Finding Fibonacci

Paul Fendley

“Fibonacci anyons” exhibit the simplest kind of non-abelian braiding.

It proved to be quite tricky to find models whose quasiparticles are Fibonacci anyons and which have time-reversal symmetry (i.e. no external magnetic field).

Non-abelian braiding means that the wavefunction changes under braiding depending on the order in which the particles are braided.

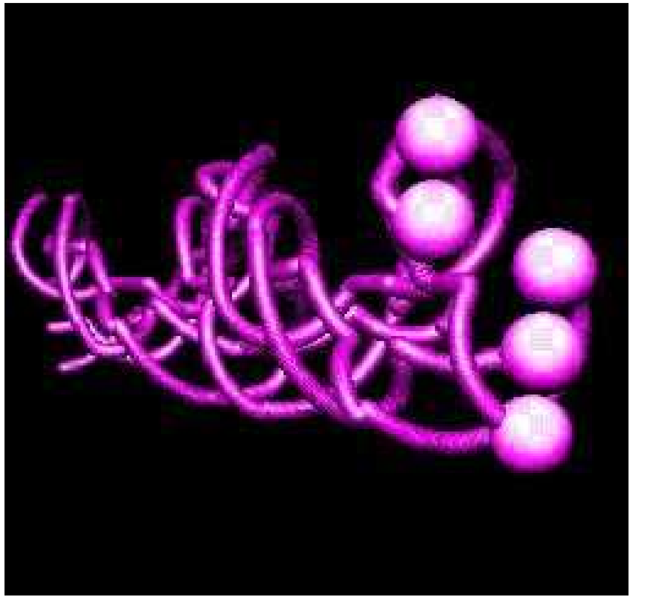


figure from Skjeltorp et al

The lines are the world-lines of the particles in 2+1 dimensional spacetime.

Here I'll describe two different quantum loop models with Fibonacci anyons.

Outline:

1. What are Fibonacci anyons?
2. Quantum loop models
3. The $d = \sqrt{2}$ barrier
4. Pushing back the barrier by including intersections (studying closed crossing curves)

The Potts model and the BMW algebra: 2002, with N. Read

Branching curves: 2005, with E. Fradkin

New loop models: 2006

Squared loop models: to appear very soon

The BMW algebra, the chromatic polynomial, and all that: to appear almost as soon, with V. Krushkal

What are Fibonacci anyons?

$$\phi \cdot \phi = I + \phi$$

$$\phi \cdot \phi = I + \phi$$

Precise meaning in field theory: The operator product expansion of two ϕ fields contains both the identity field I and the field ϕ itself .

Precise meaning in conformal field theory: the “fusion coefficients” for the primary fields I and ϕ $N_{\phi\phi}^{I} = N_{\phi\phi}^{\phi} = 1$.

Precise meaning for this talk: There are two possible quantum states for two Fibonacci anyons. One has trivial statistics, the other has the braiding properties of a single Fibonacci anyon.

This fusion algebra allows us to count the number of quantum states for $2N$ quasiparticles:

$$
\begin{aligned}
1 &: \quad I \\
1 &: \quad \phi \\
2 &: \quad \phi \cdot \phi = I + \phi \\
3 &: \quad \phi \cdot \phi \cdot \phi = I + \phi + \phi \\
5 &: \quad \phi \cdot \phi \cdot \phi \cdot \phi = \phi + 2(I + \phi) \\
8 &: \quad \phi \cdot \phi \cdot \phi \cdot \phi \cdot \phi = 3I + 5\phi
\end{aligned}
$$

The number of states for n particles is the nth Fibonacci number.

For n large, there are thus $\approx \tau^n$ states, so the quantum dimension is the golden mean

$$\tau \equiv \frac{1+\sqrt{5}}{2}$$

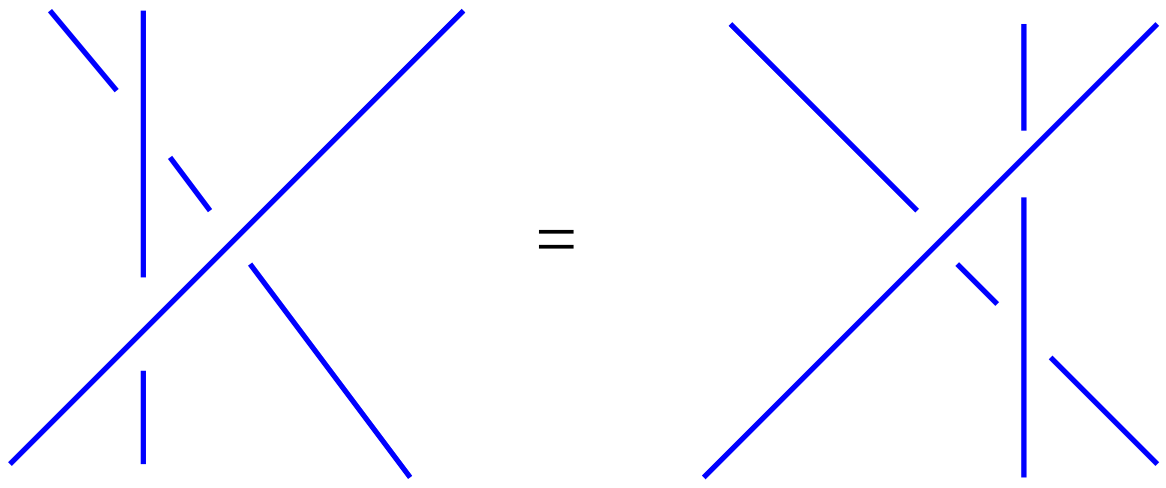
Think of the quantum dimension as the degeneracy of a particle. It must be > 1 to have non-abelian statistics: braiding entangles degenerate states.

Satisfying the Moore-Seiberg axioms allow one to find a complete and consistent set of braiding and fusing rules for Fibonacci anyons.
c.f. Preskill's lecture notes

A convenient way of describing braiding is to project the world line of the particles onto the plane.

Then the braids become overcrossings and undercrossings

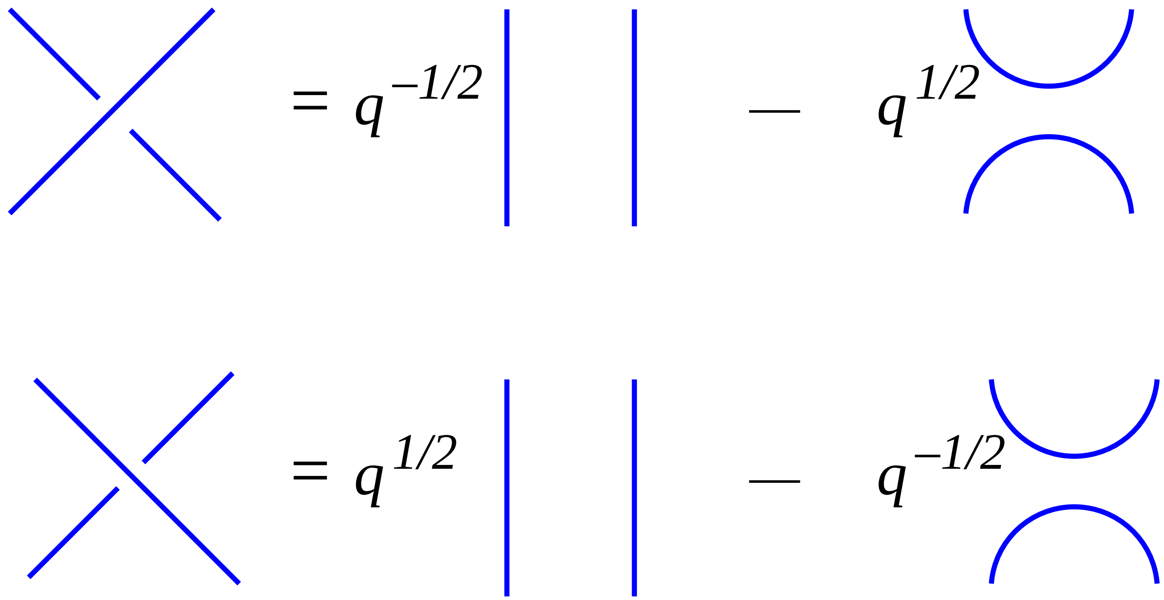
The braids must satisfy the consistency condition



which in closely related contexts is called the Yang-Baxter equation.

A simple way of satisfying the consistency conditions leads to the Jones polynomial in knot theory.

Replace the braid with the linear combination



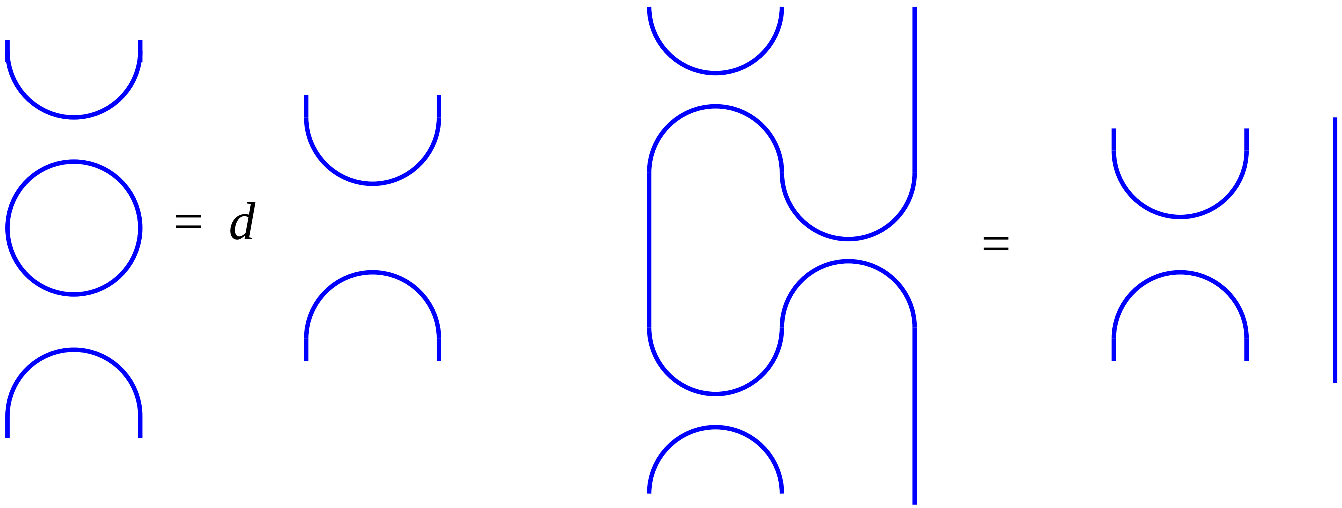
so that the lines no longer cross. q is a parameter which is a root of unity in the cases of interest: the Fibonacci case corresponds to $q = e^{i\pi/5}$.

This satisfies the consistency conditions if each closed loop receives a weight

$$d = q + q^{-1}$$

relative to the configuration without the loop.

These relations can all be treated in terms of the Temperley-Lieb algebra, which graphically is



Such loops are said to satisfy d-isotopy.

We now wish to construct a quantum loop model whose quasiparticles have these braiding properties.

A (hopefully not unique) procedure to do so is

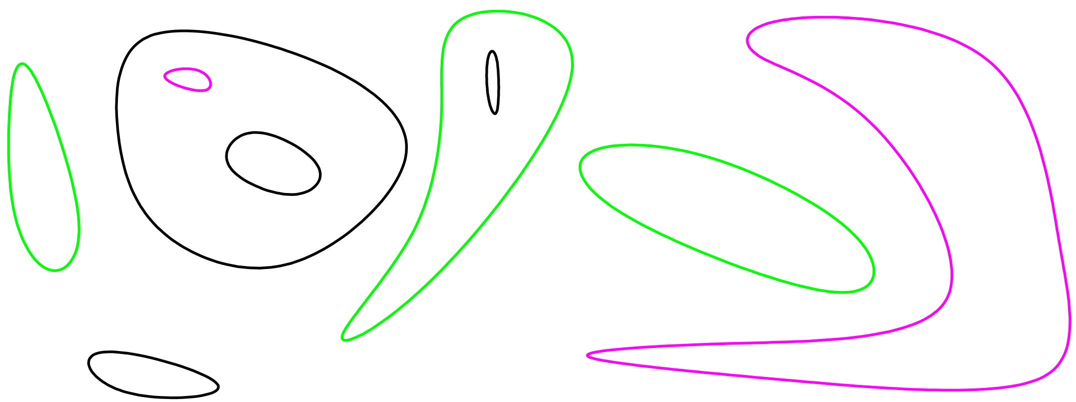
1. find a 2d classical loop model which has a critical point

2. use each loop configuration as a basis element of the quantum Hilbert space

3. use a Rokhsar-Kivelson Hamiltonian to make the quantum ground state a sum over loop configurations with the appropriate weighting.

Kitaev; Freedman

To realize this form of braiding, each loop in the ground state gets a weight d ($= \tau$ for Fibonacci)

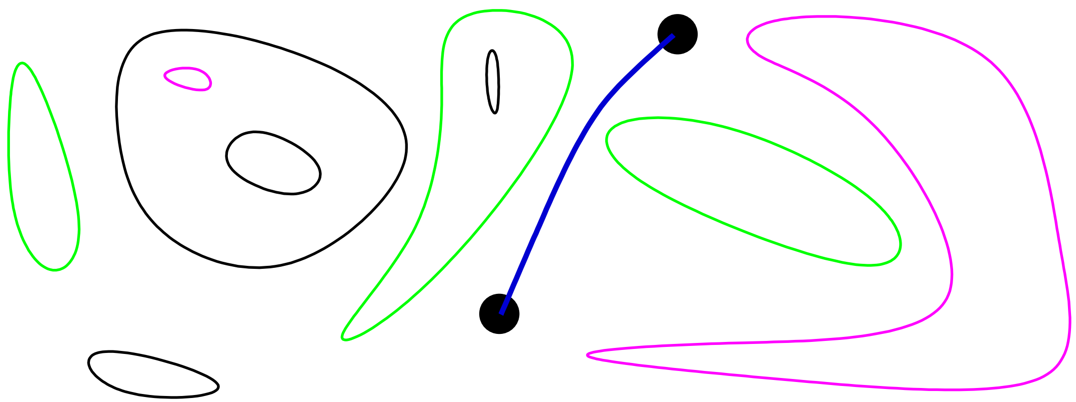


i.e. the ground state is the sum over all loop configurations

$$|g.s.\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

where $n_{\mathcal{L}}$ is the number of loops in configuration $\mathcal{L}$.

The excitations with non-abelian braiding are defects in the sea of loops.



When the defects are deconfined, they will braid with each other like the loops in the ground state.

Freedman, Nayak, Shtengel, Walker and Wang

When

$$d = 2\cos[\pi/(k+2)] \qquad \text{i.e.} \quad q = e^{i\pi/(k+2)},$$

these are the statistics of Wilson loops in $SU(2)_k$ Chern-Simons theory

Witten

To have non-abelian braiding, the quantum loop model should have topological order, which means it should be gapped.

However, for this all to work, the classical loop model needs to have a critical point.

The classical models being discussed have partition functions of the form

$$Z = \sum_{\mathcal{L}} w(\mathcal{L}) K^{L(\mathcal{L})}$$

where

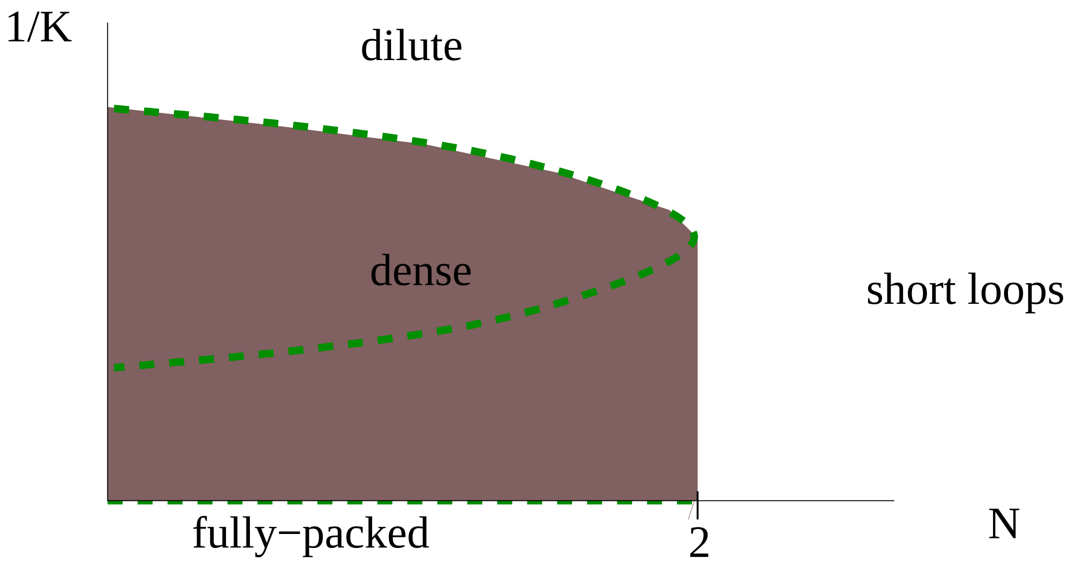
$w(\mathcal{L})$	is the topological weight of configuration $\mathcal{L}$,
$L(\mathcal{L})$	is the length of all the loops in $\mathcal{L}$,
K	is the weight per unit length

For closed loops which do not touch or cross, we have

$$w(\mathcal{L}) = N^{n_{\mathcal{L}}}$$

for some parameter N. This is usually called the $O(N)$ loop model.

For the $O(N)$ loop model in two dimensions, the phase diagram is



Typically, a critical point can occur when $K \approx 1$ (for the honeycomb lattice, the dilute-dense critical line occurs at $K = K_c = [2 + \sqrt{2 - N}]^{-1/2}$). The dense critical line is stable throughout the shaded region.

For $N > 2$, the model is not critical for any K – the partition function is dominated by short loops and so is not scale-invariant.

Important point:

At a critical point, loops of all sizes contribute the partition function in the long-distance limit. This behavior is necessary to get topological order – otherwise a length scale appears.

Thus to build a quantum loop model from the classical $O(N)$ loop model, we must have $N \leq 2$.

I said before that we want each loop to have weight $d = q + q^{-1}$, and that q is a root of unity, so $d \leq 2$.

However: This is quantum mechanics!

In any correlation function, each configuration is weighted by the probability amplitude squared.

Thus we must have

$$N = d^2$$

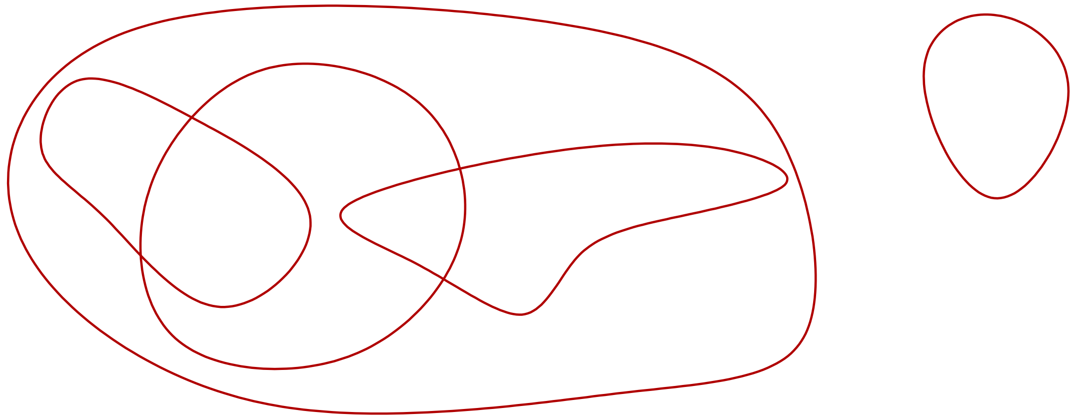
We must have $d \leq \sqrt{2}$ for this construction to work!

Fibonacci anyons have $d = \tau = 2\cos(\pi/5) > \sqrt{2}$.

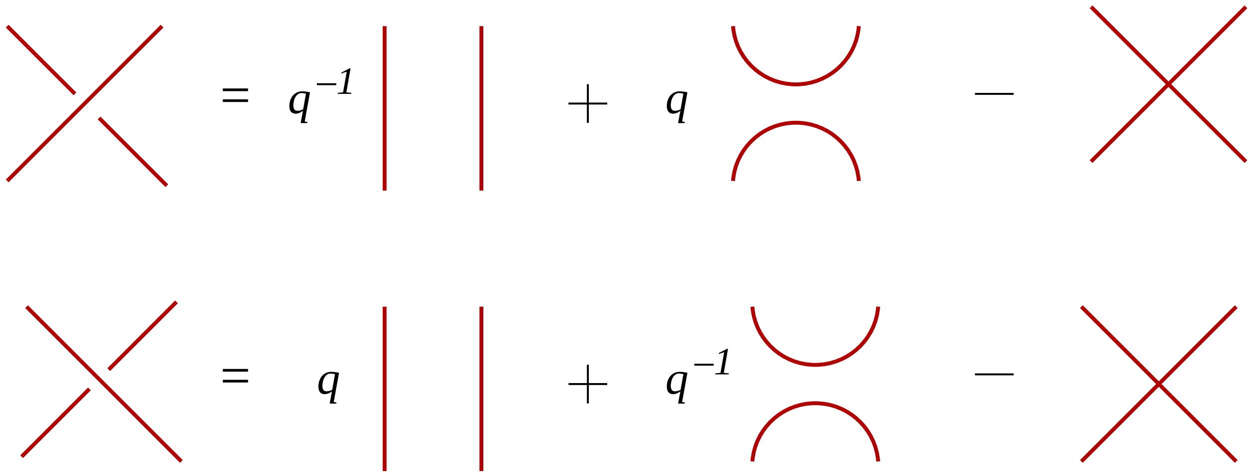
So how do we find a quantum loop model whose excitations are Fibonacci anyons?

Allow the loops to cross!

Our “loop” model consists of closed crossing curves.



We need to ensure that the braiding obeys the consistency conditions. For crossing curves, this can be done by using the Birman-Murakami-Wenzl algebra.

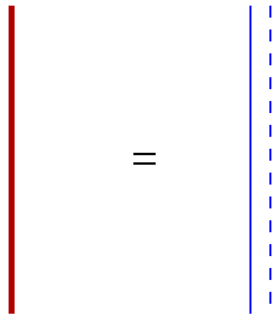


We use the BMW algebras associated with the $SO(n)$ Lie algebras and their quantum deformations.

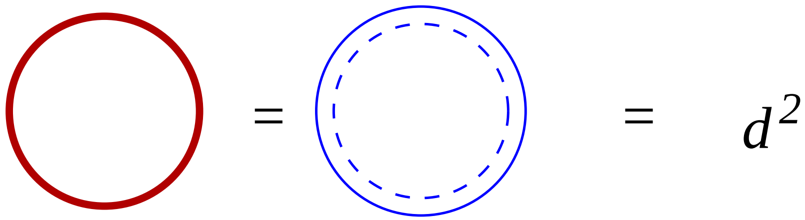
For both the $SO(3)$ and $SO(4)$ cases, we have found loop models which crash through the $d = \sqrt{2}$ barrier.

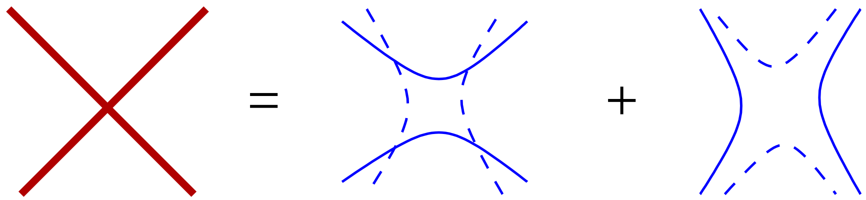
Both give models for Fibonacci anyons. The latter enables us to “square” the loops I discussed before.

The $SO(4)$ case is simpler, because the algebra $SO(4) = SU(2) \times SU(2)$. We can thus represent each line as a double line:



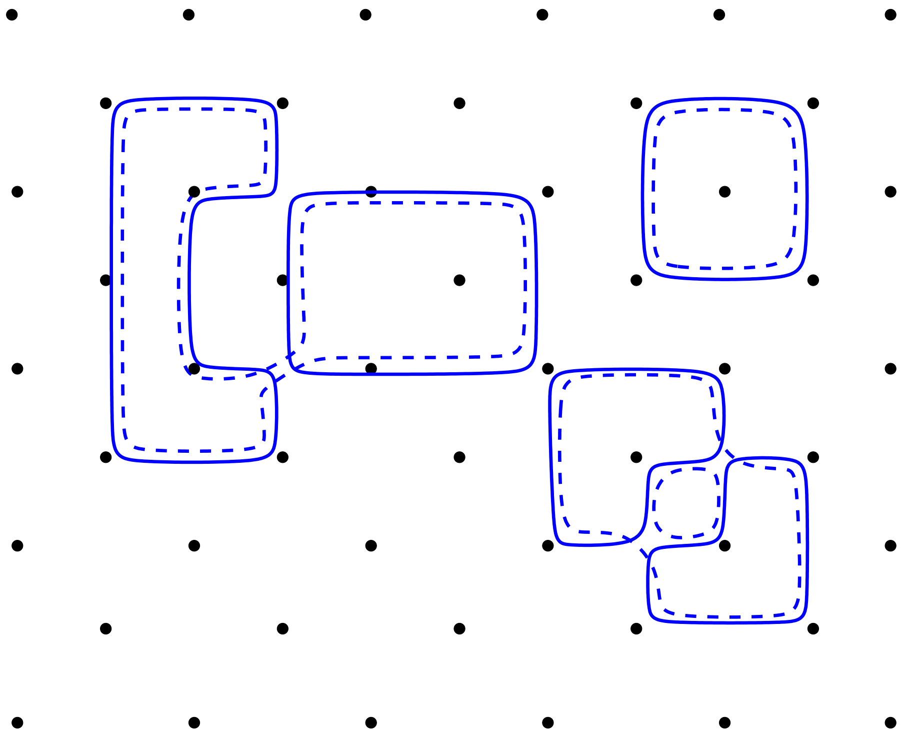
Closed loops now get a weight $d^2 = (q + q^{-1})^2$. We resolve the intersections as:





Each type of blue line behaves as a non-crossing loop!

Configurations in this $SO(4)$ loop model are a sum over configurations like



The appropriate braid relations follow from weighting each closed blue loop (either solid or dashed) by d.

These "doubled" loop models have critical points for all $d \leq 2$. This follows from

- an indirect argument using fused RSOS lattice models. These critical points are presumably between the dilute and dense phase, and are described by the

$$\frac{SU(2)_k \times SU(2)_k}{SU(2)_{2k}}$$

conformal field theories. Fendley 2006

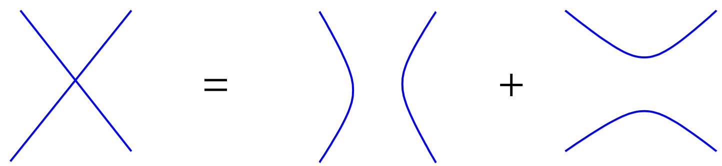
- a direct proof using "dilute BMW" models, which gives dense critical points too. Fendley 2007

Thus it is natural to conjecture that the doubled loop models have a phase diagram qualitatively similar to that of the O(N) model.

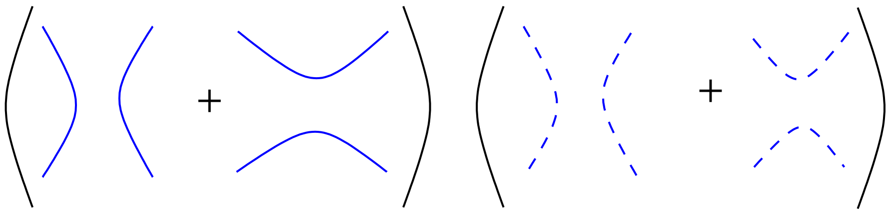
original work on the fused RSOS models: Date, Jimbo, Kuniba, Miwa, Okado
on the dilute BMW models: Nienhuis, Grimm, Warnaar

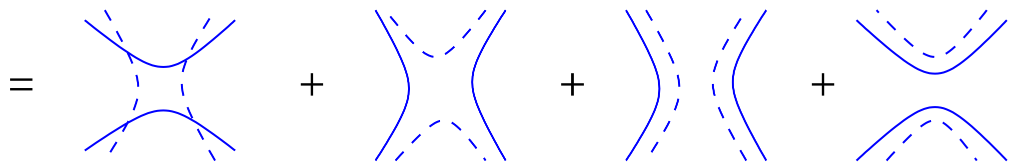
We can use the doubled models to find a "squared" version of the original loop model which remains critical for $2 \geq d > \sqrt{2}$.

Namely, allow crossings, but when evaluating the weight of each configuration, resolve the crossing as



Then the squared wavefunction can be written in terms of the doubled loops via





This change in the original loop model is just a change in the regularization, i.e. purely a short-distance modification. It should not change the topological behavior.

Thus this should realize the $SU(2)_k$ topological phases when

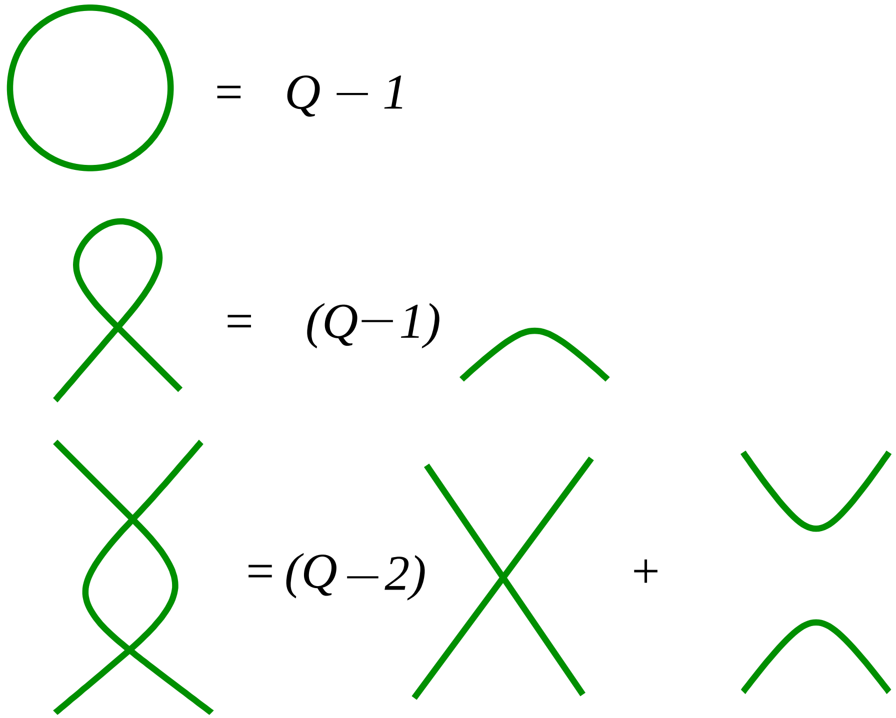
$$d = 2\cos(\pi/(k+2))$$

for any positive integer k.

For $k = 3$, we have Fibonacci anyons!

There's a second way to obtain a critical loop model with Fibonacci anyons.

The $SO(3)$ BMW algebra depends on a parameter Q, and is written graphically as



plus a three-strand relation.

To get Fibonacci anyons, one needs to take $Q = \tau + 1$, so closed loops still get weight τ.

The question is then: can one “square” this model and still get a critical theory?

The conjecture: yes

Fendley and Fradkin 2005

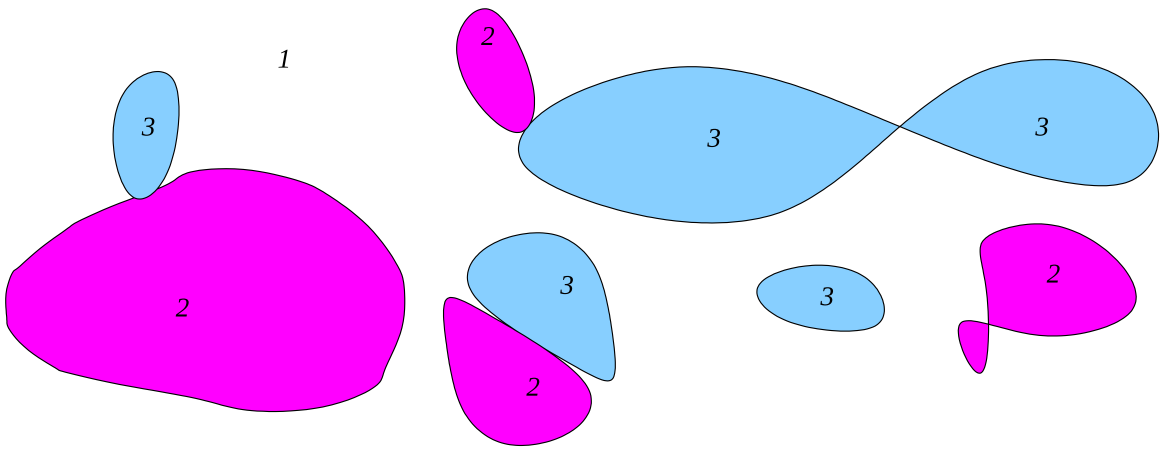
The proof:

Tutte 1969

Fidkowski, Freedman, Nayak, Walker and Wang 2006

To prove this, note that the $O(3)$ BMW relations are precisely those for the chromatic polynomial χ_Q!

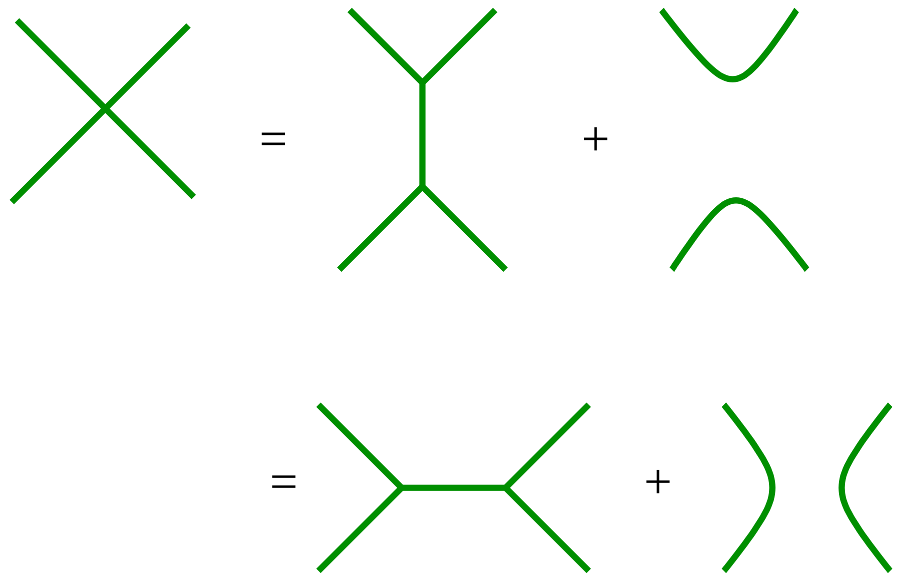
When Q is an integer, χ_Q is the number of coloring each region with Q colors such that adjacent regions have different colors.



For Q not an integer, χ_Q is defined recursively.

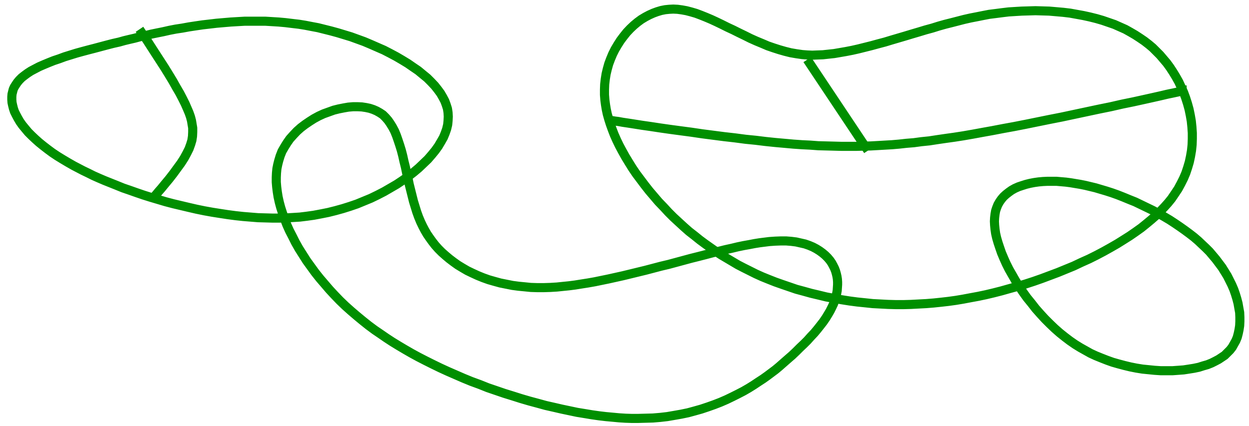
Ingredient 1: The chromatic polynomial is the topological weight of loops in the low temperature expansion of the Q-state Potts model

Ingredient 2: These crossing curves can be rewritten by replacing the crossings with trivalent vertices via



Think of these lines as behaving as spin-1 particles of SO(3) (or more precisely, the quantum deformation $U_q(SO(3))$.

This rewriting gives a loop gas with configurations looking like



This is an example of a string net theory.

Kuperberg; Kitaev; Levin and Wen; Fidkowksi *et al*; ...

The proof:

Use an amazing identity by Tutte relating

$$(\chi_{\tau+1})^2 = \frac{1}{\sqrt{5}} \tau^{3-N_{tri}/2} \chi_{\tau+2}$$

where N_{tri} is the number of trivalent vertices. This is true for any graph on the sphere with trivalent vertices.

This means that the low-temperature expansion of the Potts model with $Q = \tau + 2 = (5+\sqrt{5})/2$ is equivalent to the low-temperature expansion of the Potts model with $Q = \tau + 1$ with the topological weight of each configuration squared.

The Potts models have a critical point when $Q \leq 4$. Fibonacci anyons!

This quantum loop gas with trivalent vertices presumably corresponds to the $SO(3)_k$ topological field theory.

The only relations one needs to impose (on the sphere) are the “H-I” relation, and that “tadpoles” vanish. For higher genus surfaces, one can construct the necessary additional relation (the Jones-Wenzl projector) from the $SU(2)$ ones.

We don’t yet have a proof that there is a “squared” critical point for $k > 3$, but in light of the results for the loops, it seems very possible.

Conclusions

- There are lattice models and field theories which exhibit topological order and conformal quantum critical points. For $SO(3)_k$, these are based on the string nets or Potts models; for $SU(2)_k$, on a modified loop model.

- One technical comment: in all these cases, one can construct a representation of the BMW algebra which yields a critical model even with the Jones-Wenzl projector imposed.

- These models are naturally defined on the square lattice. Might there exist simpler lattice models with the same critical points?