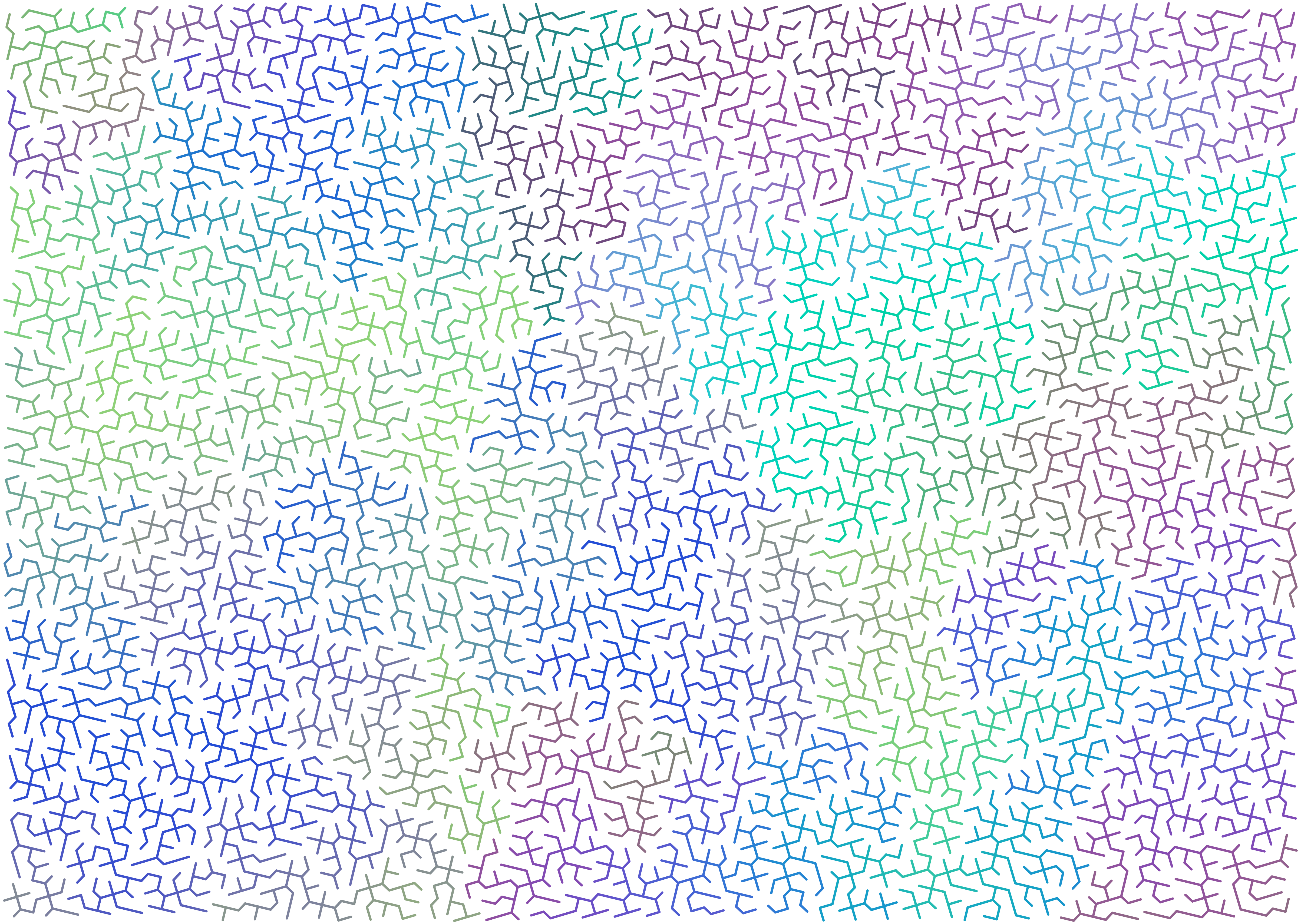


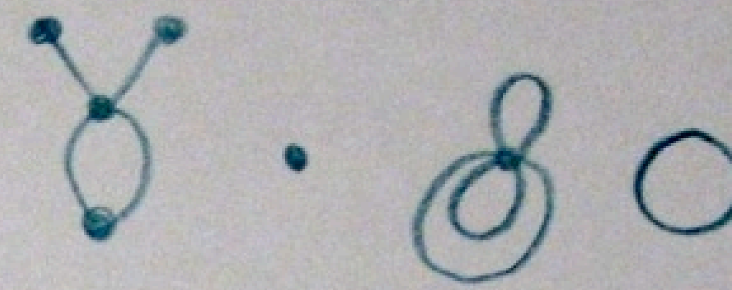
1. Deletion-contraction relations

2. Hard hexagons in the shadow world (and beyond)

3. Most topological state sums are instances of a more general construction

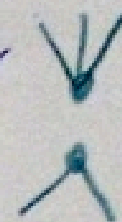
Deletion - Contraction

G = planar graph $\rightsquigarrow$



$I(G)$ = graph invariant

$I(G) \in \mathbb{C}$

$A \cdot I($  $) + B \cdot I($  $) + C \cdot I($  $) = 0$

$A, B, C \in \mathbb{C}$, $A, B, C \neq 0$

Vertex + edge fugacities:

$$I'(G) = I(G) \cdot \alpha^{\#e(G)} \cdot \beta^{\#v(G)}$$

$$\rightsquigarrow (A', B', C') = (A, \alpha B, \alpha\beta C)$$

In order to fully evaluate a graph, we further assume:

- $I(G_1 \amalg G_2) = I(G_1) \cdot I(G_2)$
-  $= \lambda \cdot$ 
- $\bullet = \rho \cdot \emptyset$
- $\bigcirc = \ell \cdot \emptyset$

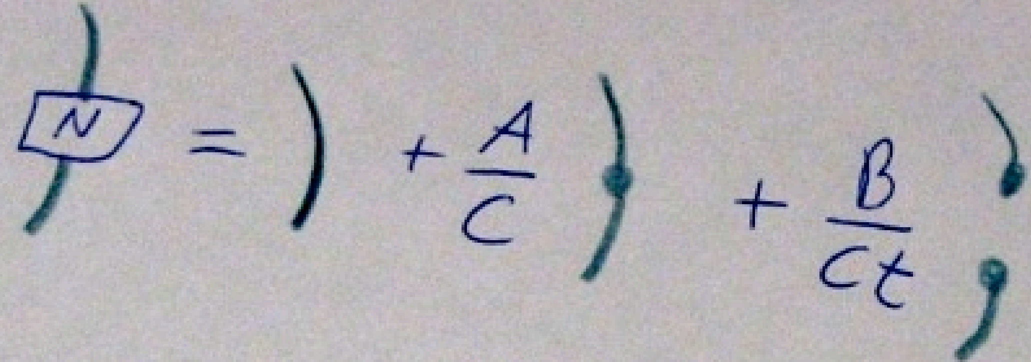
Also,

 $= -\frac{B}{A} \cdot \Big($  $\bullet \Big) - \frac{C}{A} \cdot \Big($  $\Big) = \frac{-B\rho - C}{A} \cdot$ 

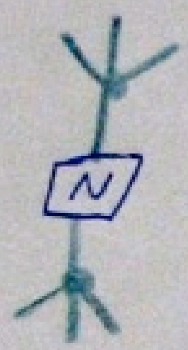
so for convenience, define

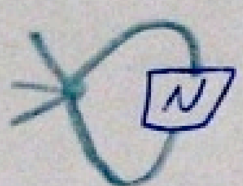
$$t = \frac{-B\rho - C}{A}$$

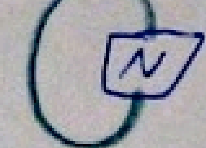
Consider



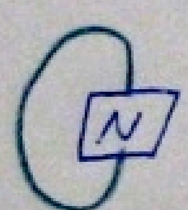
N is almost nilpotent:

 $=0$,

 $=0$, but

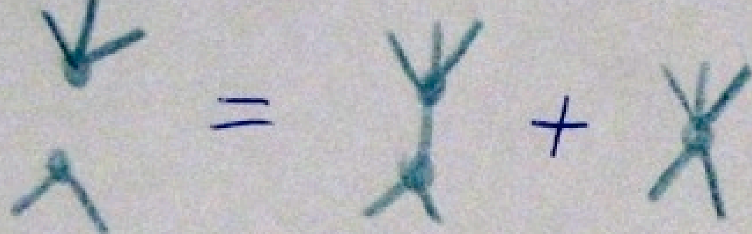
 $\neq 0$

So we impose further relation $N=0 \Rightarrow$

 $= \ell + \frac{A\lambda\rho}{C} + \frac{B\rho}{C} = 0$

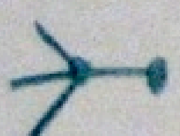
(Can be thought of as a relation between ℓ and λ.)

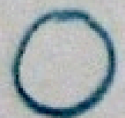
Chromatic polynomial, $\chi_K(G)$ = # of vertex colorings with K colors

 $\Rightarrow \quad A=1,\ B=-1,\ C=1$

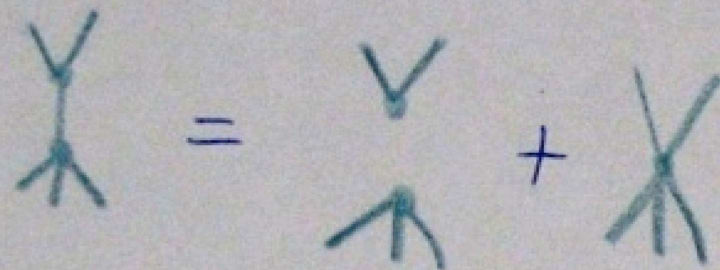
 $=0 \quad \Rightarrow \quad \lambda=0$

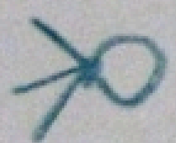
 $=K \quad \Rightarrow \quad p=K$

 $=(K-1)\cdot$  $\quad \Rightarrow \quad t=K$

( $= \ell = \dfrac{-A\lambda p}{C} - \dfrac{Bp}{C} = K$)

Tutte polynomial (for connected graphs)

 $\Rightarrow A=1, B=-1, C=-1$

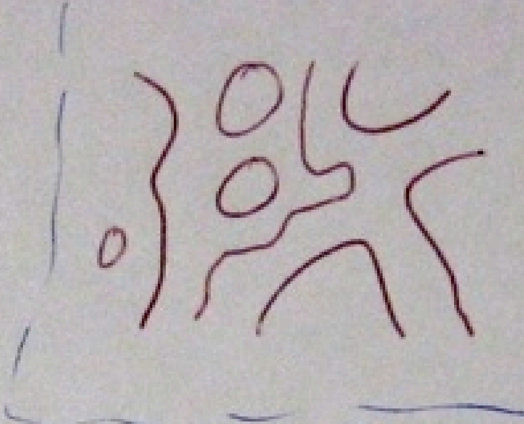
 $= y \cdot$  $\Rightarrow \lambda = y$

 $= x \cdot$  $\Rightarrow t = x$

$\Rightarrow \rho = 1-x$

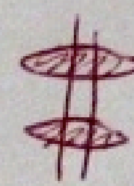
$\left(\right.$  $= \ell = \frac{-A\lambda\rho}{C} - \frac{B\rho}{C} = (y-1)\cdot(1-x) \left.\right)$

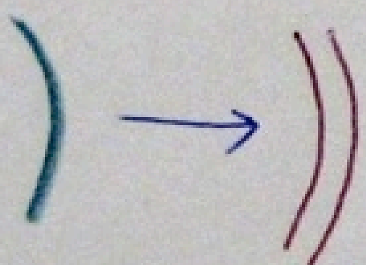
Temperly-Lieb

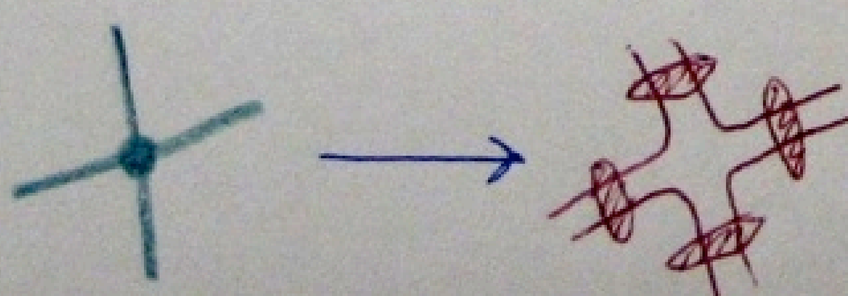


$\bigcirc = d\cdot\phi$

("d-isotopy")

- Choose $u, v \in \mathbb{C}$, $u, v \neq 0$
- Choose  such that  $= u\cdot)(+ v\cdot$ 
- choose $\beta \in \mathbb{C}$, $\beta \neq 0$ (vertex fugacity)
- Define graph invariant by these substitutions:



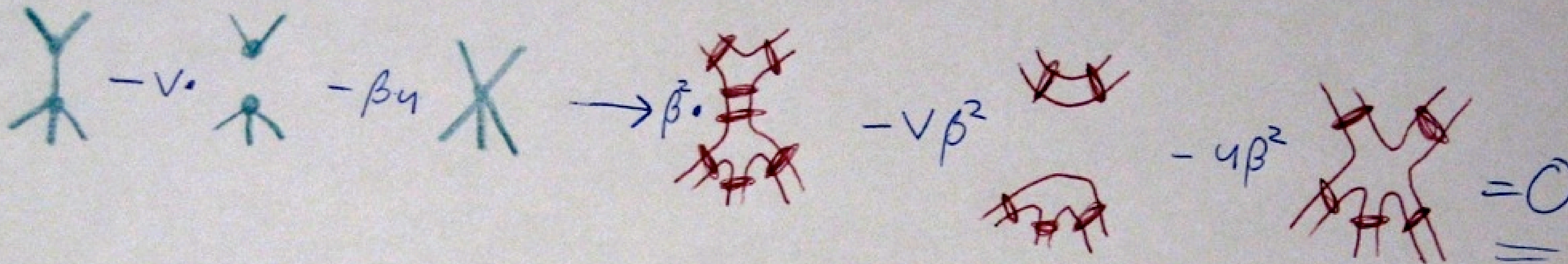
 $\cdot\beta$

(Special cases:

$\bullet \longrightarrow \beta\cdot\bigcirc = \beta d$

 $\longrightarrow \beta\cdot$ )

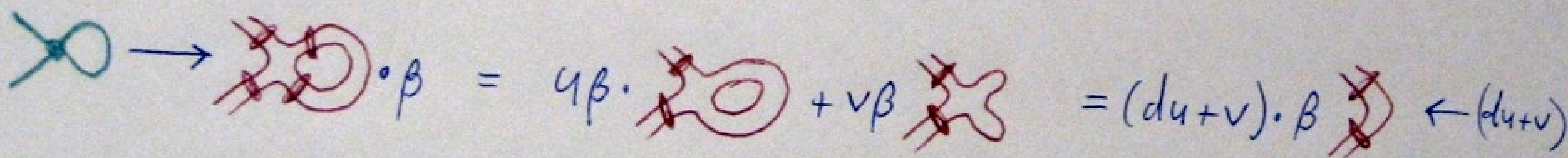
Then



$\Rightarrow A=1,\ B=-v,\ C=-\beta u$

$\bullet \to \beta\cdot \bigcirc = \beta d \quad \Rightarrow \quad \rho = \beta d$

$\Rightarrow \quad \ell = d^2$



$\Rightarrow \lambda = du+v$

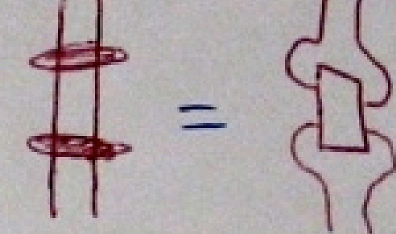
(check that $\ell + \frac{A\lambda\rho}{C} + \frac{B\rho}{C} = 0$)

Tutte polynomial via T-L:

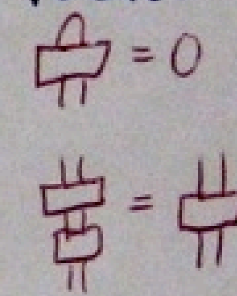
$$\left.\begin{aligned} A &= 1 \quad \checkmark \\ B &= -v = -1 \\ C &= -\beta u = -1 \\ \lambda &= du + v = y \\ \rho &= \beta d = 1 - x \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} d^2 &= (y-1)(1-x) \\ \beta^2 &= \frac{1-x}{y-1} \\ u^2 &= \frac{y-1}{1-x} \\ v &= 1 \end{aligned}$$

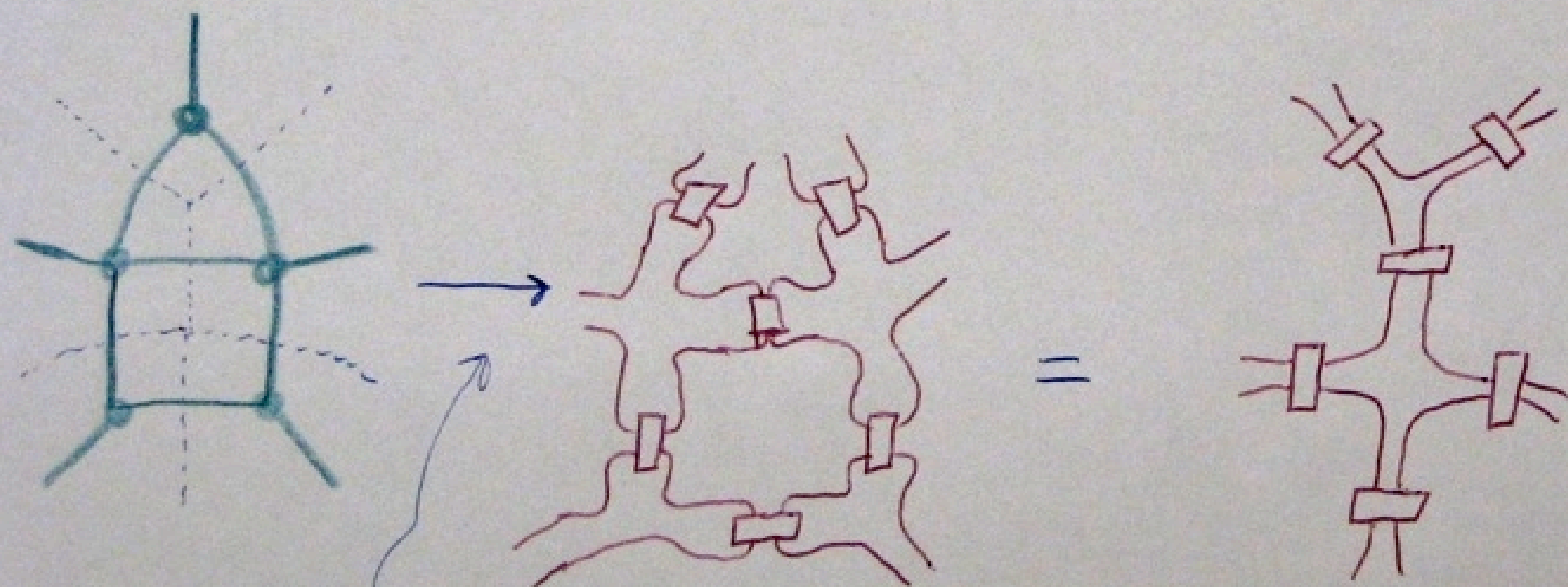
Chromatic polynomial via T–L

$$\left.\begin{aligned} A &= 1 \checkmark \\ B &= -v = -1 \\ C &= -\beta u = 1 \\ p &= \beta d = K \\ \ell &= d^2 = K \end{aligned}\right\} \Rightarrow \quad \begin{aligned} d^2 &= K \\ \beta &= d \\ u &= -1/d \\ v &= 1 \end{aligned} \quad \Rightarrow$$



recall:



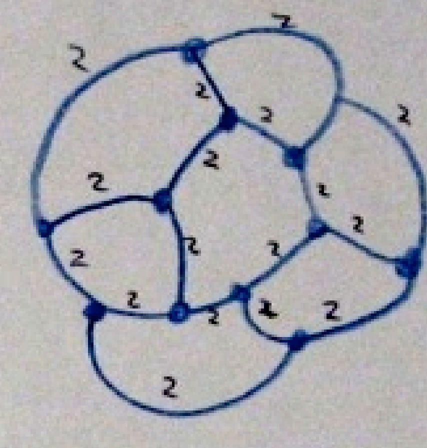


(up to factors of $\beta = d$)

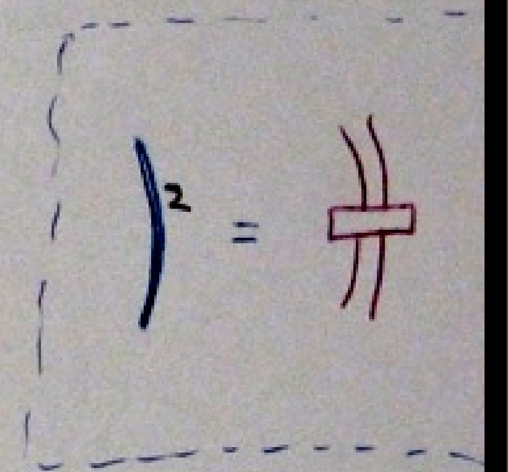
$$\boxed{\chi_{d^2}(G) = \mathrm{Yamada}(\hat{G}) \cdot d^{\#v(G)}}$$

Hard Hexagons in the Shadow World

Want to evaluate "spin network":



(assume $\hat{G}$ is trivalent)



{Shadow world / Turaev-Viro} -state sum:

$$\sum_{\substack{\text{restricted}\\ \text{face}\\ \text{labelings}}} \left[\left(\prod_{\text{verts}} \circledast\right)\cdot\left(\prod_{\text{edges}} \ominus\right)^{-1}\cdot\left(\prod_{\text{faces}} \bigcirc\right)\right]$$

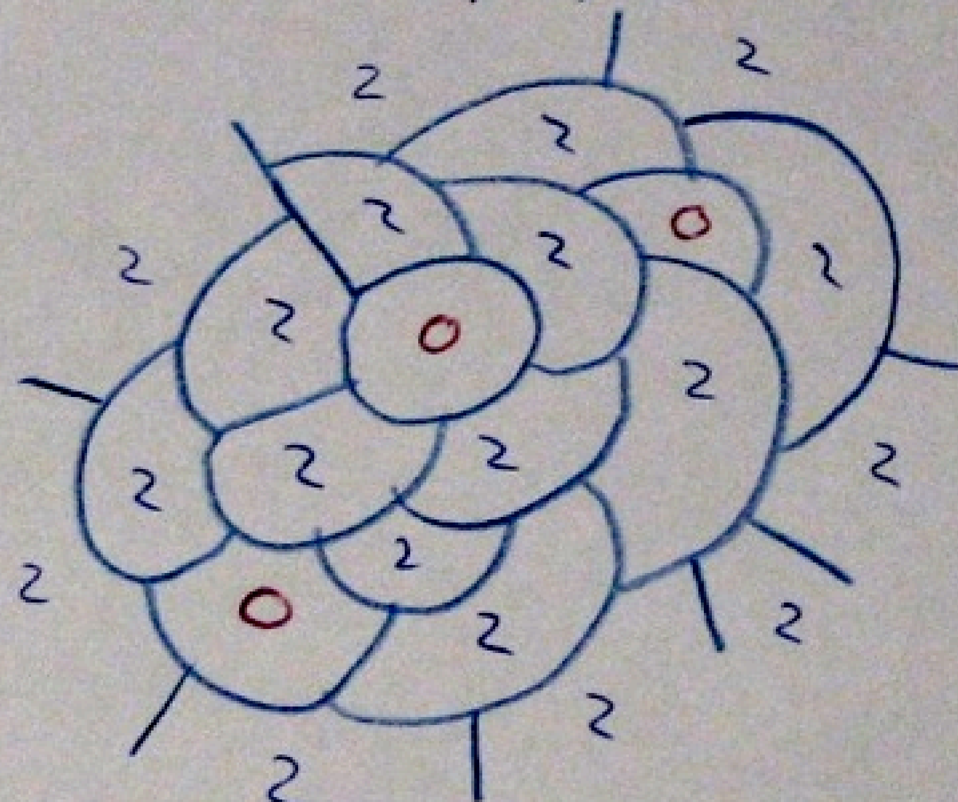
zero at ∞

$$= \sum_{\substack{\text{all}\\ \text{face}\\ \text{labelings}}} \left[\prod_{\text{verts}} \circledast \prod_{\text{edges}} \ominus^{-1} \prod_{\text{faces}} \bigcirc\right]\cdot \frac{1}{\sum_i \bigcirc_i^2}$$

$D = \sum d_i^2$

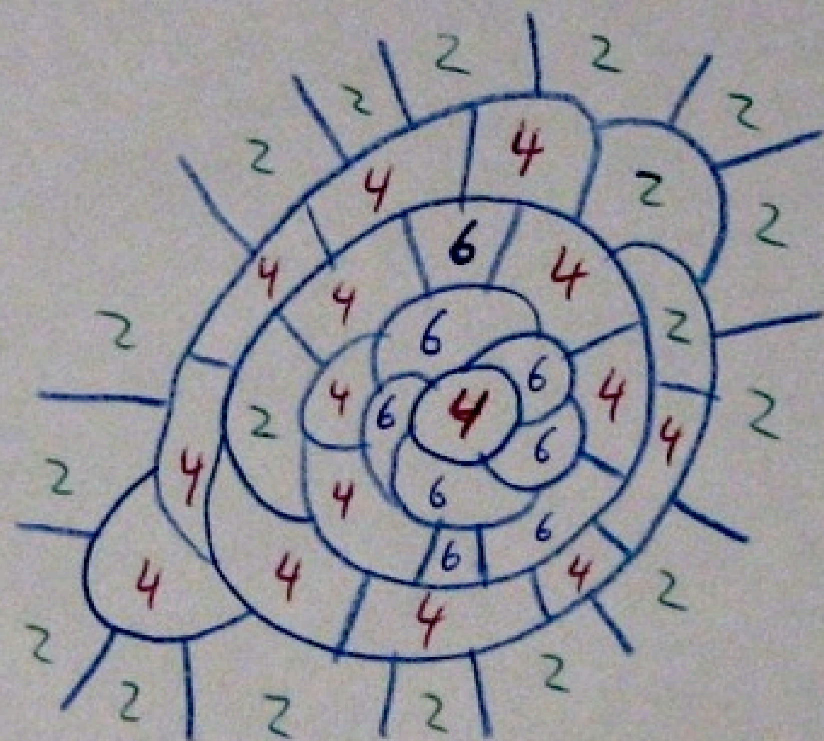
If $d = \tau = \frac{1+\sqrt{5}}{2}$, then label set $= \{0, 2\}$

$(0,2,0)$ is not an admissible triple, so...



$\Rightarrow$ summation is over $\underbrace{\text{"hard"}}_{\text{configurations of}}$ polygons.

For general d (but edge labels still all 2), label set $= \{0, 2, 4, 6, \ldots\}$ and adjacent face labels differ by $-2, 0$ or 2:



$\rightsquigarrow$ Some sort of height model

State sums via handle decompositions

- Turaev-Viro state sum
- Witten-Reshetikhin-Turaev state sum
- Crane-Yetter state sum (4-dim'l)
- Dijkgraaf-Witten state sum (any dimension)
- Turaev "shadow" state sum

Goal: Derive all off the above in a unified framework

More specifically, show that all of the above arise from computing the path integral of a semi-simple TQFT in terms of a handle decomposition.

Ingredients for a TQFT:

- top dimension $n+1$
- system of "fields" (e.g. pictures) for manifolds of dimension $\leq n$
- local relations (at least as strong as isotopy) for fields on n-manifolds

Now define

finite $\mathbb{C}$-linear combinations

$$A(Y^n; c) = \mathbb{C}[\text{fields on } Y \text{ which restrict to } c \text{ on } \partial Y] / \langle \text{local relations} \rangle$$

boundary condition

(skein module)

Also define cylinder categories

$$A(X^{n-1}) = \begin{cases} \text{objects} = \{\text{fields on } X\} \\ \text{morphisms } a \to b = A(X \times I; \hat{a}, b) \\ \text{composition} = \text{gluing} \end{cases}$$

$A(\partial Y)$ acts on $\{A(Y^n; c)\}_{\text{all } c}$ via gluing of collars

$\rightsquigarrow$ gluing formula for n-manifolds

(For $n=2$, "particles" are irreps of $A(S^1)$.)

$(n+1)$-dimensional part

What we want:

- $Z(W^{n+1}) : A(\partial W) \to \mathbb{C}$

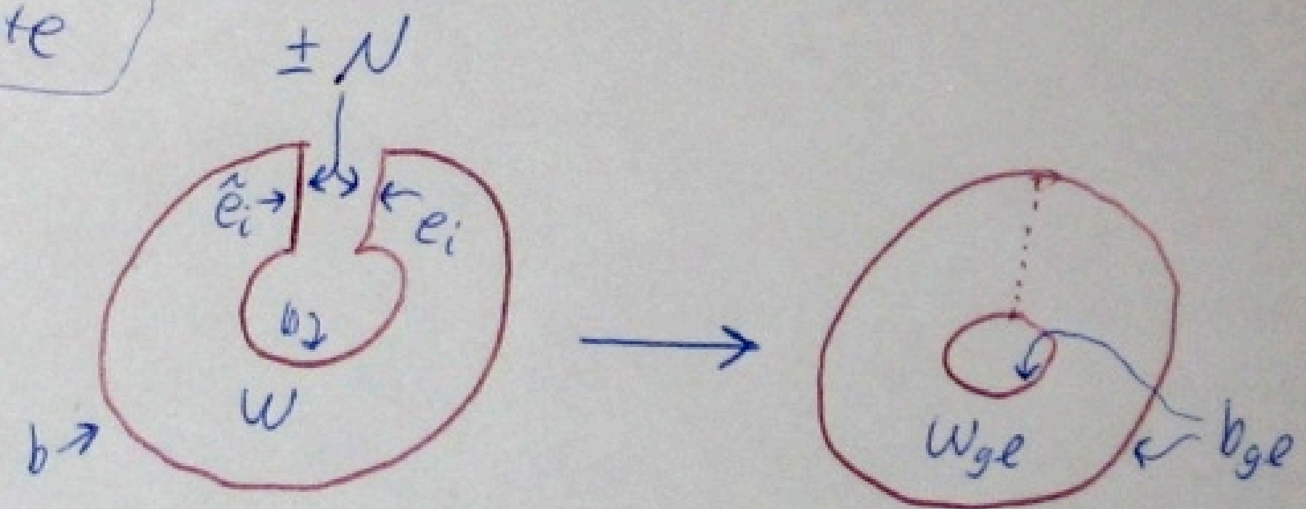
- can define inner product

$$\langle \cdot , \cdot \rangle : A(Y^n) \times A(Y) \to \mathbb{C}$$

$$\boxed{\langle a, b \rangle \overset{\text{def}}{=} Z(Y \times I)(\hat{a} \cup b)}$$

this I.P. should be non-degenerate

- gluing formula:



$$\boxed{Z(W_{gl})(b_{gl}) = \sum_i Z(W)(b \cup \hat{e}_i \cup e_i) \cdot \frac{1}{\langle e_i, e_i \rangle}}$$

($\{e_i\}$ = orthogonal basis of $A(N)$)

candidate $Z(B^{n+1})$

Thm. Choose $z \in A(S^n)^*$. If

① induced I.P. on $A(B^n; c)$ is positive definite $\forall c$, and

② $\dim(A(Y^n; c)) < \infty \quad \forall (Y, c)$

then there is a unique partition function Z such that

$Z(B^{n+1}) = z \in A(\partial B^{n+1})^*$.

n

Proof:

$z \rightsquigarrow$ I.P. on $A(B^n; c) \rightsquigarrow$ gluing formula for 1-handles

$\rightsquigarrow$ I.P. on $A(B^{n-1} \times S^1; c) \rightsquigarrow$ gluing formula for 2-handles

$\rightsquigarrow$ I.P. on $A(B^{n-2} \times S^2; c) \rightsquigarrow$ gluing formula for 3-handles ....

Show independence under:

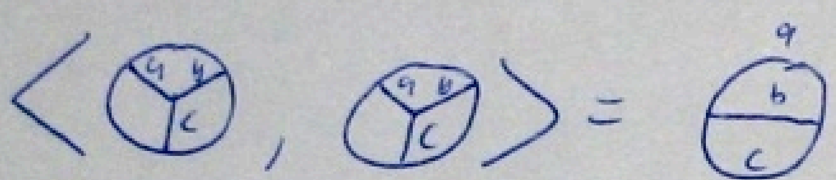
ⓐ handle slides (easy)

ⓑ handle cancelation (not hard)

Example:

$n=2$, fields = multi-curves, local relations = d-isotopy
(Temperly-Lieb)

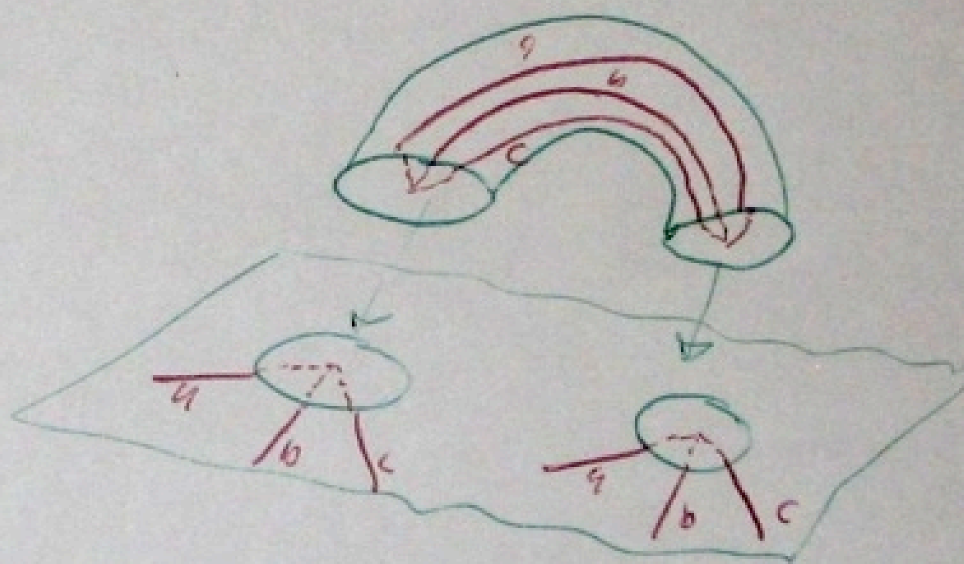
- I.P. on B^2



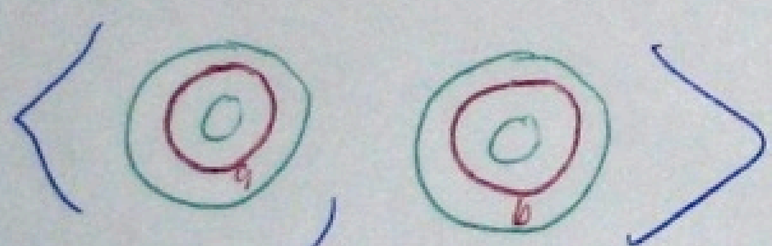
$= \theta_{abc}$

- gluing a 1-handle

factor of $\dfrac{\theta_{abc}}{\theta_{abc} \cdot \theta_{abc}} = \theta_{abc}^{-1}$

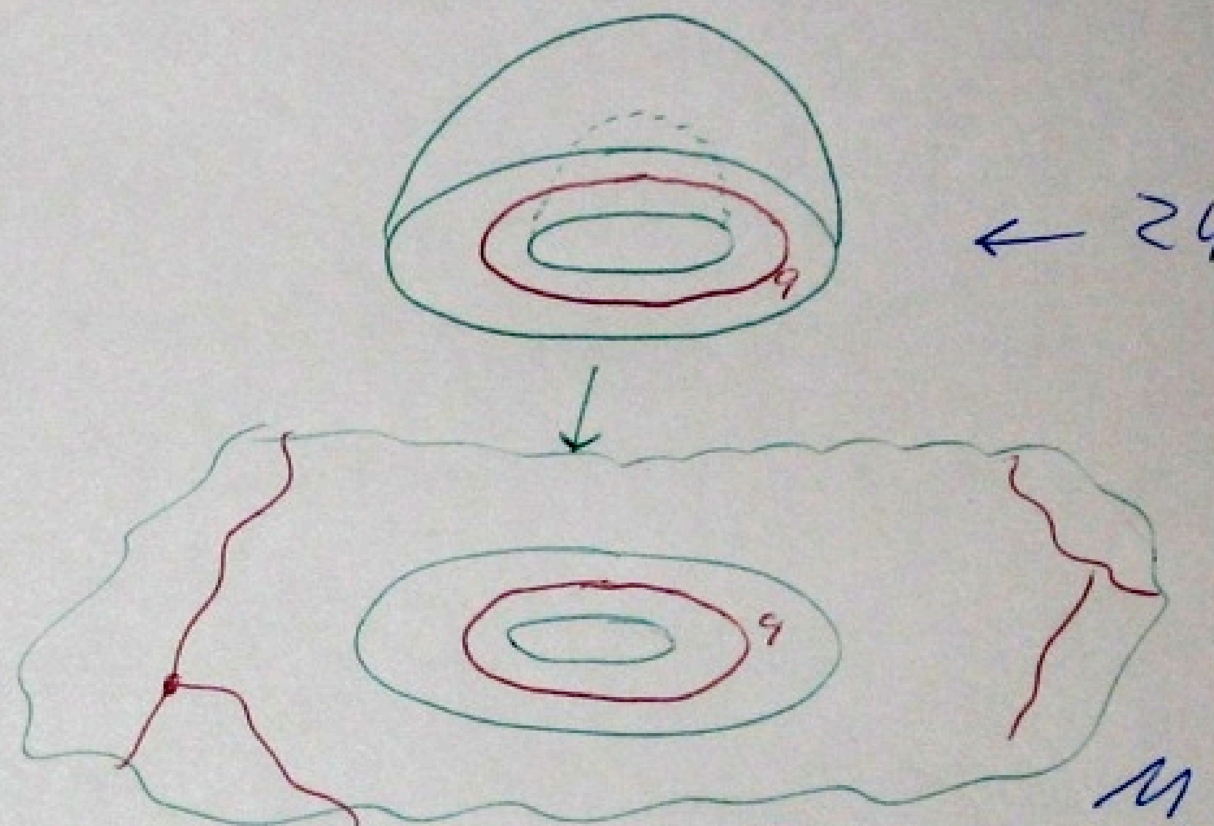


- IP on $S^1 \times I$



$= \delta_{ab}$

- gluing a 2-handle



$Z(M \cup 2h)(x)$

$$= \sum_a Z(M)(x \cup a) \cdot d_a$$

- I.P. on S^2

$$\langle \phi_{S^2}, \phi_{S^2} \rangle = Z(S^2 \times I) = Z(B^3 \cup 2h) = \sum_a Z(B^3)(O^a) \cdot d_a = \sum_a d_a^2$$

- gluing a 3-handle

$\rightsquigarrow$ factor of $\dfrac{1}{\langle \phi_{S^2}, \phi_{S^2} \rangle} = \dfrac{1}{\sum d_a^2}$

Putting it all together...

cell-decomposition

If M^3 has a generic handle decomposition (dual to a triangulation), then

$$Z(M) = \sum_{\substack{\text{labelings} \\ \text{of 2-handles}}} \left[\prod_{0\text{-h}} \bigcirc \prod_{1\text{-h}} \Theta^{-1} \prod_{2\text{h}} \overset{d_a}{\bigcirc} \prod_{3\text{h}} \frac{1}{\left(\sum d_a^2\right)} \right]$$

Fields	Local Relation	State Sum
Maps into BG (G a finite group) n = arbitrary	Homotopy of maps	Dijkgraaf-Witten sum on a triangulation
Pictures based on a disklike 2-category (e.g. a spherical category) n = 2	Isotopy plus relations coming from the category	Turaev-Viro sum
Pictures based on a ribbon category (a disklike 3-category) n=3	Isotopy plus relations coming from the category	For a generic cell handle decomposition of a 4-manifold, the Crane-Yetter state sum
[same as above]	[same as above]	For 2-handles attached to the 4-ball, the Witten-Reshetikhin-Turaev surgery formula
[same as above]	[same as above]	For a "special spine" of a 4-manifold, the Turaev shadow state sum