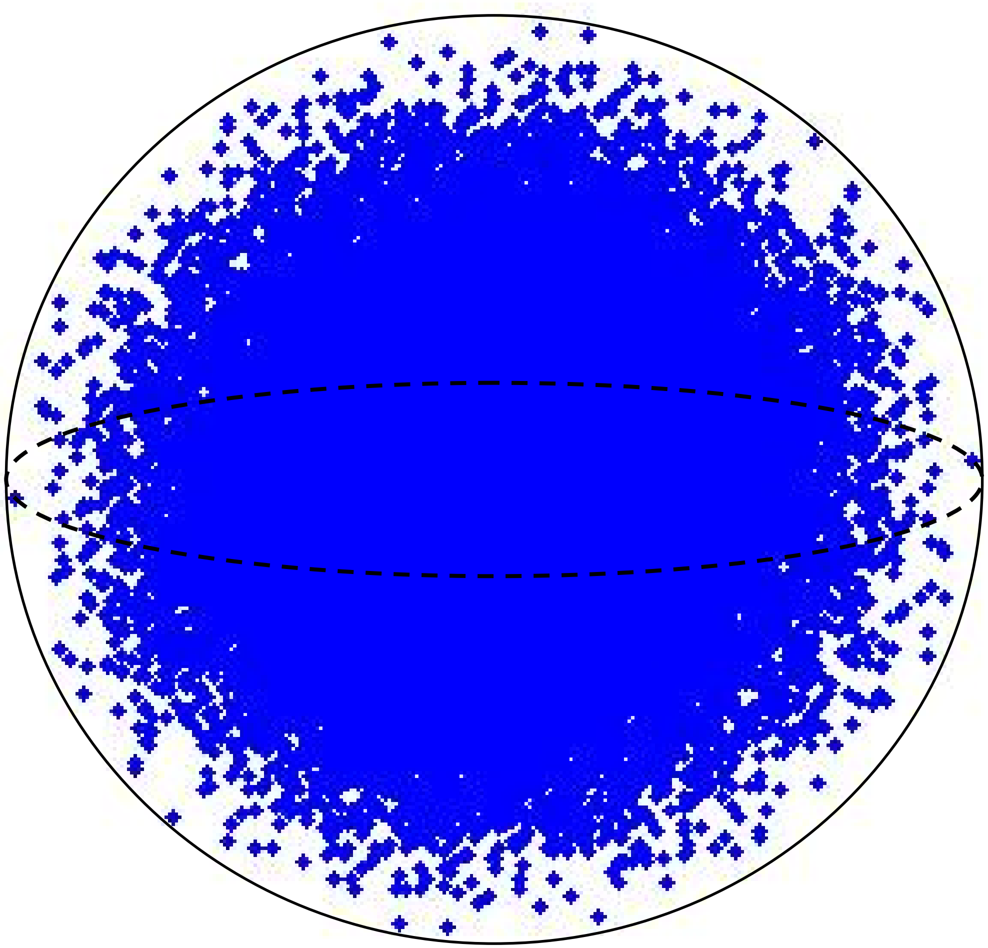
N = 7



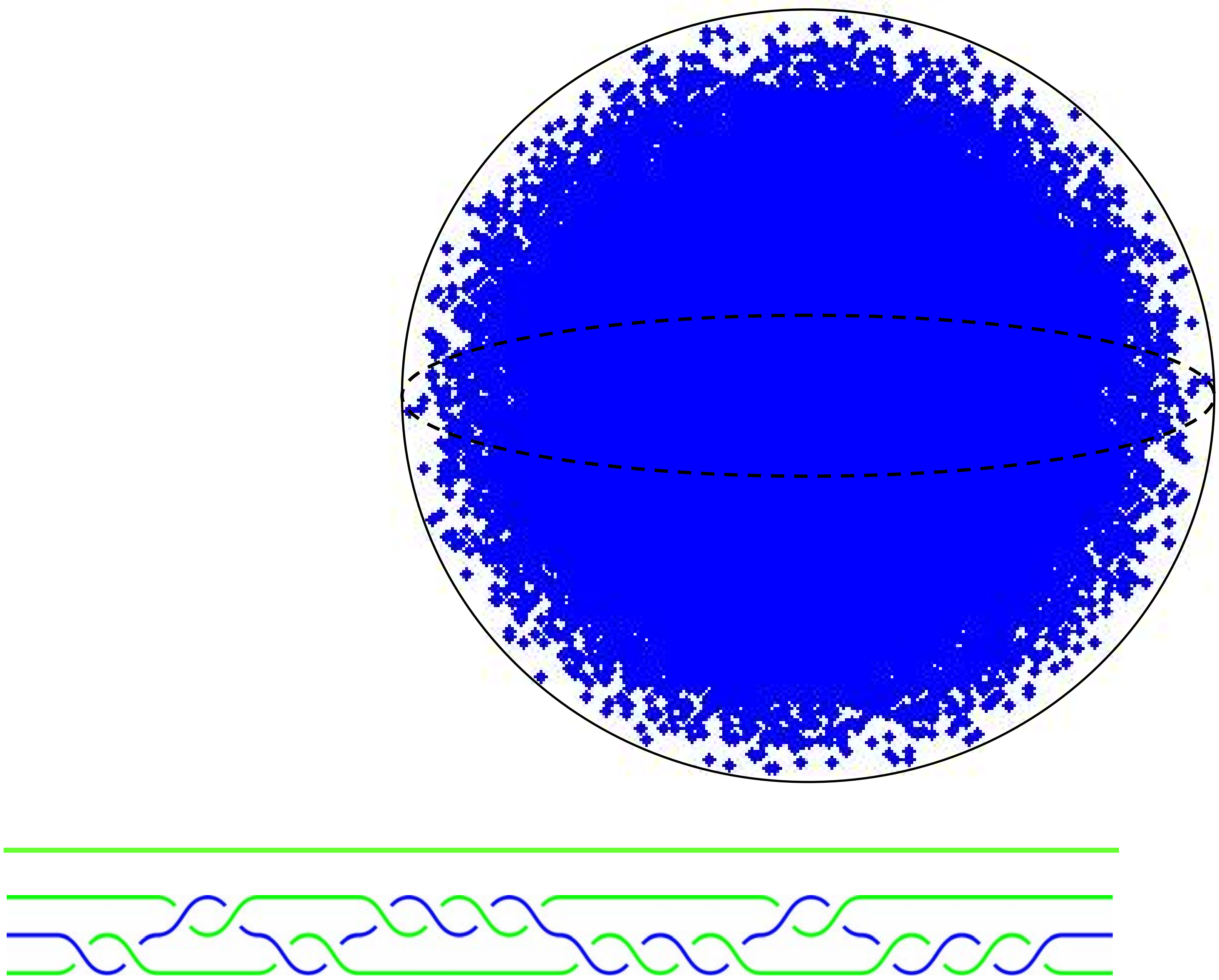
N = 8



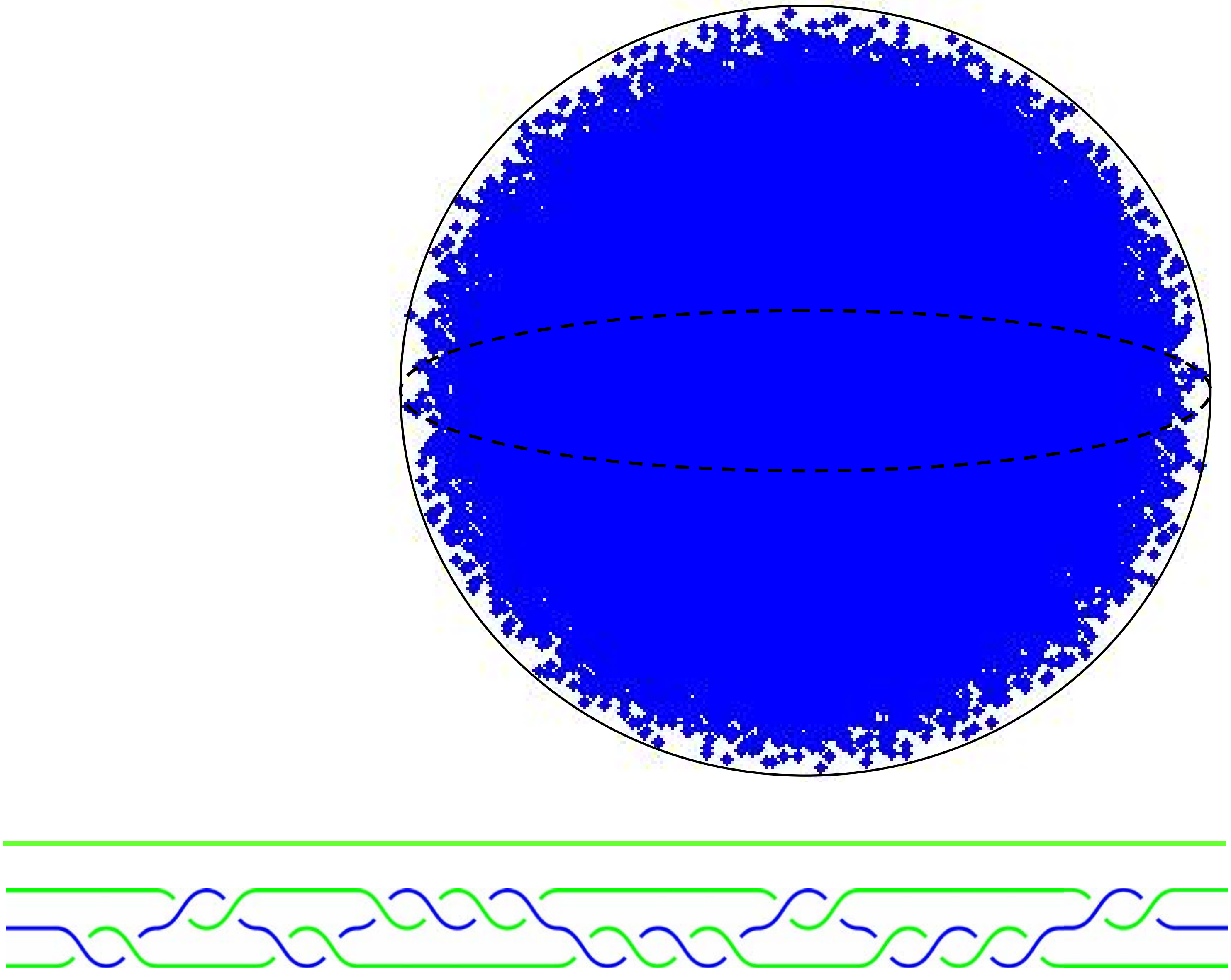
N = 9



N = 10

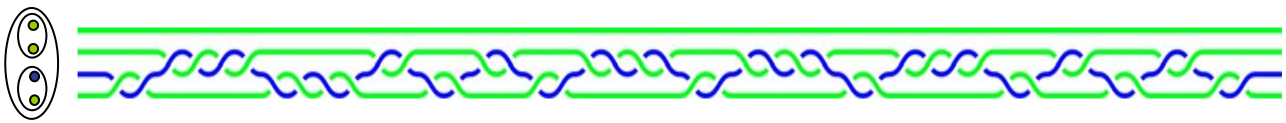


N = 11



Brute Force Search

$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + O(10^{-3})$$



Brute force searching rapidly becomes infeasible as braids get longer.

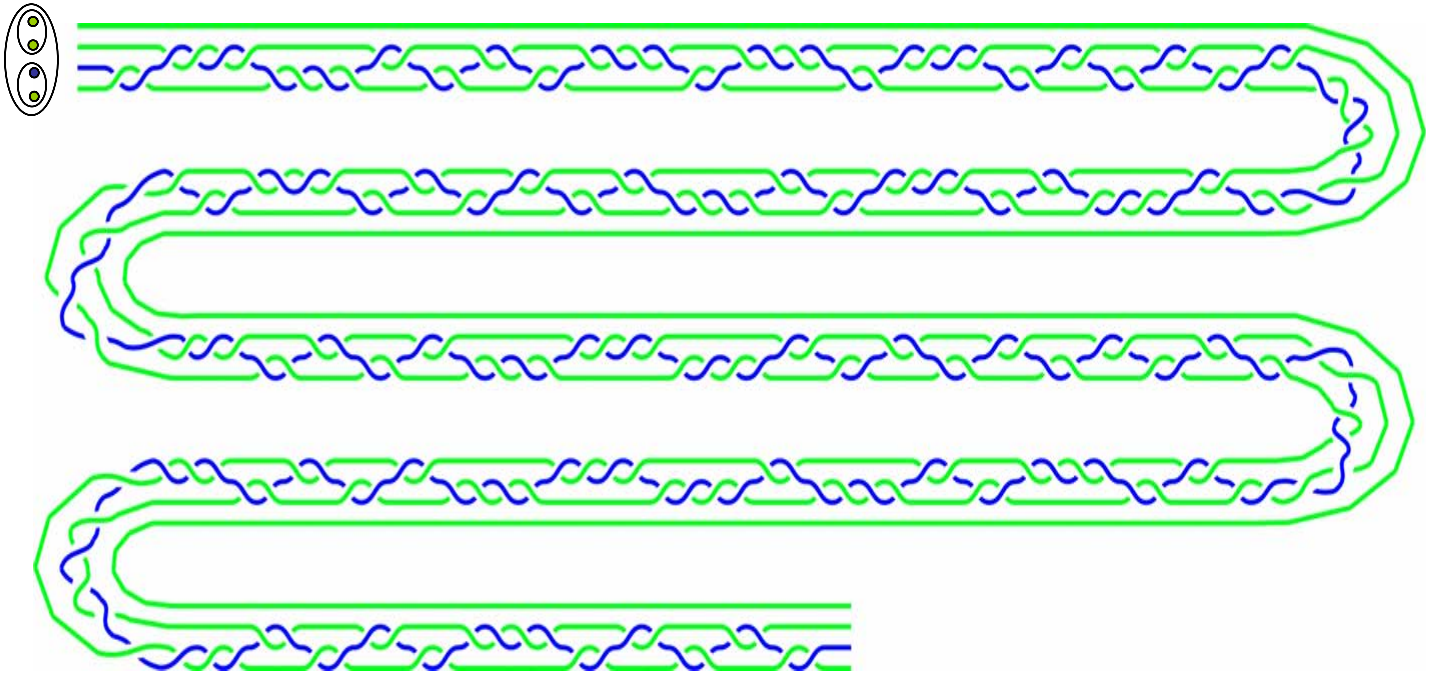
Fortunately, a clever algorithm due to Solovay and Kitaev allows for systematic improvement of the braid given a sufficiently dense covering of *SU(2)*.

Solovay-Kitaev Construction

(Actual calculation)

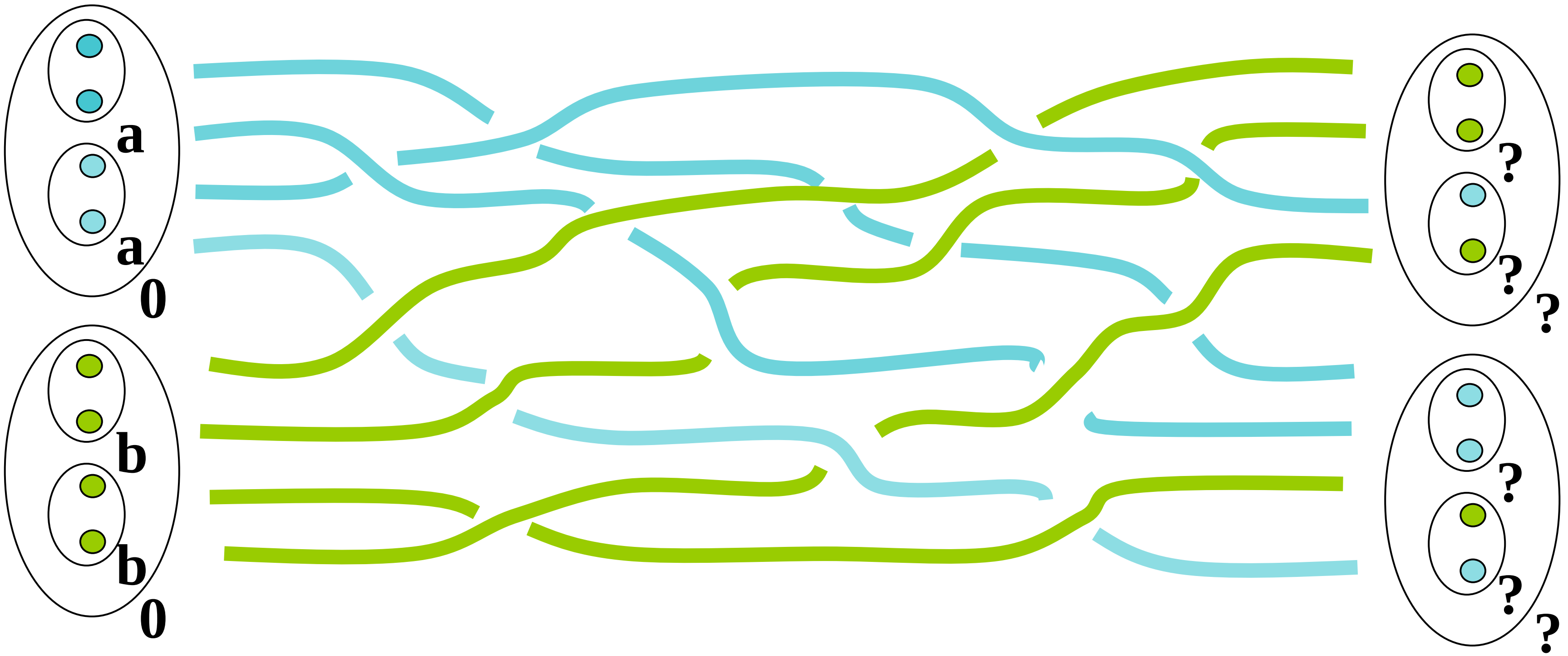
$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + O(10^{-4})$$

ε



Braid Length $|\ln \varepsilon|^c, \quad c \approx 4$

Two Qubit Gates

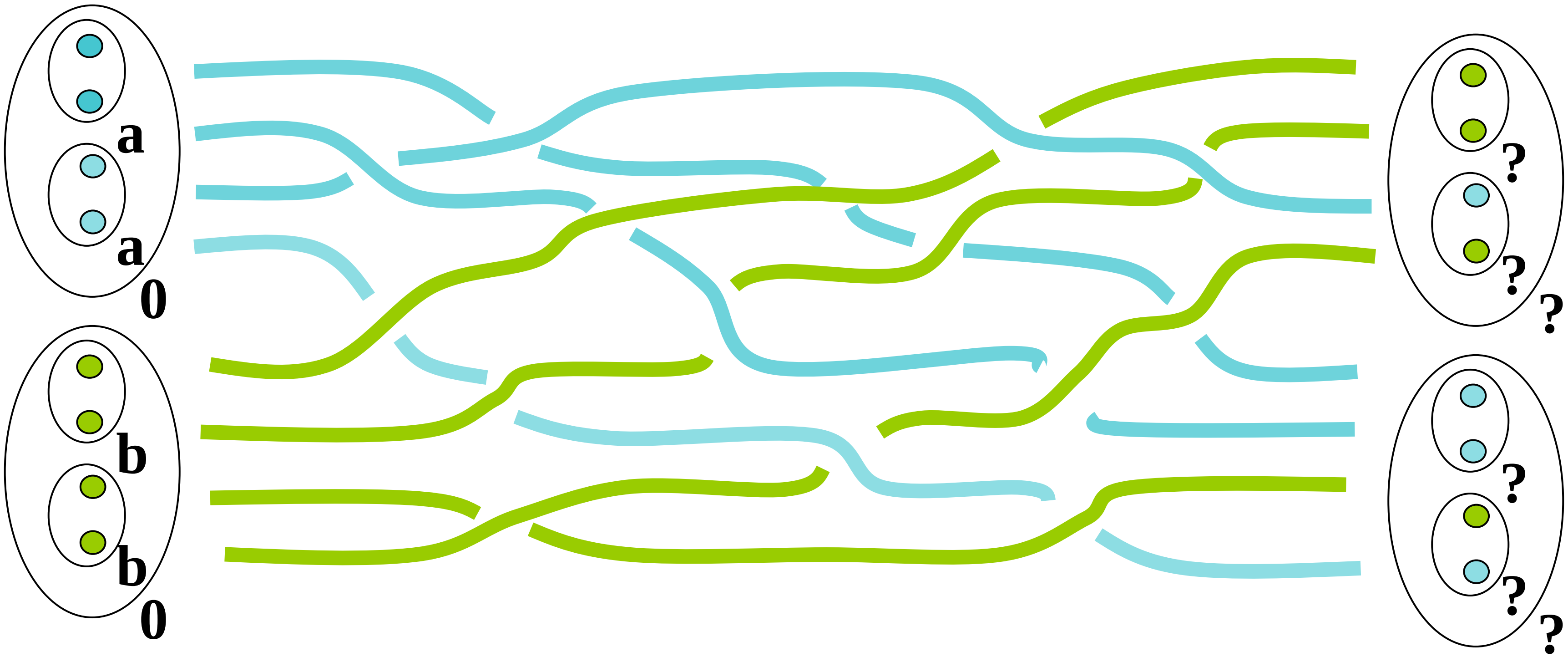


Problems:

1. We are pulling quasiparticles out of qubits: **Leakage error!**

2. **168** dimensional search space (as opposed to **3** for three-particle braids). Straightforward "brute force" search is not feasible.

Two Qubit Gates

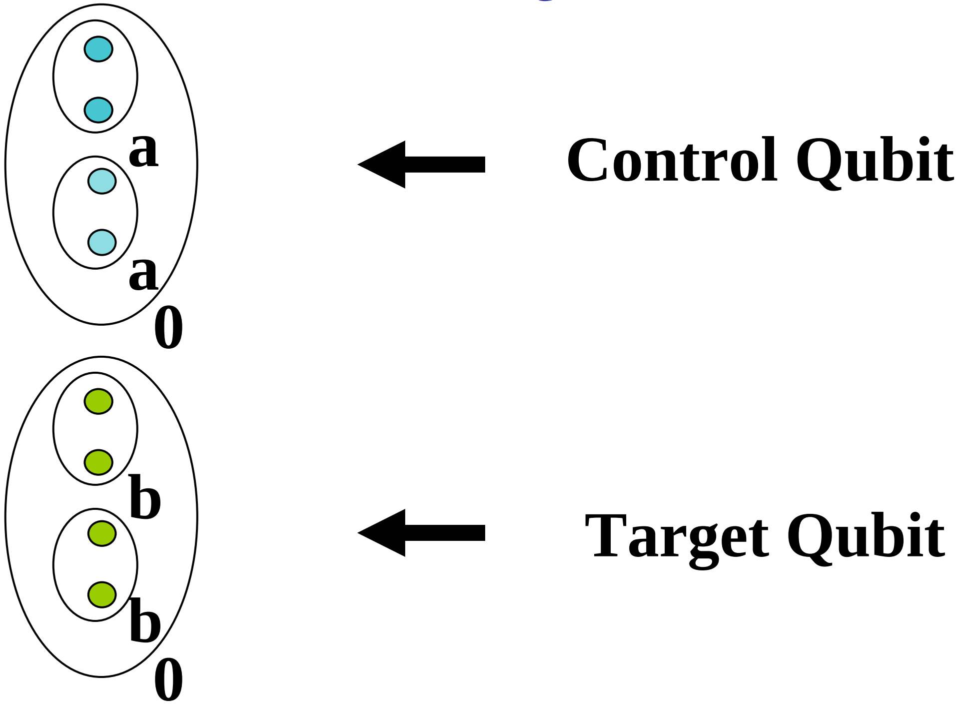


$$\begin{matrix} ab = & 00 & 01 & 10 & 11 \end{matrix}$$

$$U_{two-qubit} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

4 x **4** block acting on the logical qubit space. In general this matrix is *not unitary* due to *leakage error.*

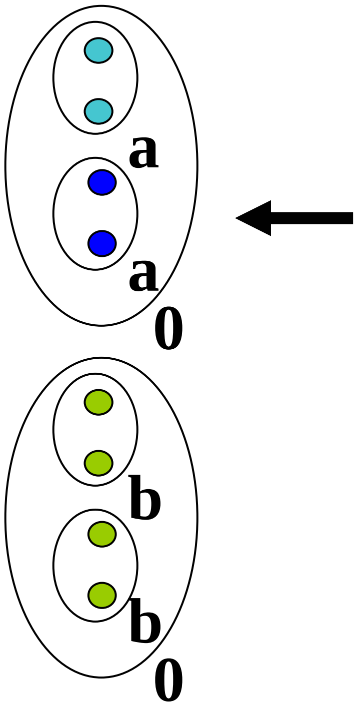
Two Qubit Controlled Gates



ab = 00 01 10 11

$$U_{two-qubit} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Two Qubit Controlled Gates

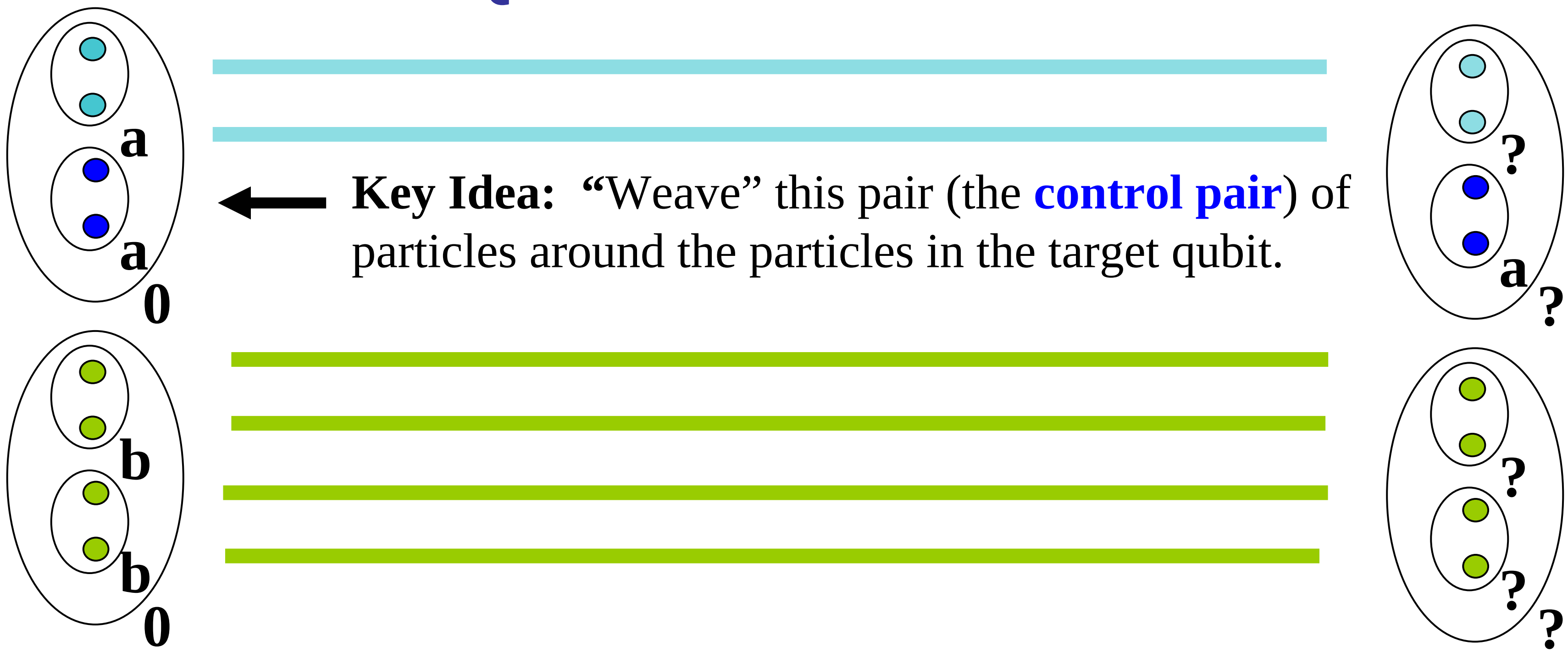


Key Idea: **"**Weave" this pair (the **control pair**) of particles around the particles in the target qubit.

ab = 00 01 10 11

$$U_{two-qubit} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Two Qubit Controlled Gates



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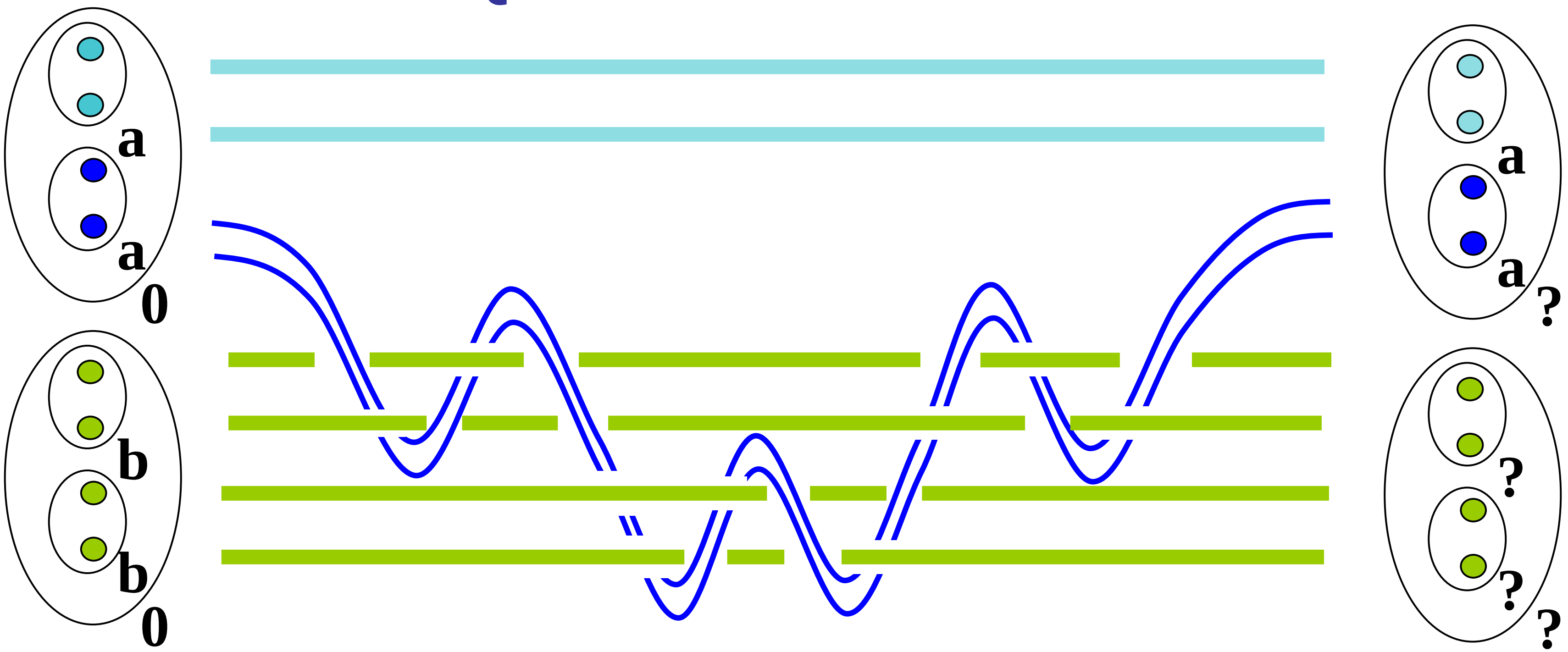
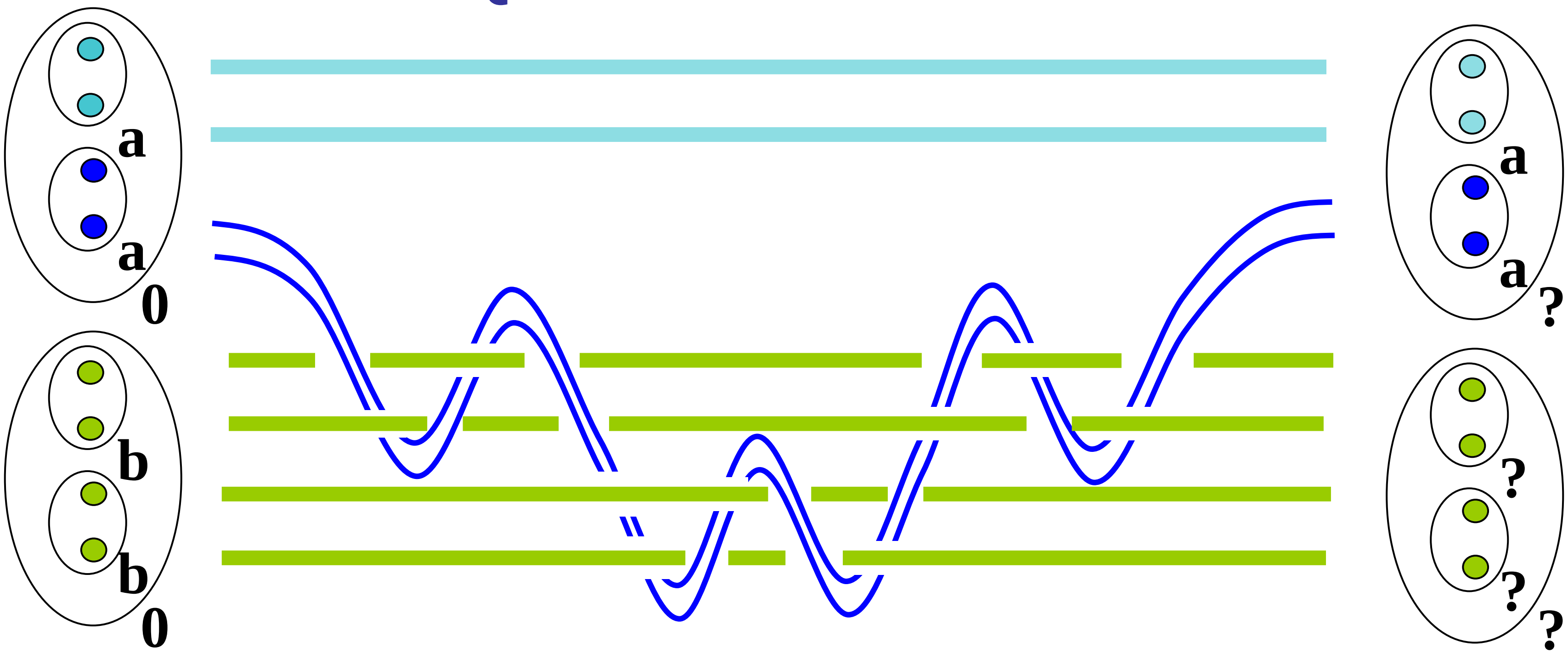
Two Qubit Controlled Gates



ab = 00 01 10 11

$$U_{two-qubit} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Two Qubit Controlled Gates



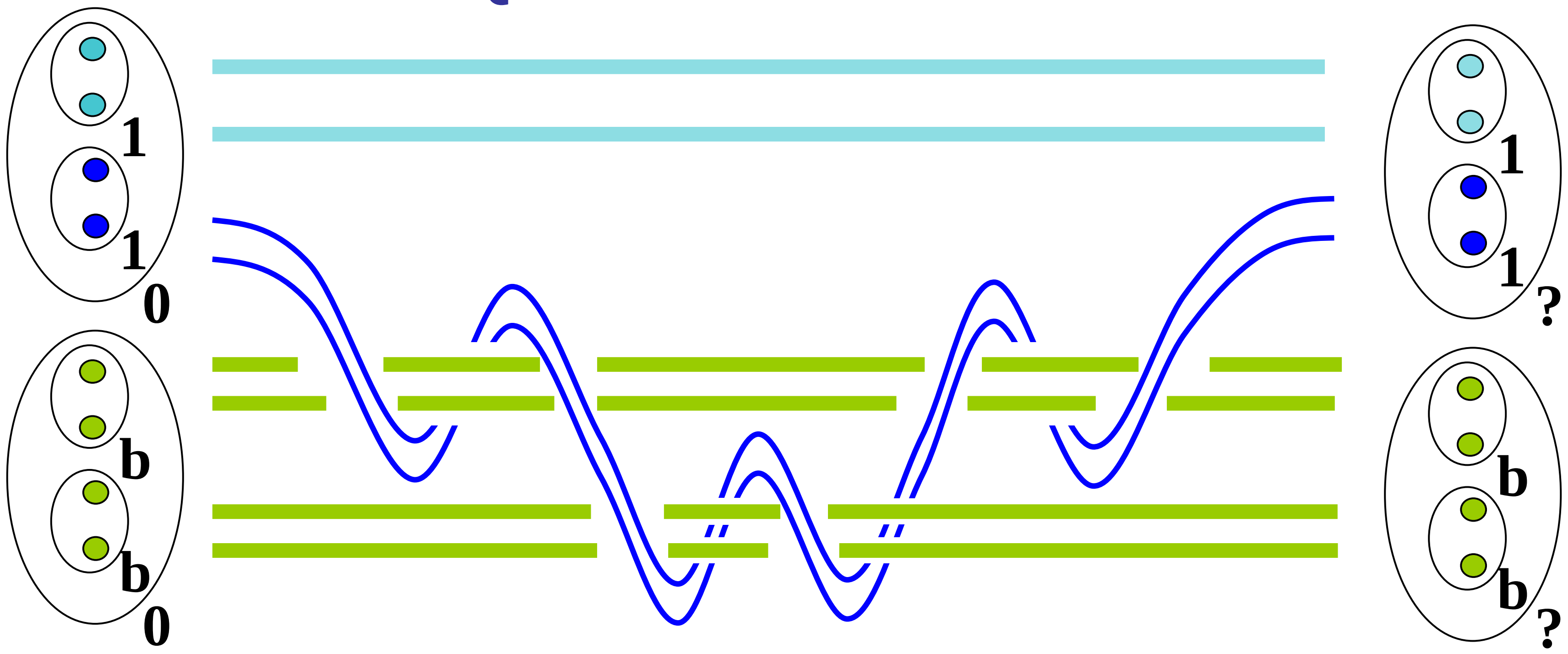
ab = 00 01 10 11

$$U_{two-qubit} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{array}\right]$$

Important Rule: Braiding an object with **topological charge 0** does not induce any transitions.

Only **a=1** sector is nontrivial.

Two Qubit Controlled Gates



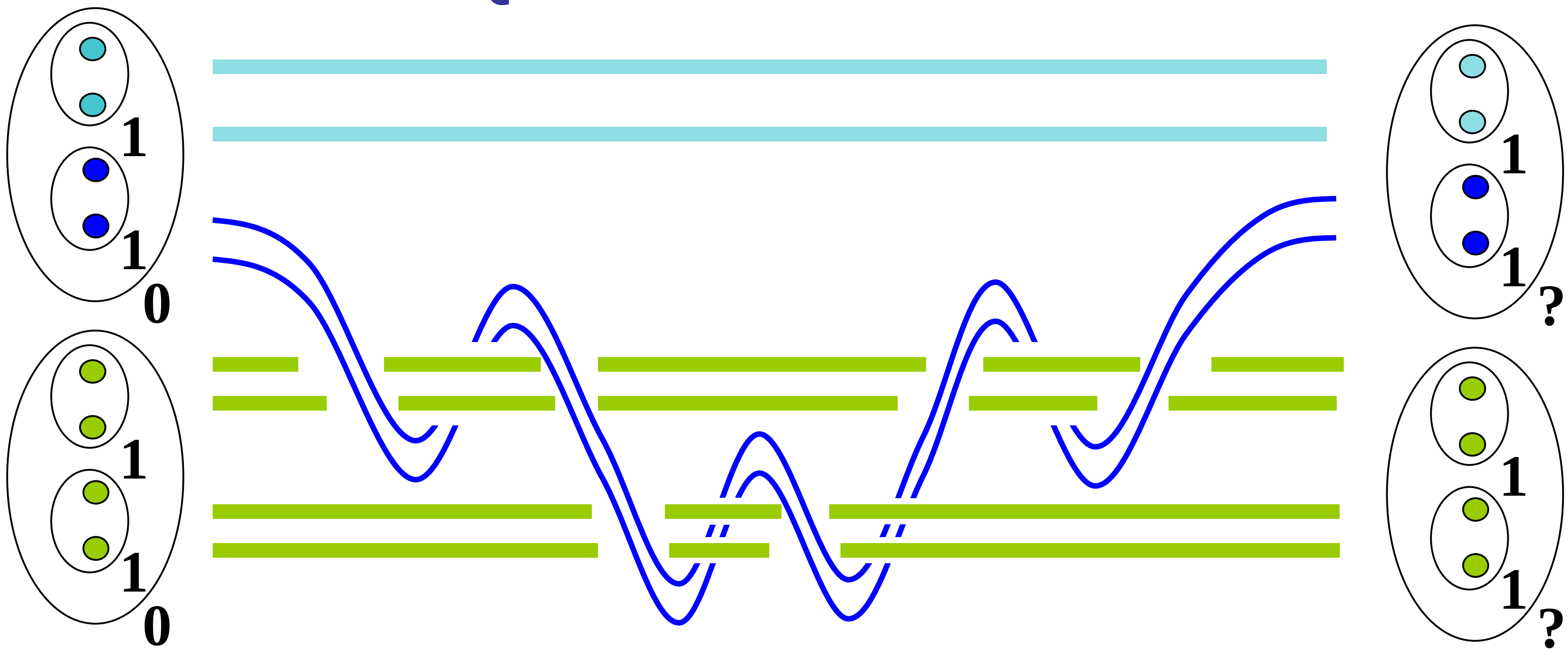
ab = 00 01 10 11

$$U_{two-qubit} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Another idea: Weave control pair around *pairs* of particles in the target qubit. If **b = 0** this braid again produces no transition.

Only **ab = 11** sector is nontrivial.

Two Qubit Controlled Gates



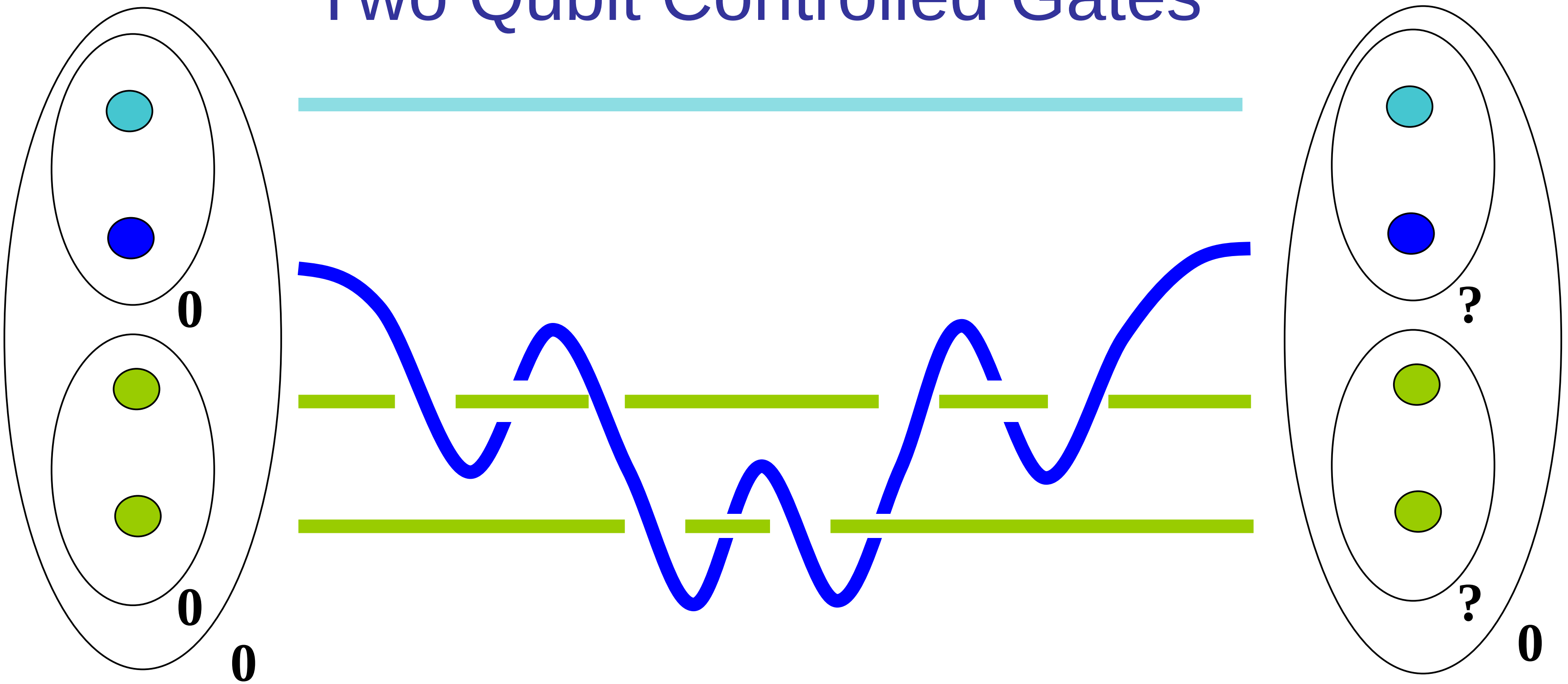
ab = 00 01 10 11

$$U_{two-qubit} = \left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a_{44} \end{array}\right]$$

Another idea: Weave control pair around *pairs* of particles in the target qubit. If **b = 0** this braid again produces no transition.

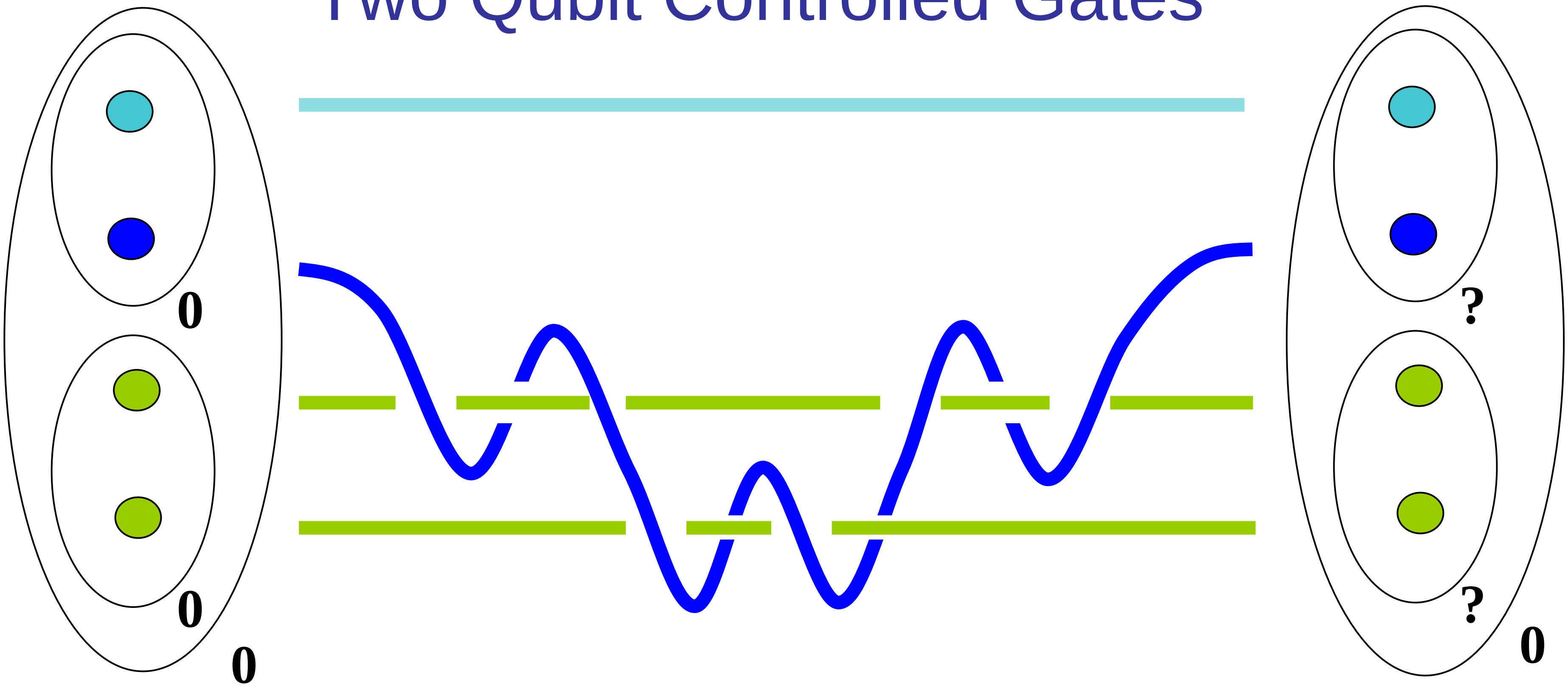
Only **ab = 11** sector is nontrivial.

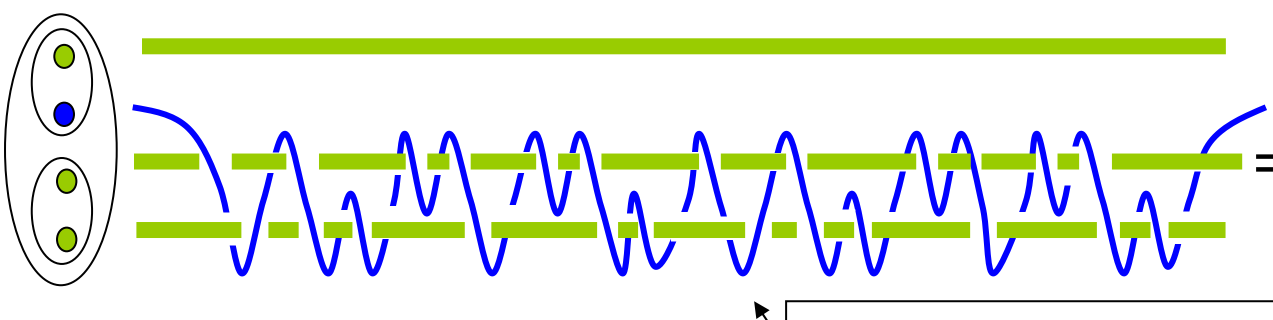
Two Qubit Controlled Gates



For Fibonacci anyons this is equivalent to finding a single qubit rotation!

Two Qubit Controlled Gates



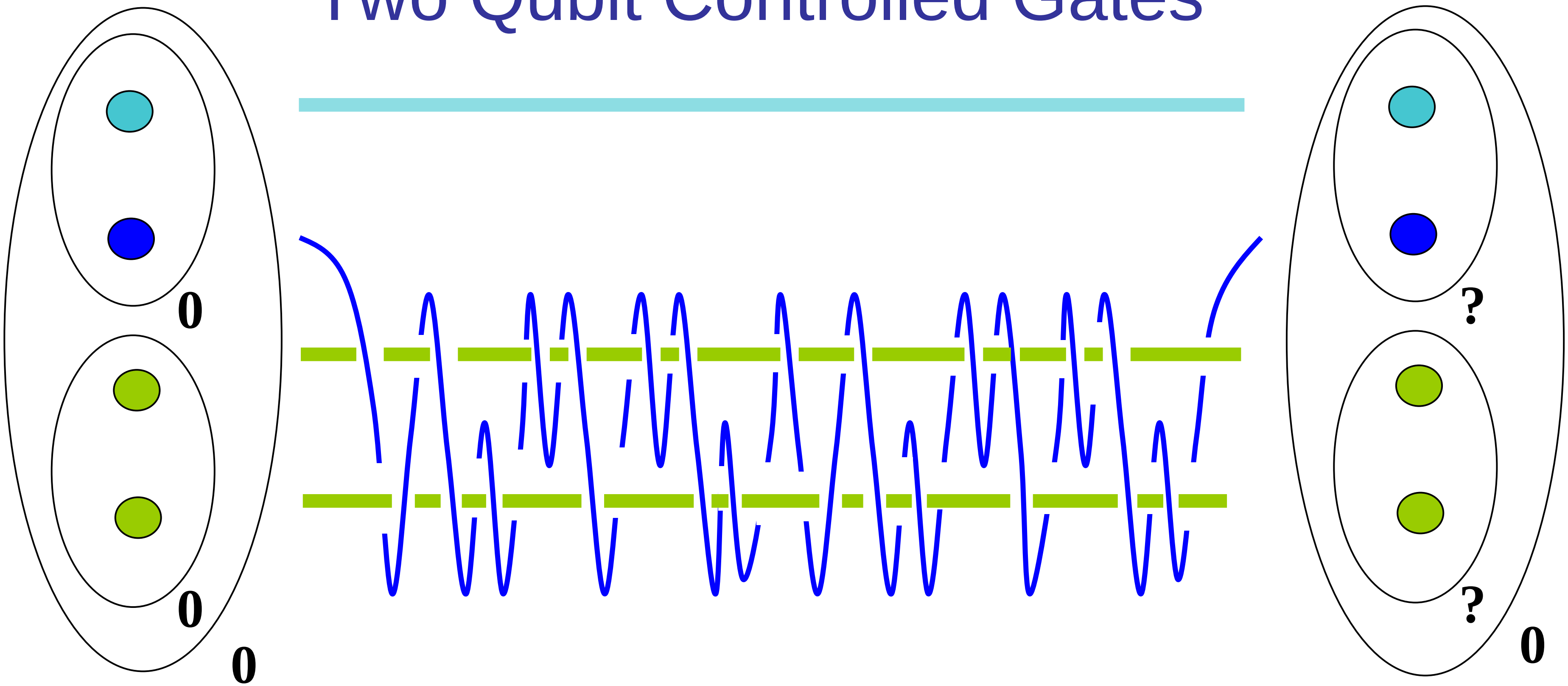


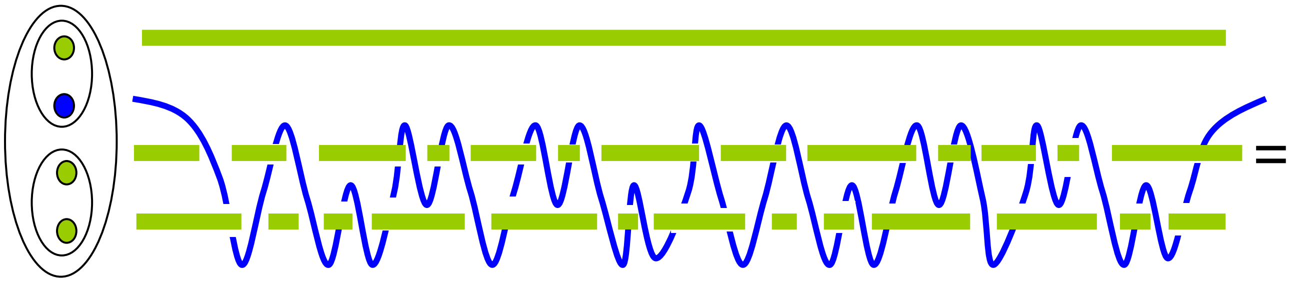
a_{44}

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + O(10^{-3})$$

Result of brute force search

Two Qubit Controlled Gates



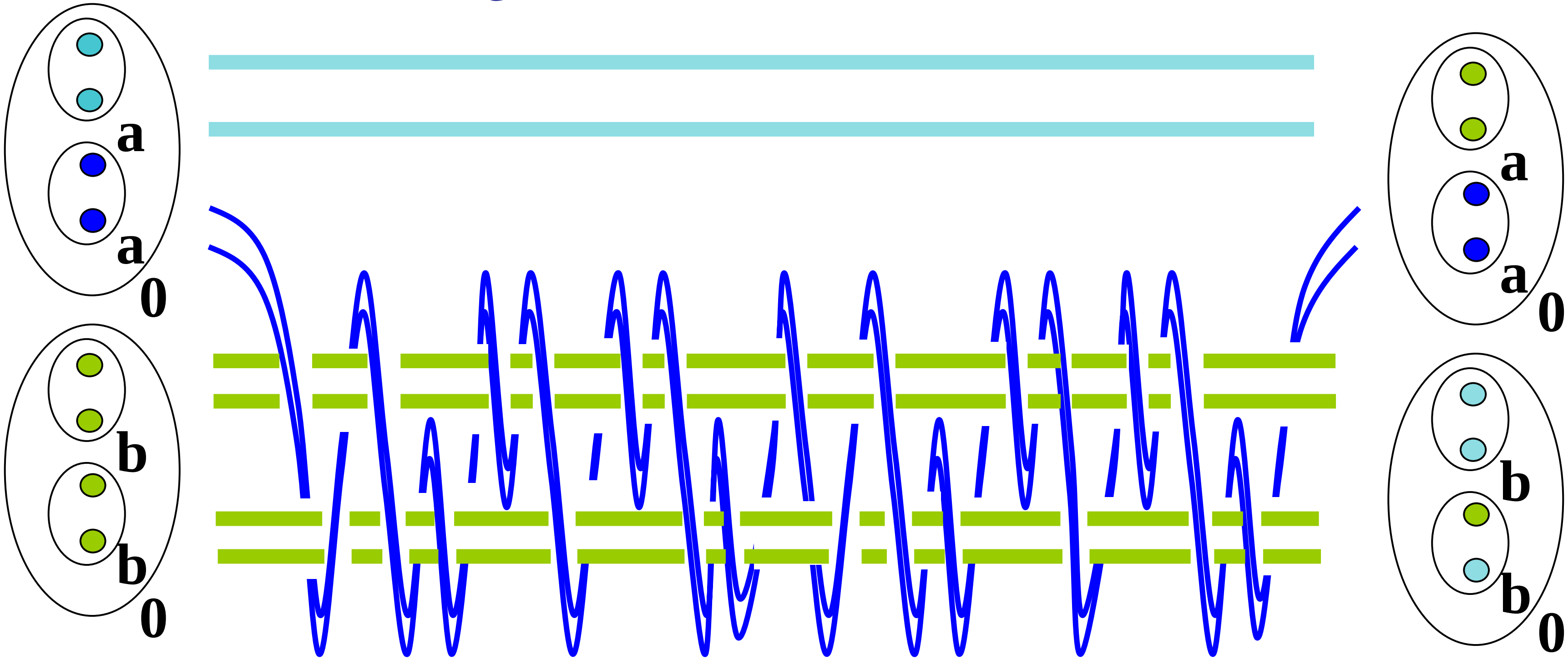


a_{44}

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + O(10^{-3})$$

Result of brute force search

Two Qubit Controlled Gates

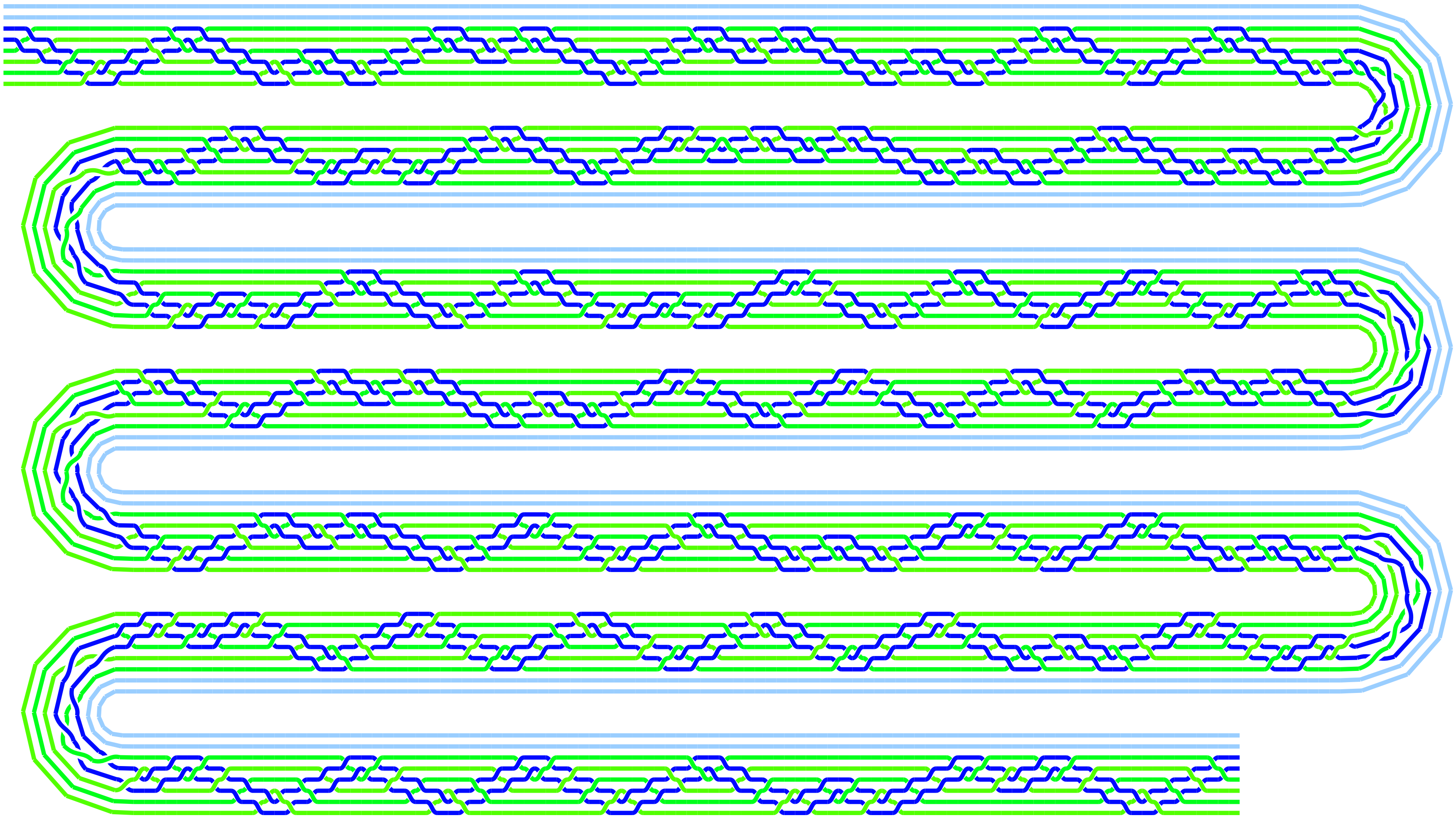


ab = 00 01 10 11

$$U_{two-qubit} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1+O(10^{-3}) \end{bmatrix}$$

Controlled-Phase Gate

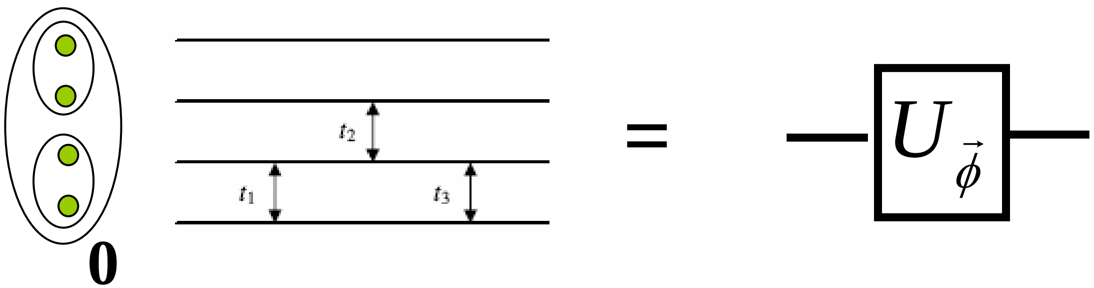
Solovay-Kitaev Improved Controlled-phase gate



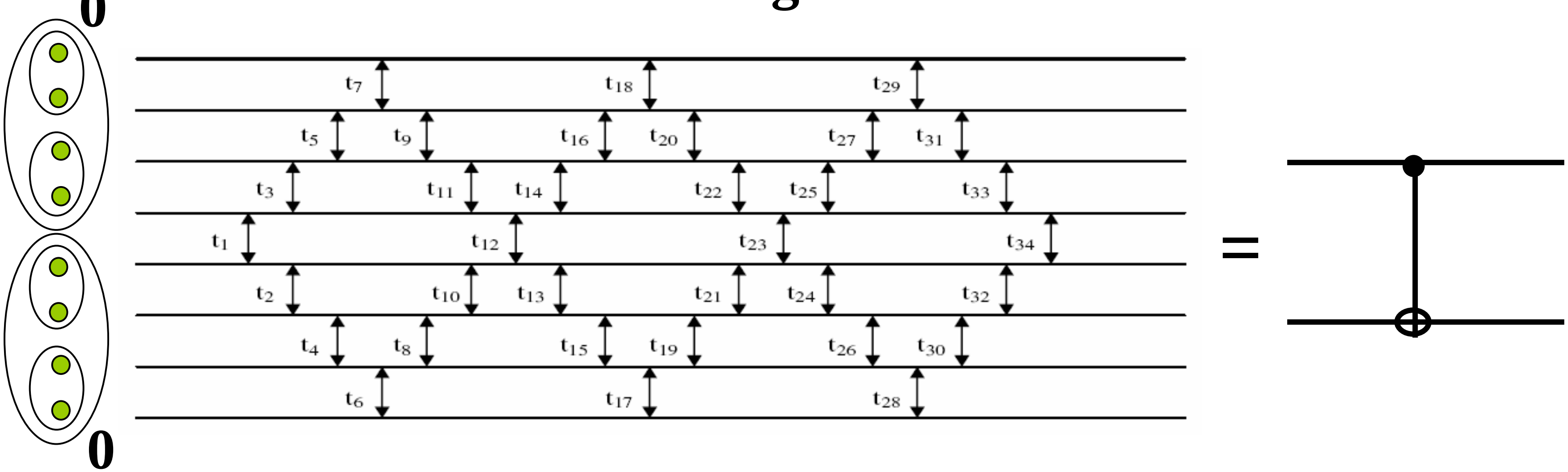
Universal Set of Gates

M.Hsieh, J. Kempe, S. Myrgren and K.B. Whaley (2003)

Single Qubit Gates

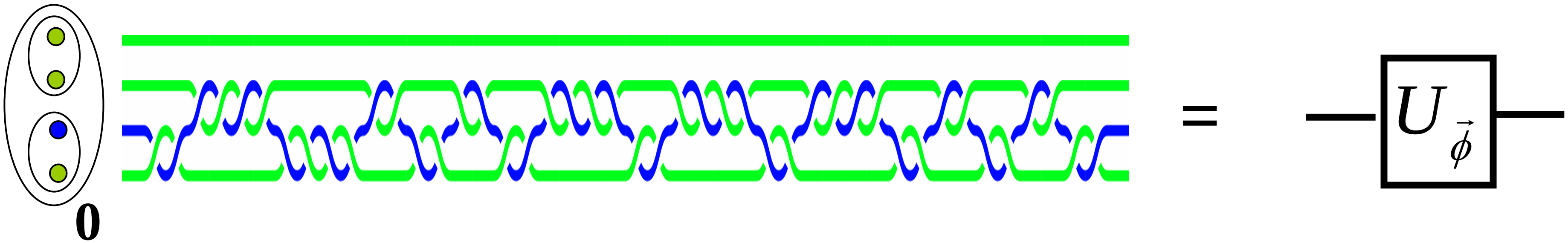


Controlled-NOT gate

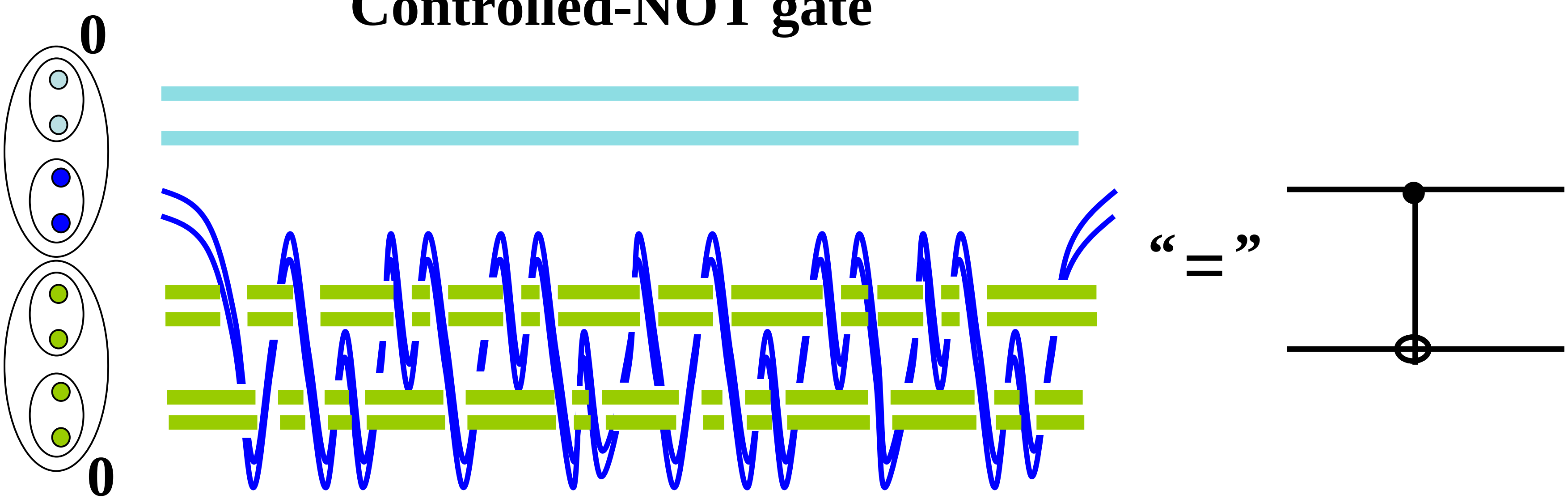


Universal Set of Gates

Single Qubit Gates



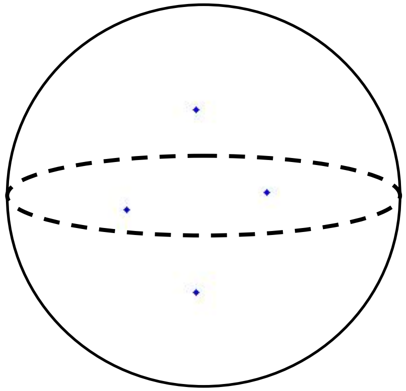
Controlled-NOT gate



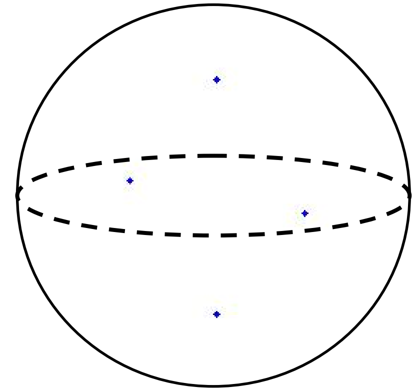
What about $k > 3$?

N = 1

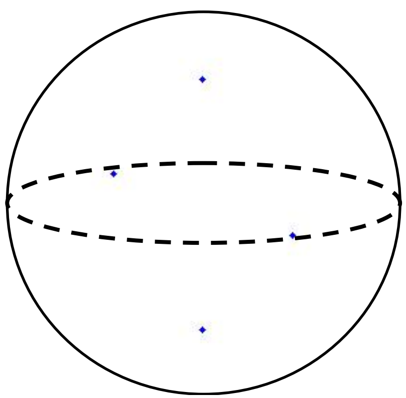
k=2



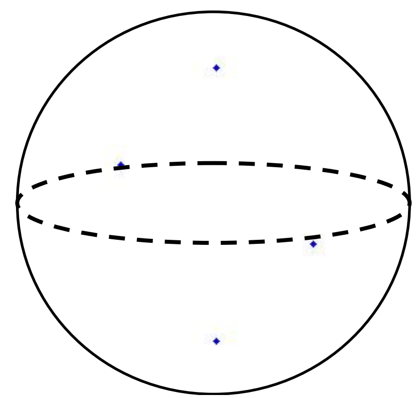
k=3



k=4

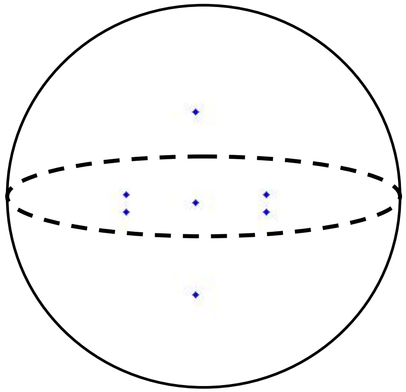


k=5

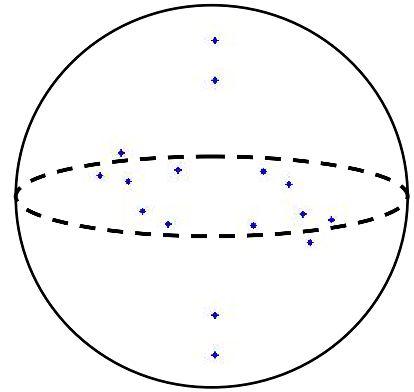


N = 2

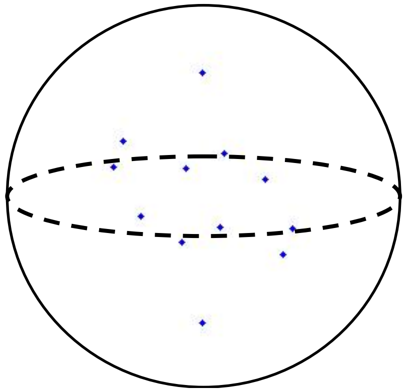
k=2



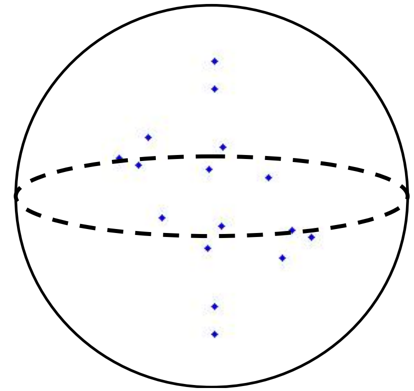
k=3



k=4

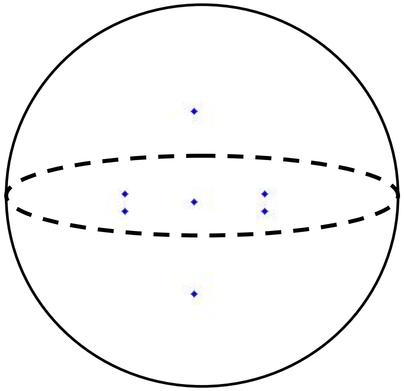


k=5

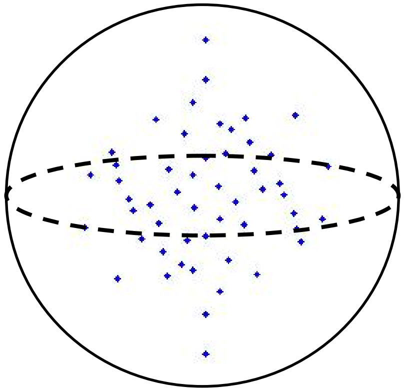


N = 3

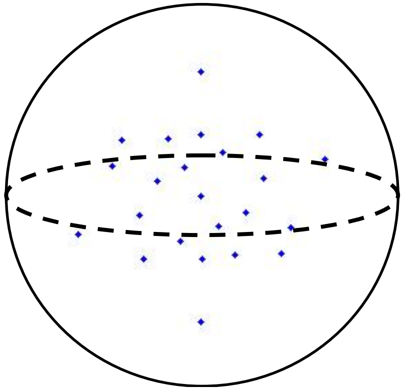
k=2



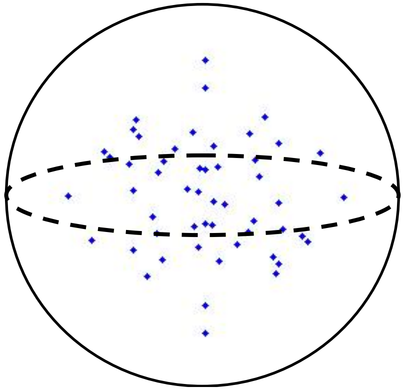
k=3



k=4

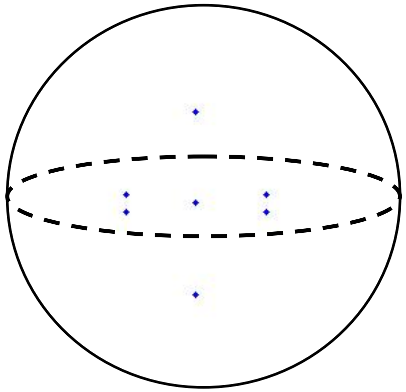


k=5

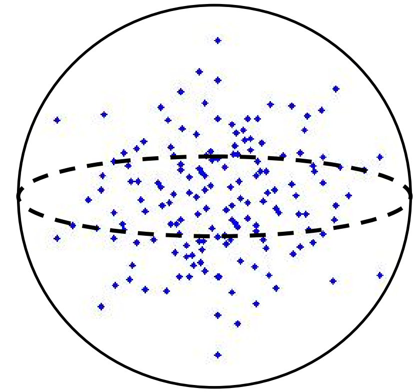


N = 4

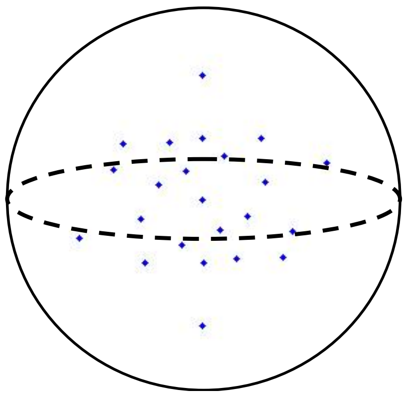
k=2



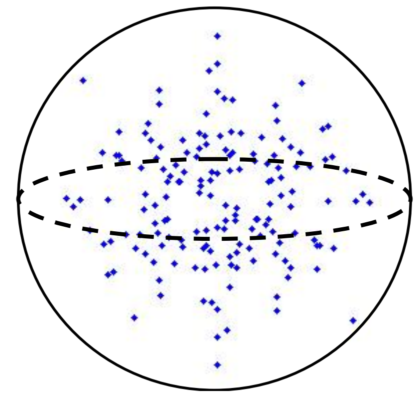
k=3



k=4

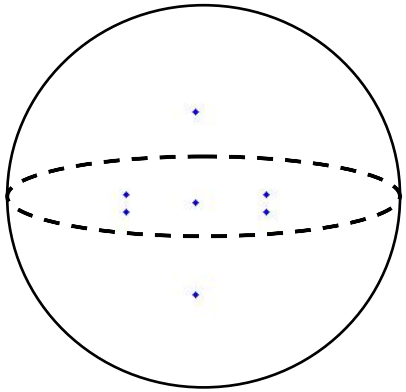


k=5

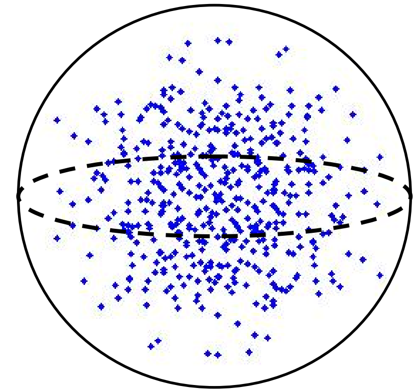


N = 5

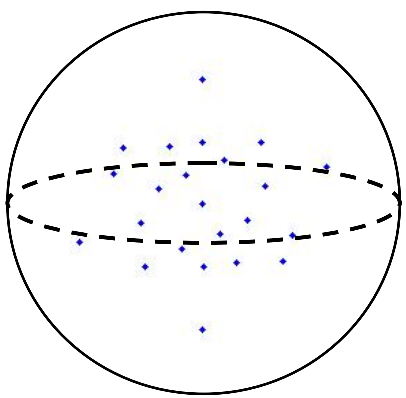
k=2



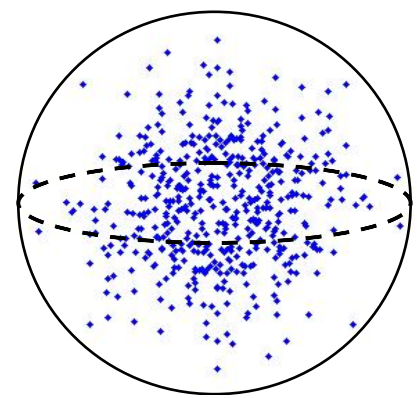
k=3



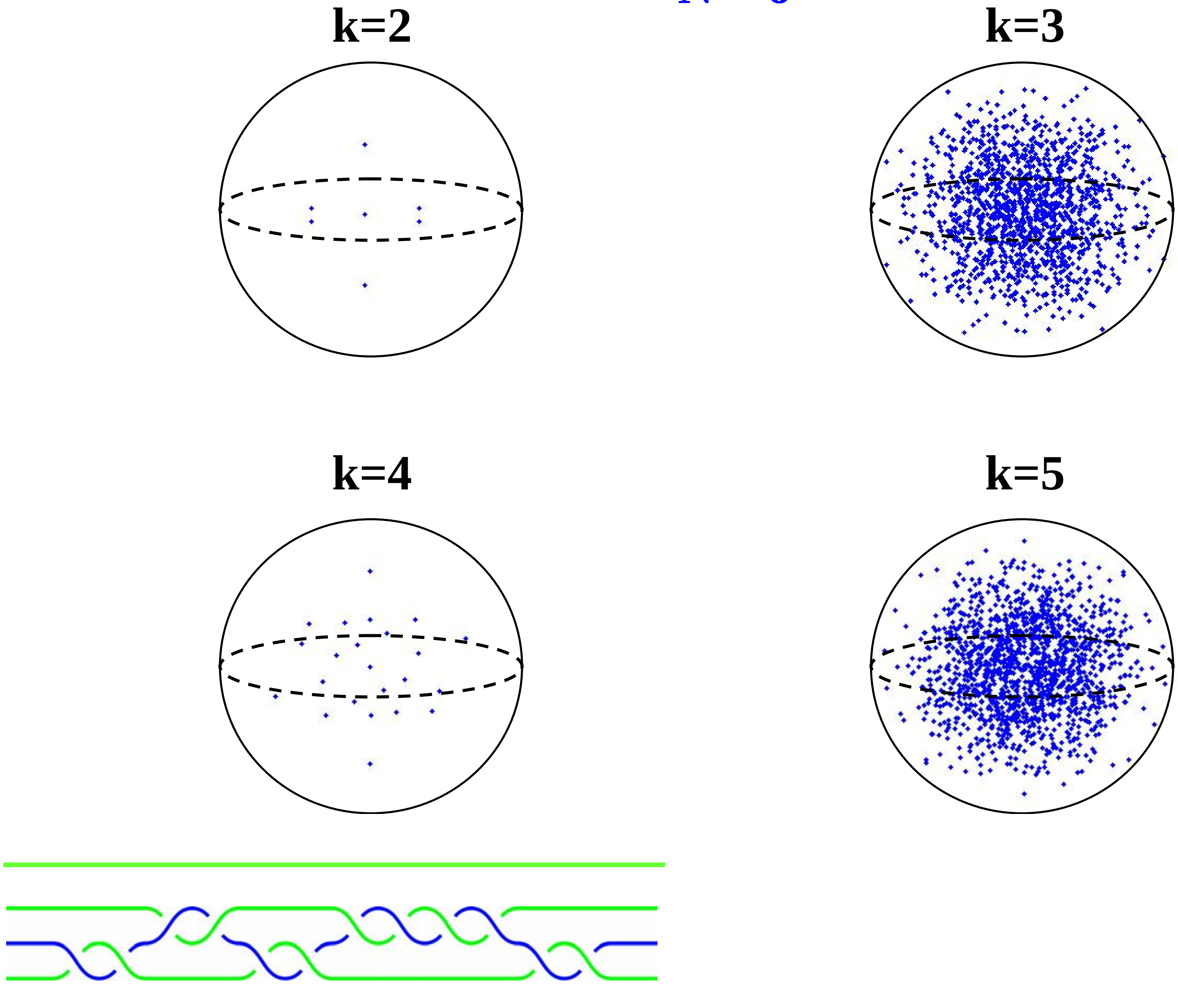
k=4



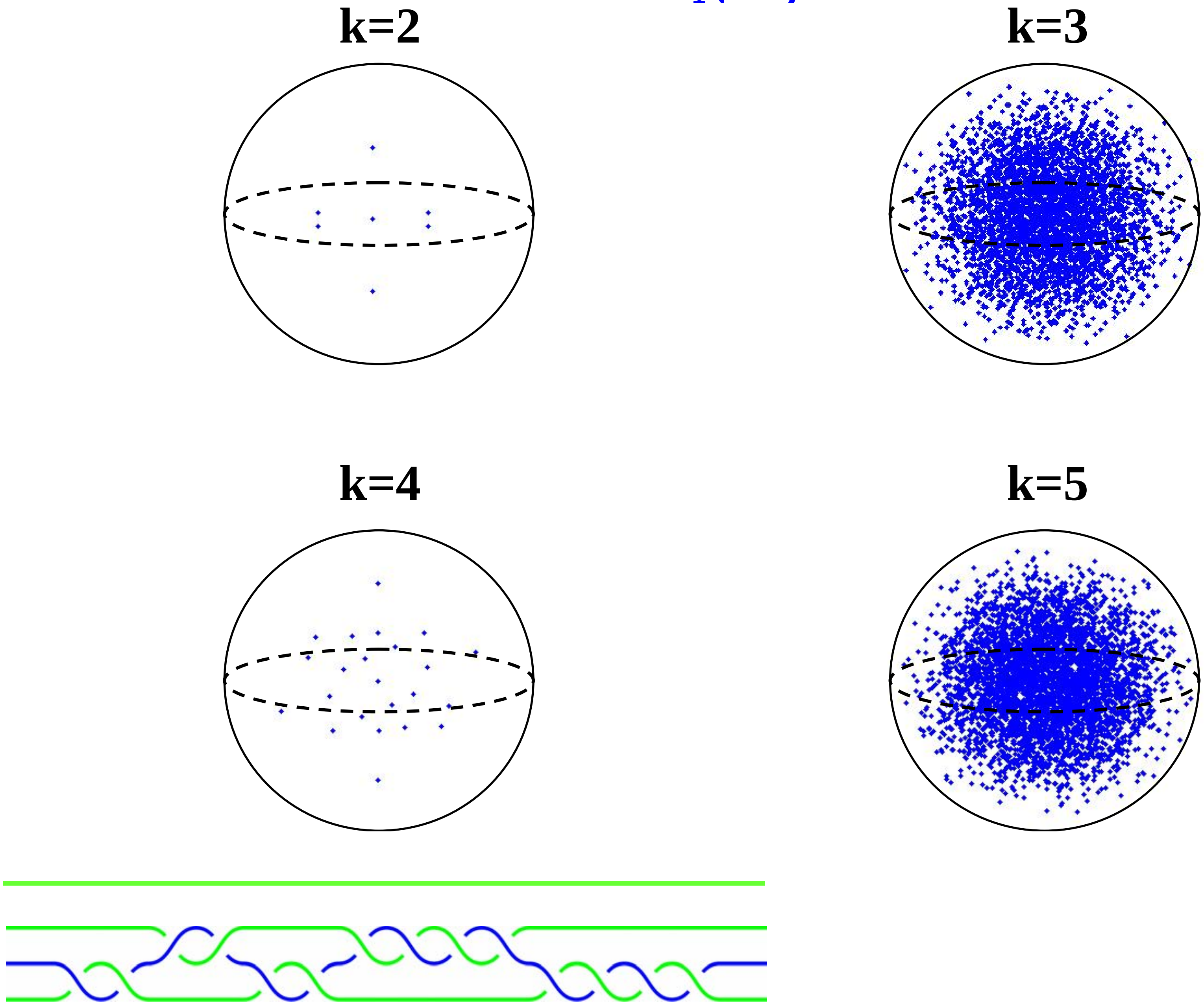
k=5

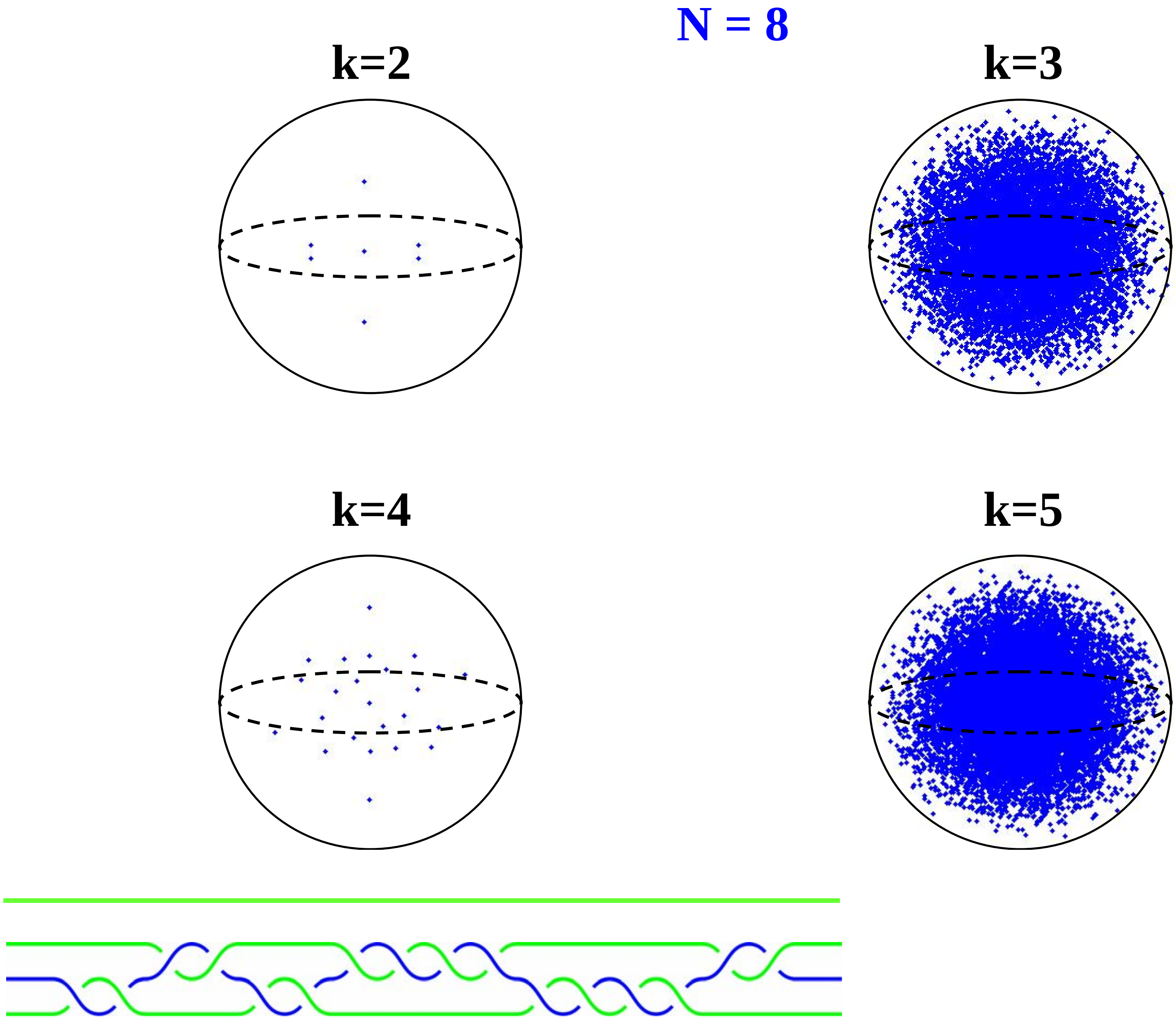


N = 6

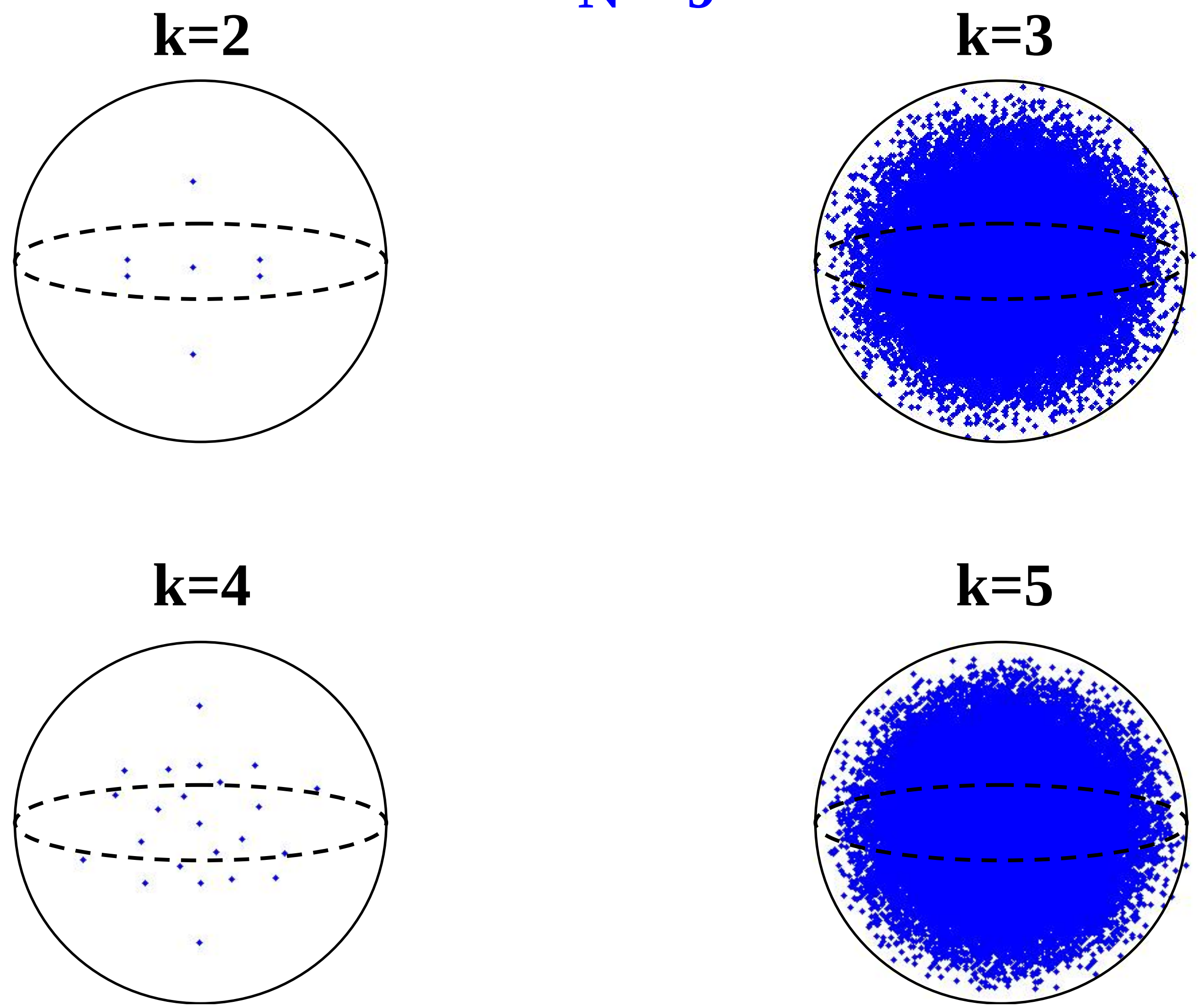


N = 7

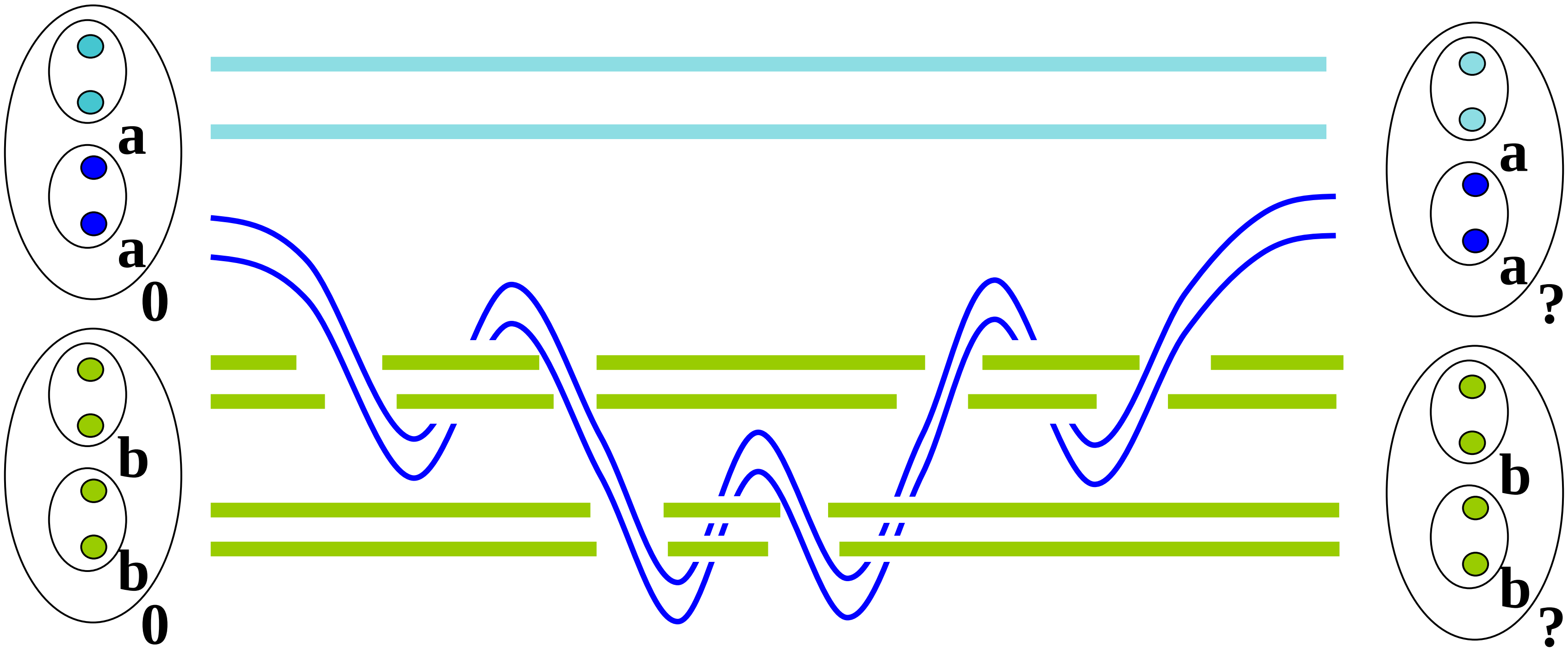



N = 8
k=2
k=3
k=4
k=5

N = 9



Does this construction work for k>4?

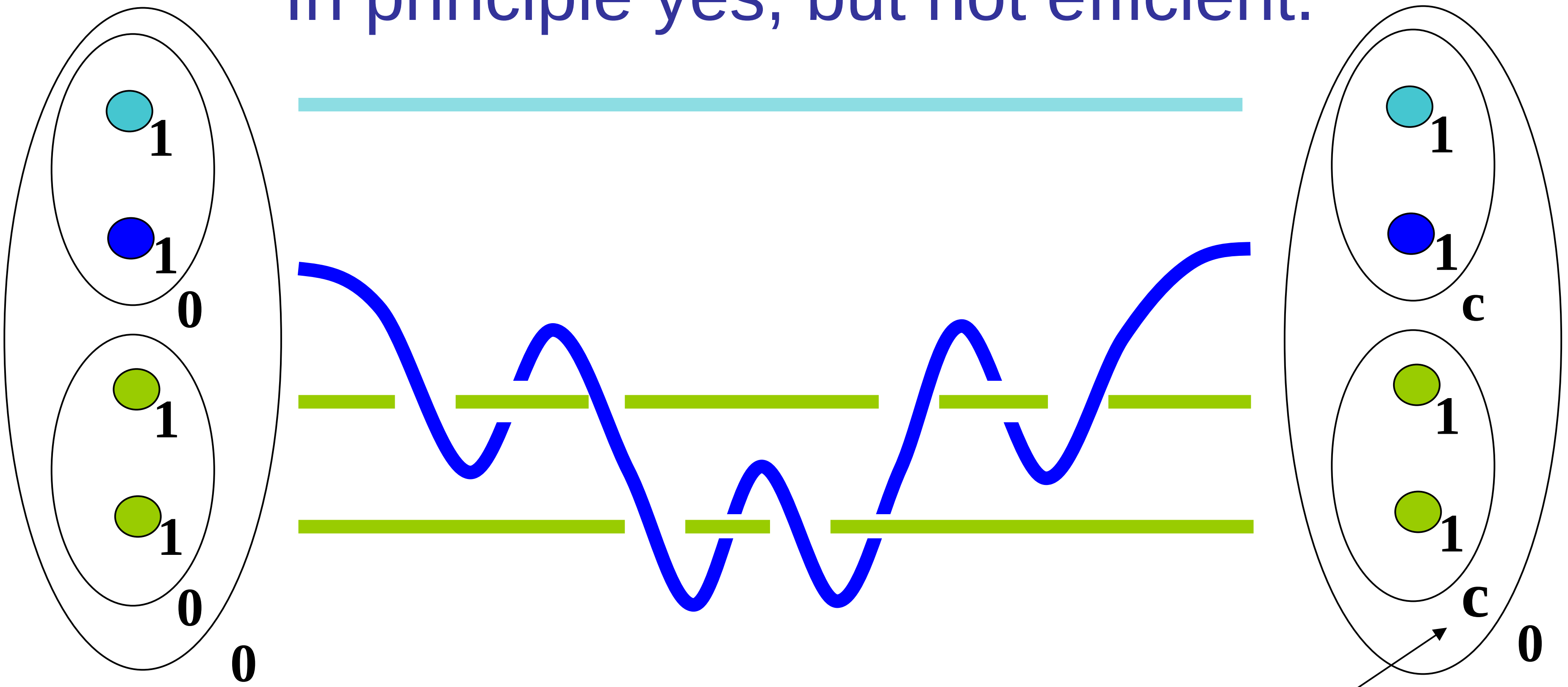


ab = 00 01 10 11

$$U_{two-qubit} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{44} \end{array}\right]$$

Only **ab = 11** sector is nontrivial.

In principle yes, but not efficient.



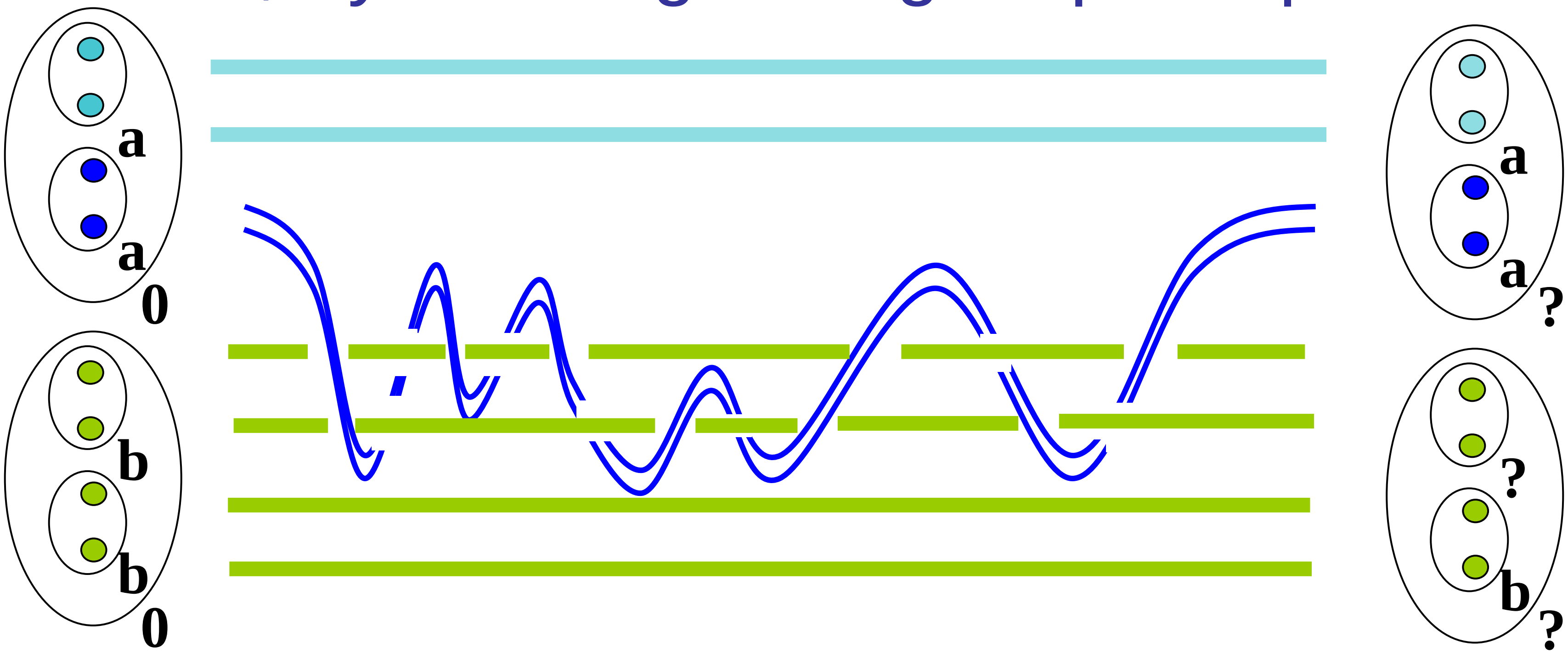
For k>3, finding approximate gates requires Solovay-Kitaev in **SU(3)**.

This is feasible, but it is more efficient to break the problem into smaller **SU(2)** problems.

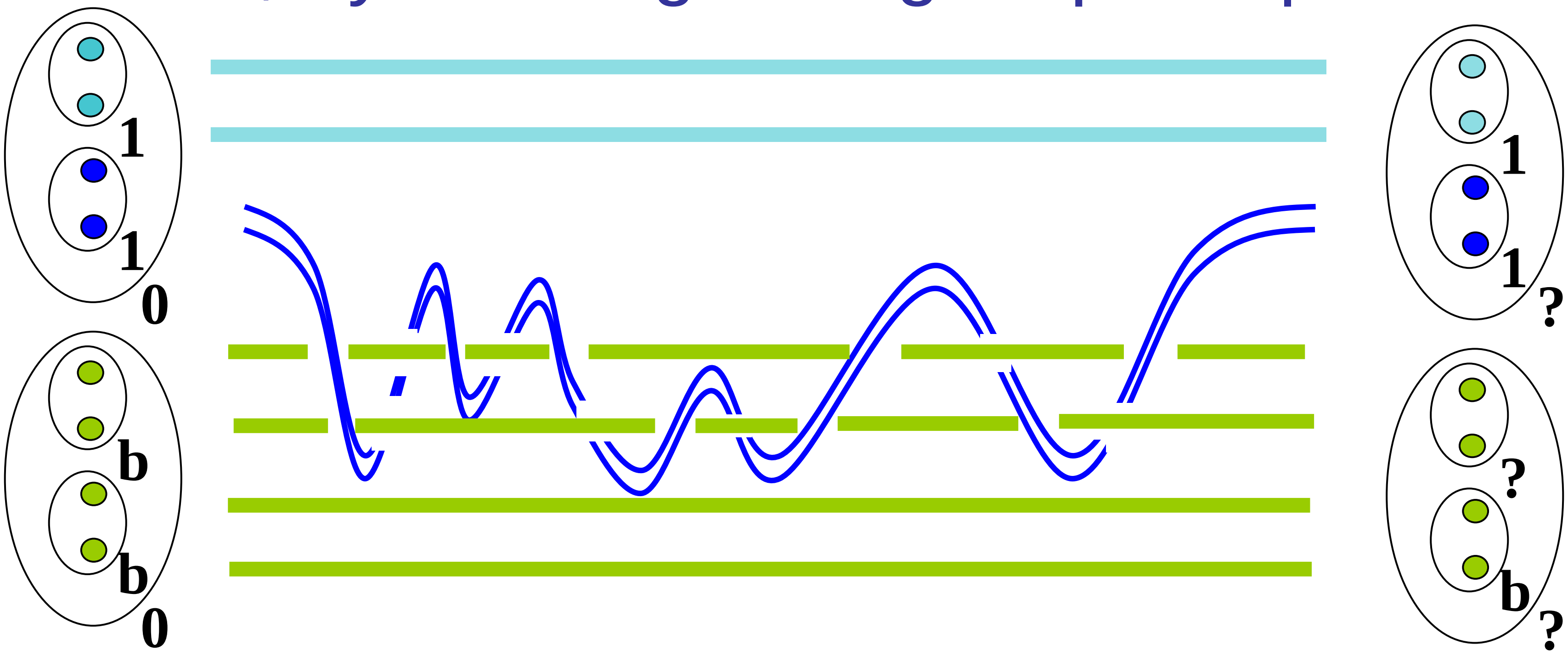
c = 0, 1, 2

New label occurs for k > 3

OK, try weaving through top two particles

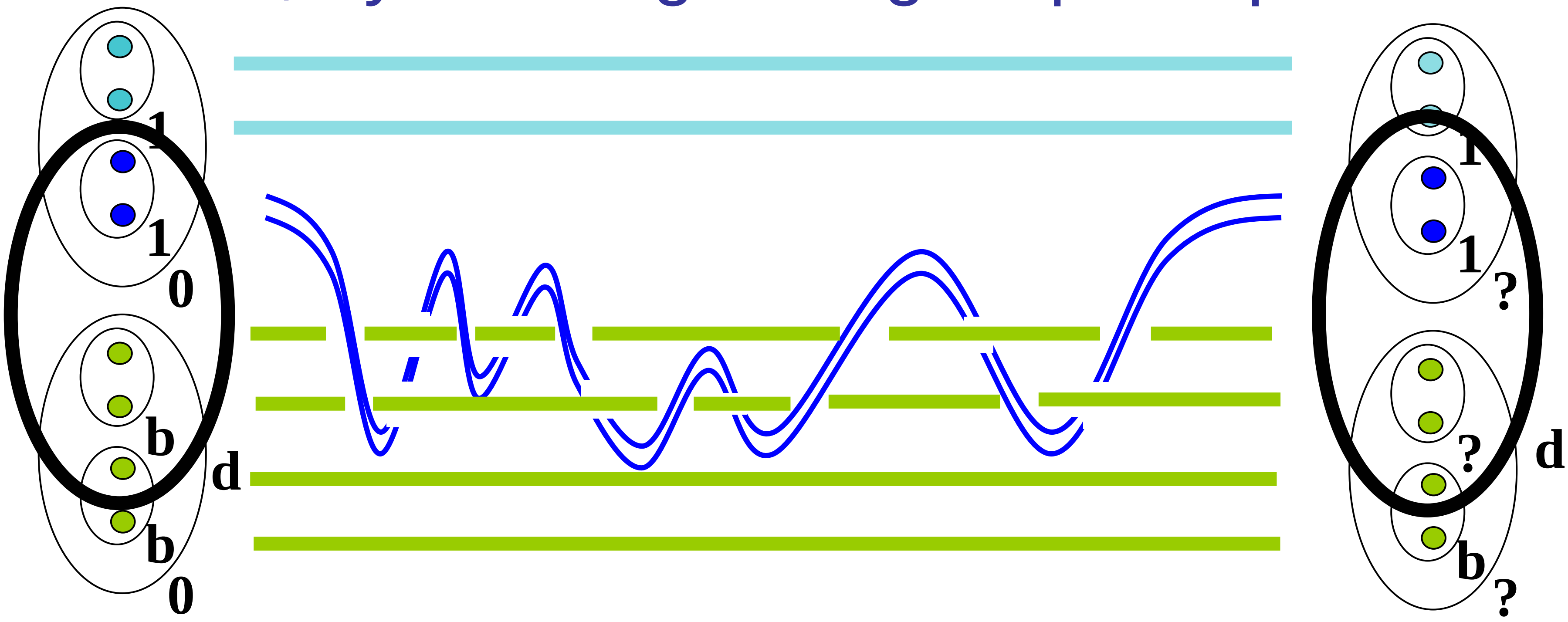


OK, try weaving through top two particles

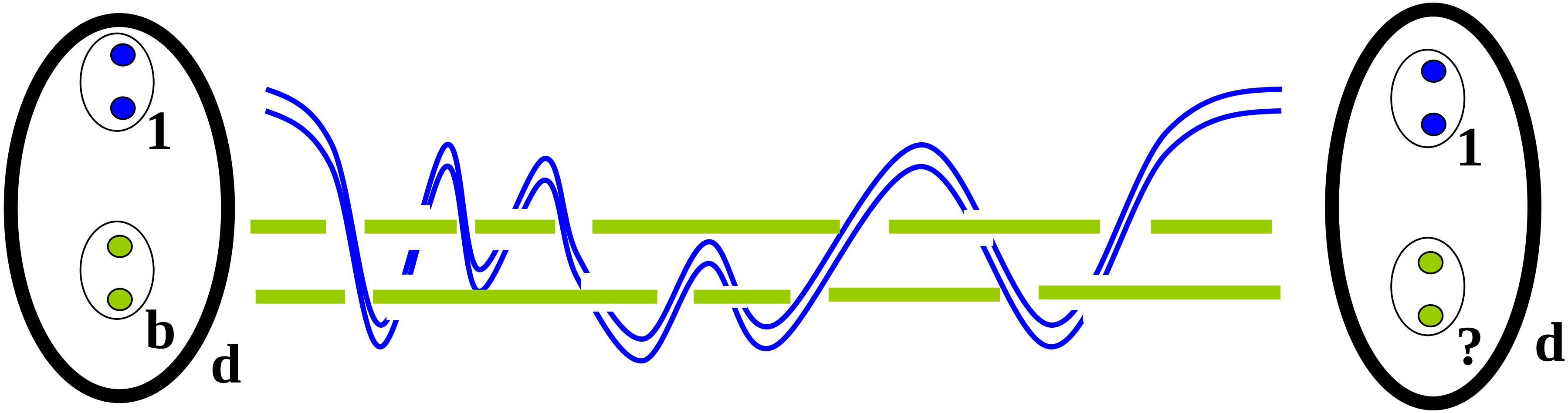


Again, if a=0, nothing happens, so only the a=1 case is relevant.

OK, try weaving through top two particles



OK, try weaving through top two particles

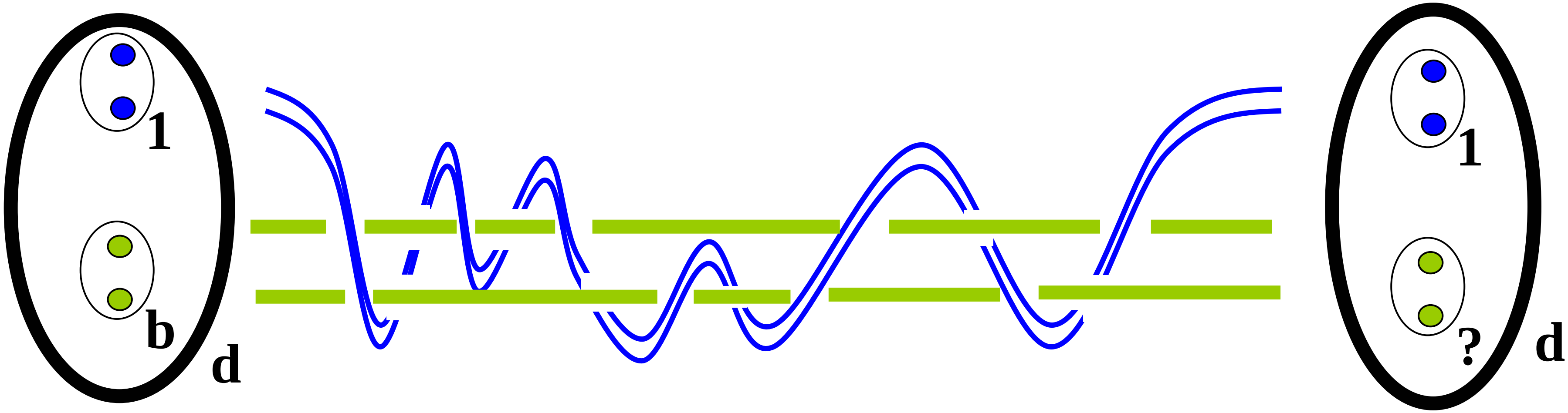


db = 01 10 11 21

$$U = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$$

These three matrix elements must be equal for there to be no leakage error.

Try weaving through top two particles



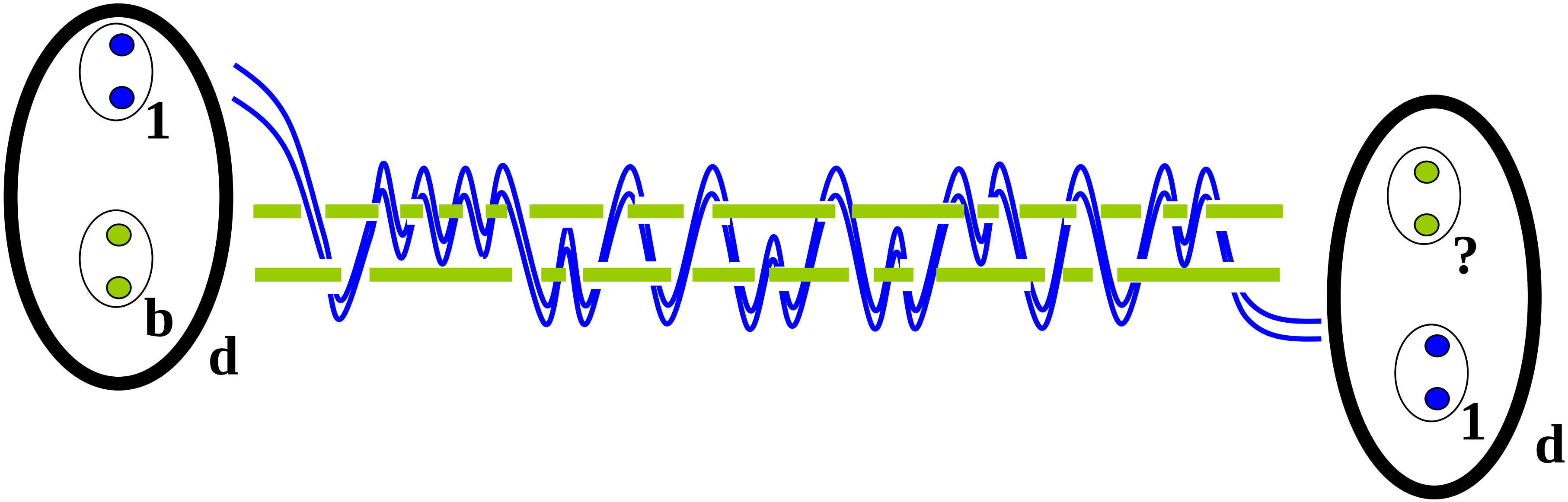
db = 01 10 11 21

$$U = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

Useful fact: For braids with zero "winding" determinant of each block must be 1

➡ Only braid which does not lead to leakage error gives the **identity operation.**

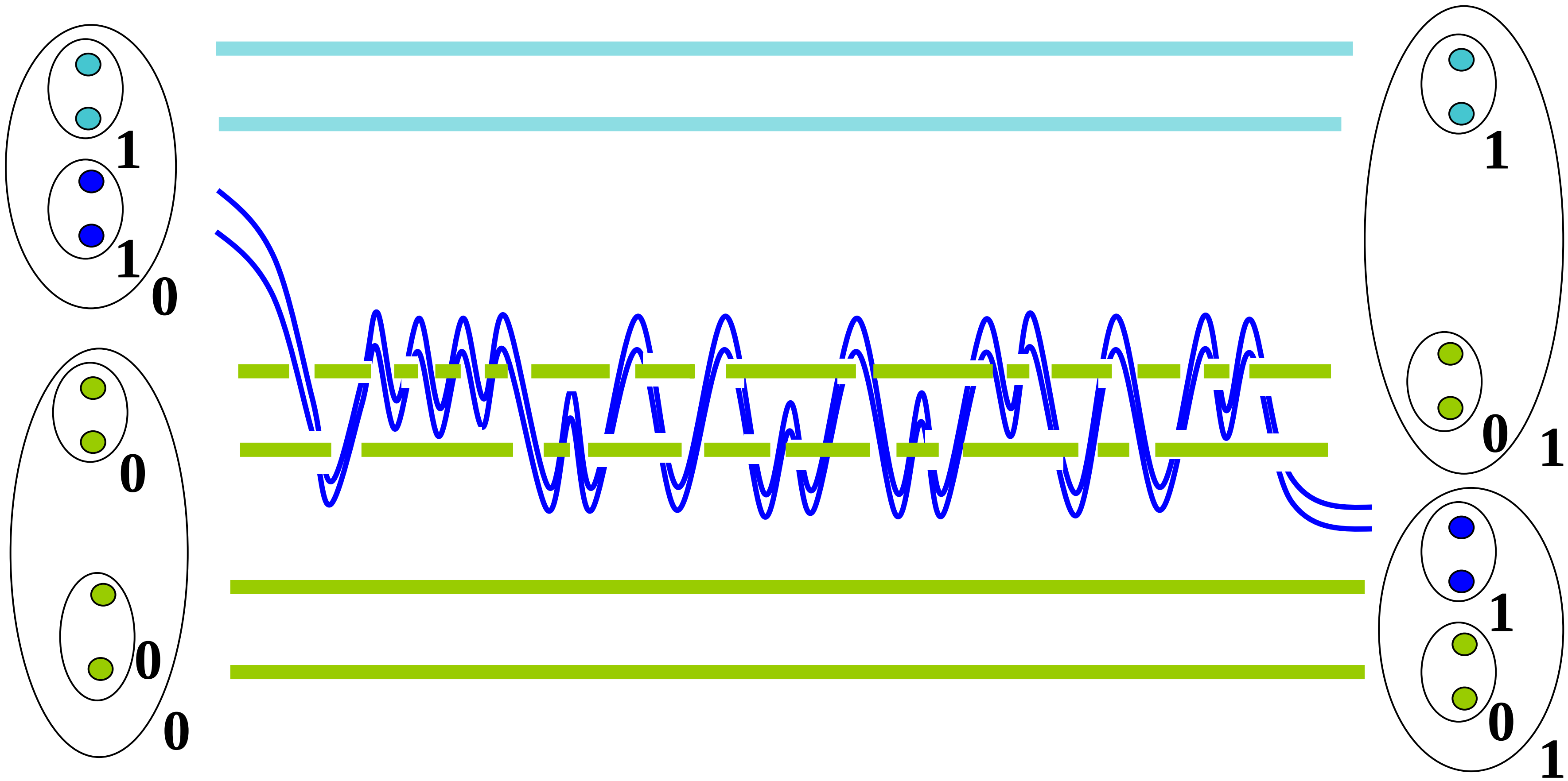
What if we don't bring the blue pair back?



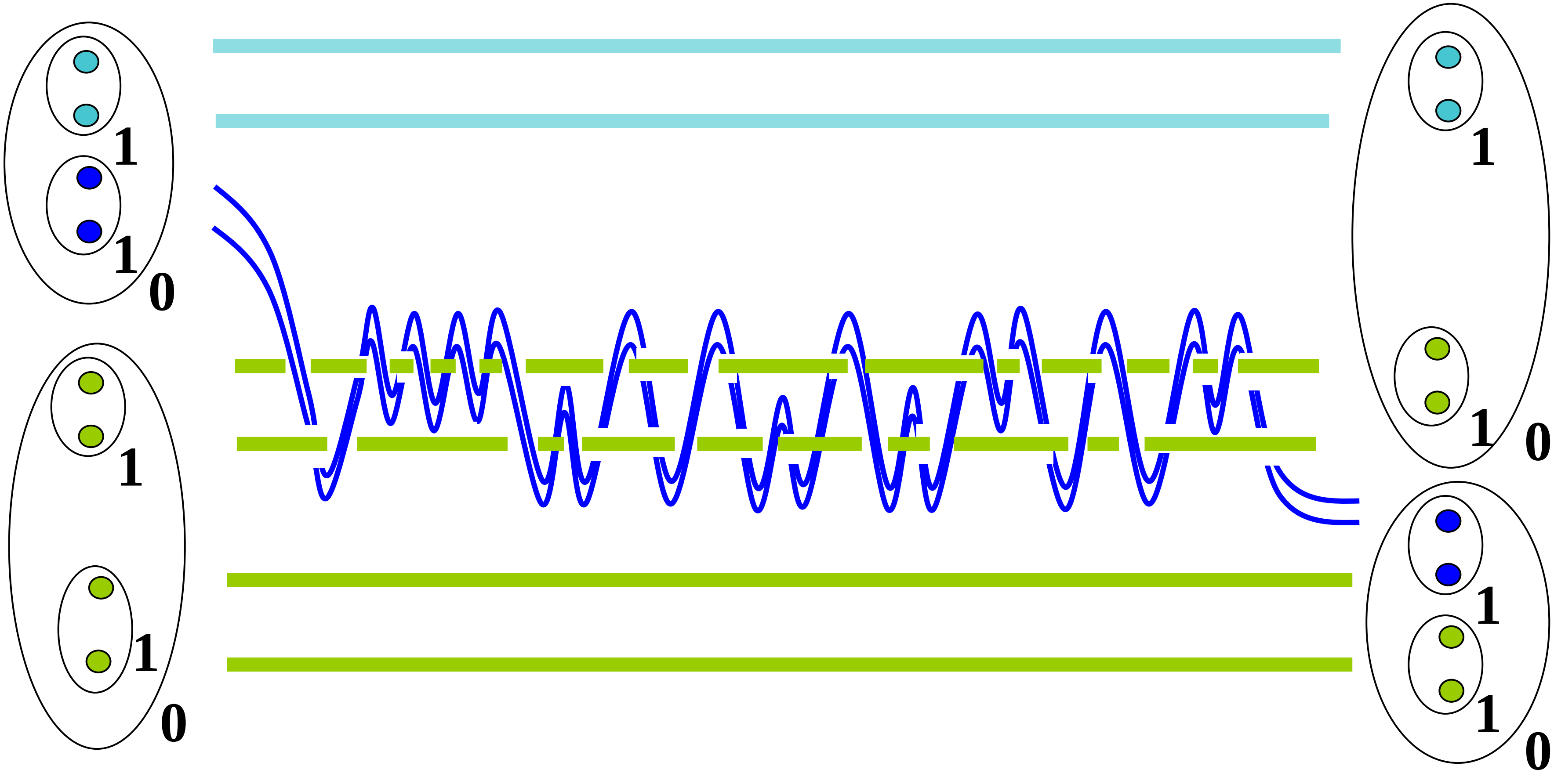
db = 01 10 11 21

$$U = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

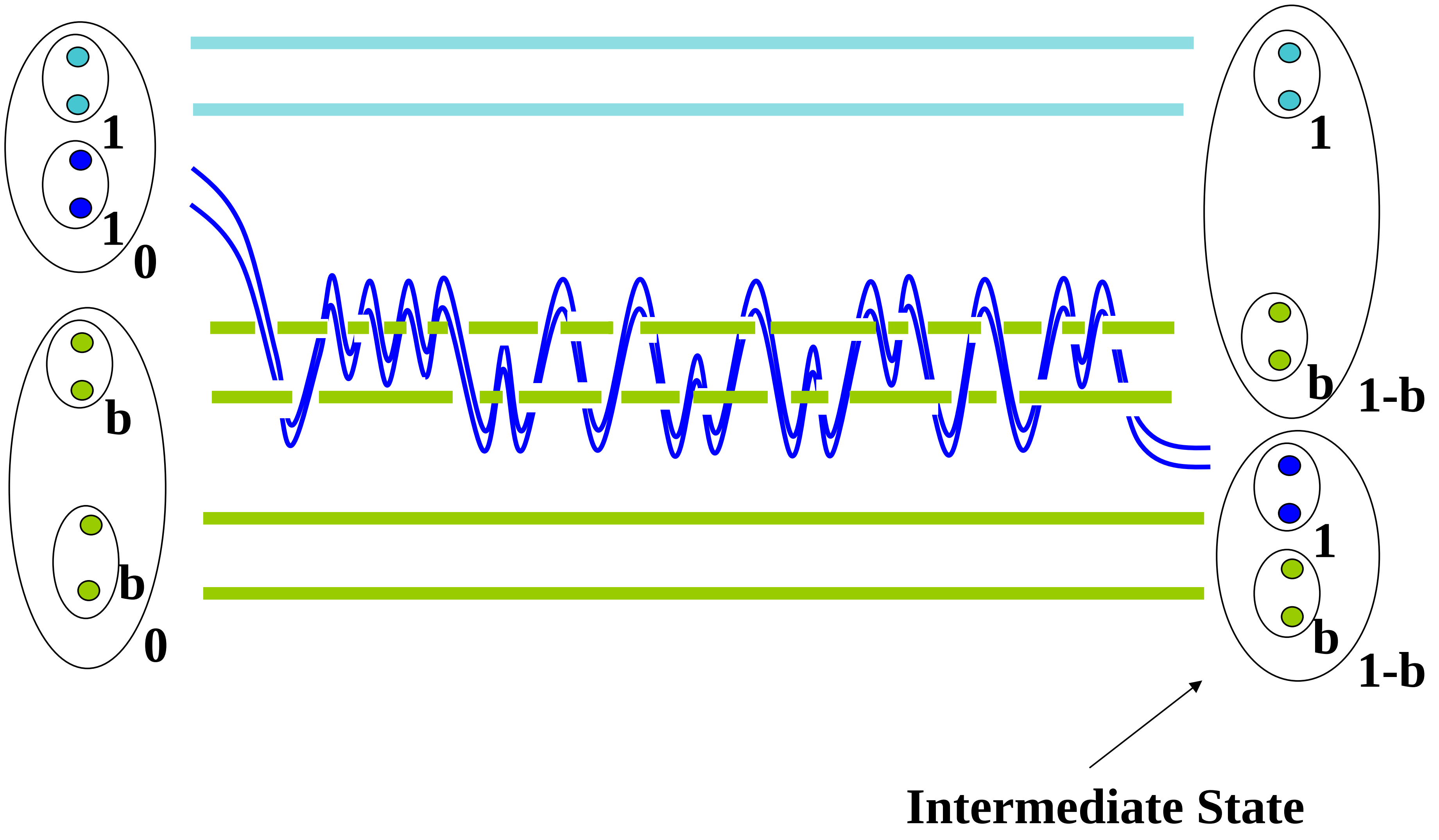
Because the three b=1 matrix elements are equal, when b=1 this operation simply swaps the blue pair with the green pair.

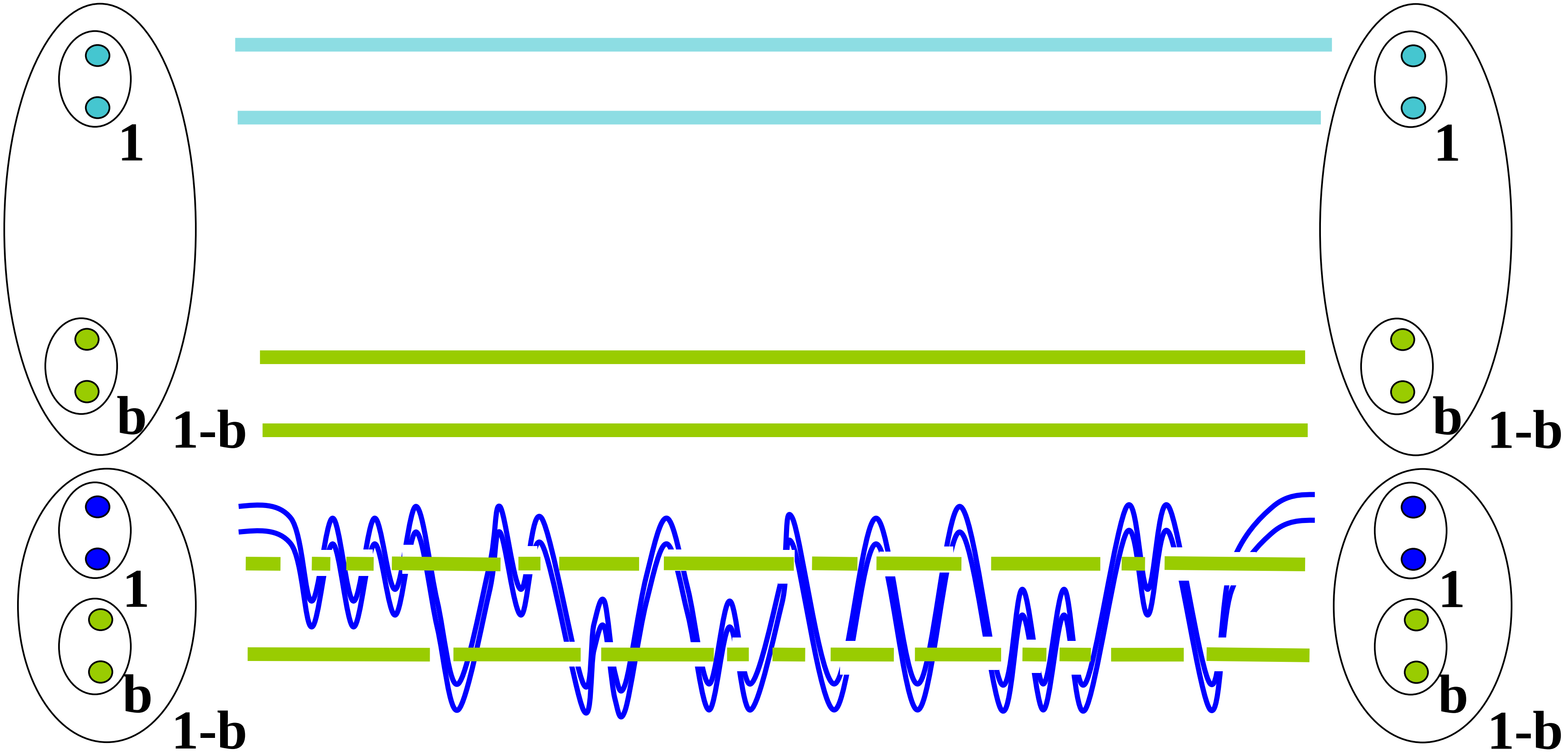


If **b=0**, the fusion rules imply that the overall label of “injected” target must be 1



If **b=1**, we simply swap the blue pair with the green pair, and the overall label of “injected” target remains 0.


1
1
0
b
b
0
1
b
1-b
1
b
1-b
Intermediate State

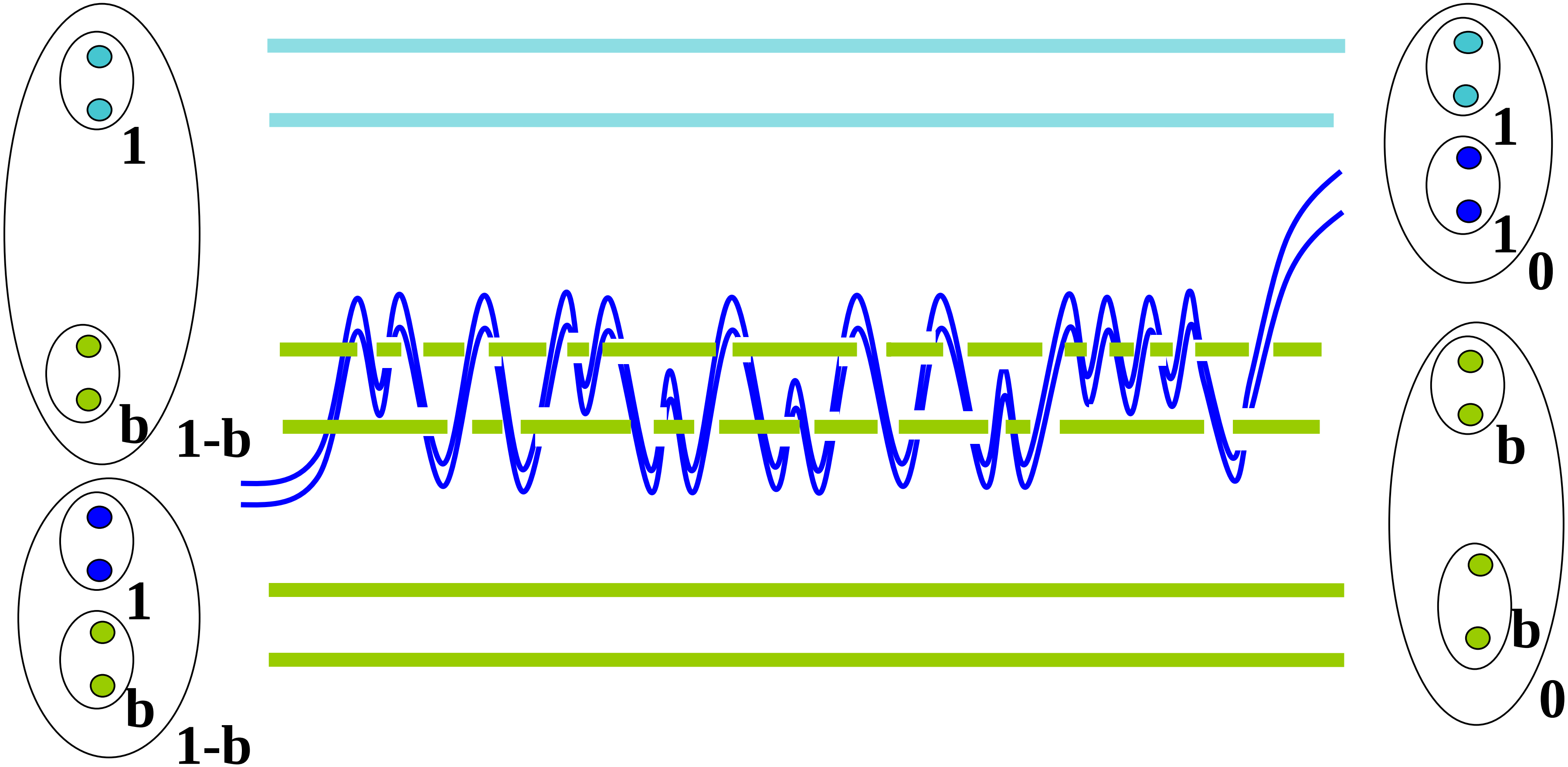


db = 01 10 11 21

b=0

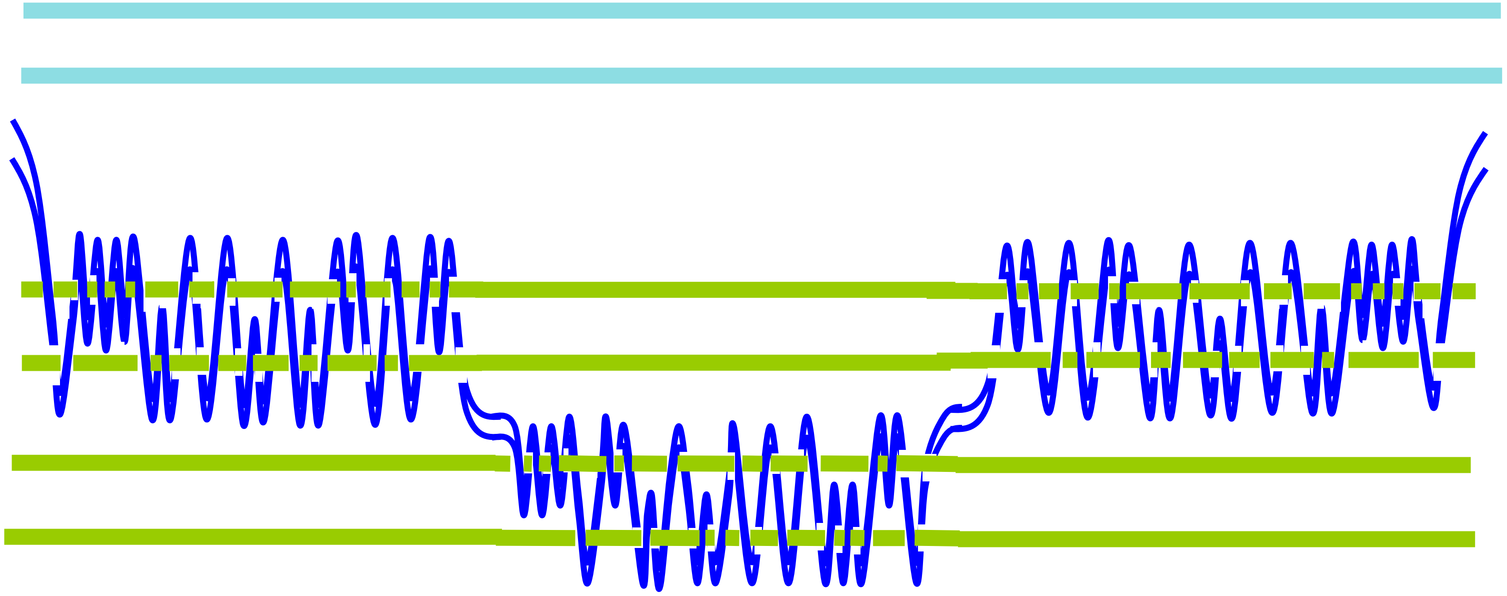
$$U = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{-1} & \mathbf{0} & 0 \\ 0 & \mathbf{0} & \mathbf{-1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

b=1



$$U_{two-qubit} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] + O(10^{-3})$$

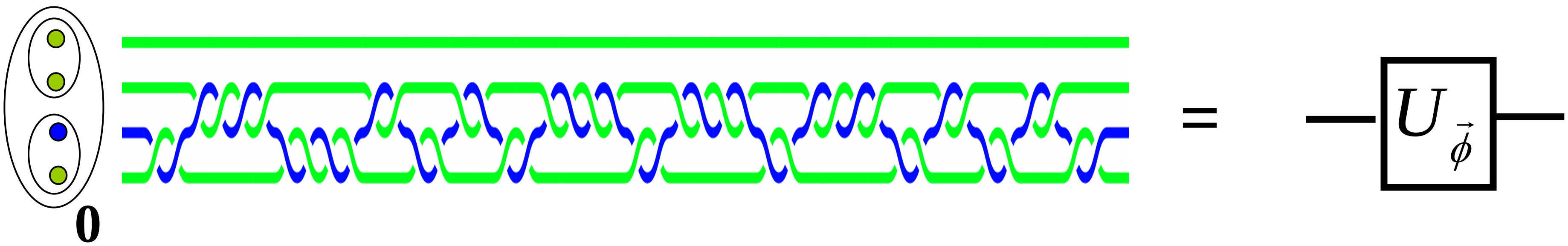
k=5 Controlled-Phase Gate



$$U_{two-qubit} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] + O(10^{-3})$$

Universal Set of Gates for k=5

Single Qubit Gates



Controlled-NOT gate