Topology and Quantum Computing

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www.math.uic.edu/~kauffman/Unitary.pdf

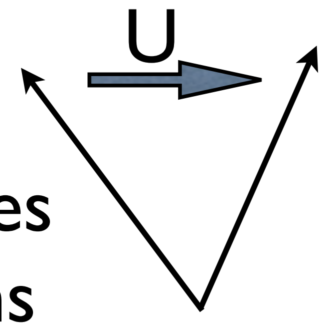
Spin Networks and Anyonic Topological Computing

Louis H. Kauffman[a] and Samuel J. Lomonaco Jr.[b]

[a] Department of Mathematics, Statistics and Computer Science (m/c 249), 851 South Morgan Street, University of Illinois at Chicago, Chicago, Illinois 60607-7045, USA
[b] Department of Computer Science and Electrical Engineering, University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, USA

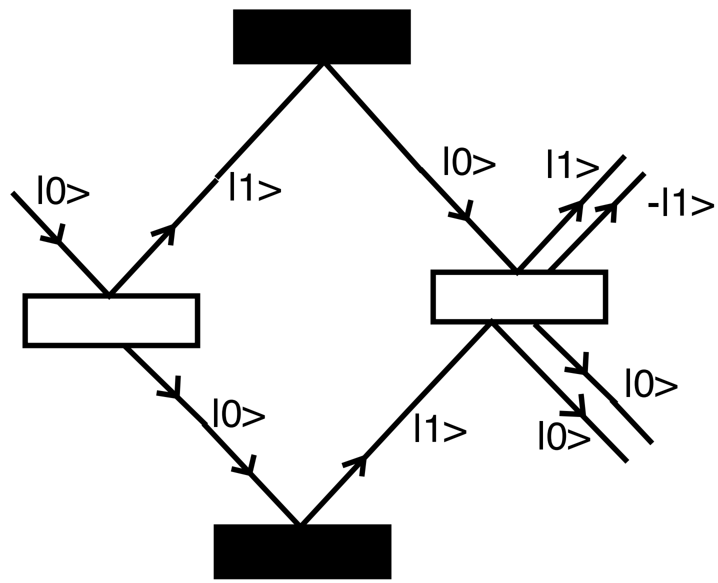
Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector |S> in a complex vector space.

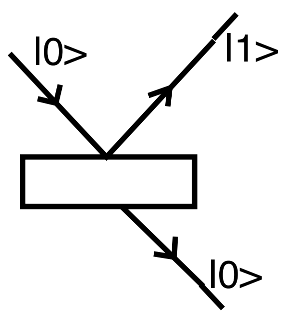


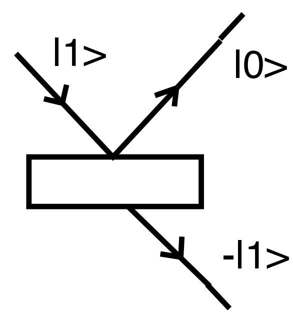
1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: |S> -----> U|S>

2. If |S> = z1|e1> + z2|e2> + ... + zn|en> in a measurement basis {e1,e2,...,en}, then measurement of |S> yields |ei> with probability |zi|^2.



Mach-Zender Interferometer





$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Quantum Gates
are unitary transformations enlisted for the purpose of computation.

CNOT =

1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

CNOT|00> = |00>
CNOT|01> = |01>
CNOT|10> = |11>
CNOT|11> = |10>

Universal Gates

A *two-qubit gate* G is a unitary linear mapping

$$G : V \otimes V \longrightarrow V \otimes V$$ where V is

a two complex dimensional vector space. We say that the gate G is *universal for quantum computation* (or just *universal*) if G together with local unitary transformations (unitary transformations from V to V) generates all unitary transformations of the complex vector space of dimension 2^n to itself. It is well-known [44] that $CNOT$ is a universal gate.

A gate G is universal
iff
G is entangling.

A gate G, as above, is said to be *entangling* if there is a vector

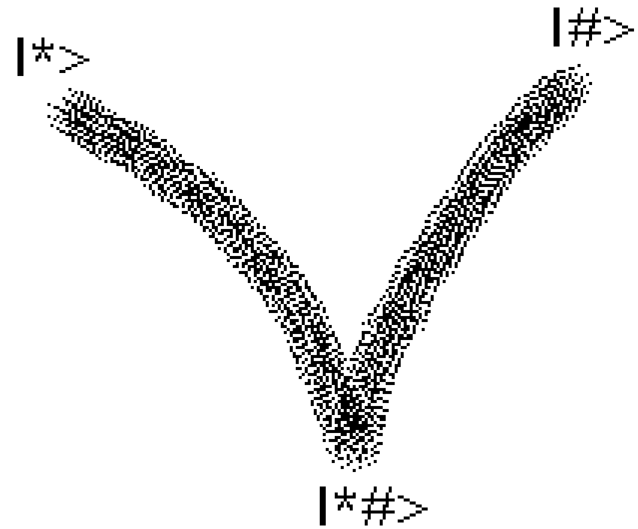
$$|\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \in V \otimes V$$

such that $G|\alpha\beta\rangle$ is not decomposable as a tensor product of two qubits. Under these circumstances, one says that $G|\alpha\beta\rangle$ is *entangled.*

In [6], the Brylinskis give a general criterion of G to be universal. They prove that *a two-qubit gate G is universal if and only if it is entangling.*

An Entangled State

The EPR State



$$|{*\#}\rangle = (|01\rangle + |10\rangle)/\mathrm{Sqrt}(2)$$

An Entanglement Criterion

Remark. A two-qubit pure state

$$|\phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

is entangled exactly when $(ad - bc) \neq 0$. It is easy to use this fact to check when a specific matrix is, or is not, entangling.

The Bell States

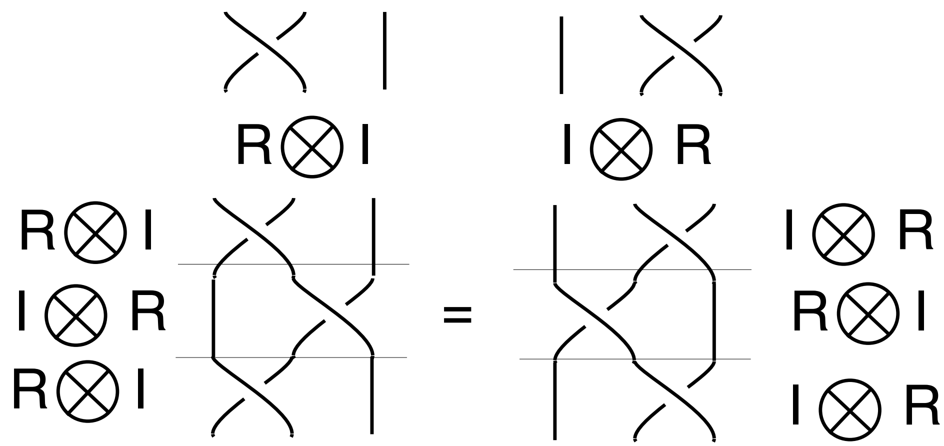
$$R|00\rangle = (1/\sqrt{2})|00\rangle - (1/\sqrt{2})|11\rangle,$$

$$R|01\rangle = (1/\sqrt{2})|01\rangle + (1/\sqrt{2})|10\rangle,$$

$$R|10\rangle = -(1/\sqrt{2})|01\rangle + (1/\sqrt{2})|10\rangle,$$

$$R|11\rangle = (1/\sqrt{2})|00\rangle + (1/\sqrt{2})|11\rangle.$$

Braiding and the Yang-Baxter Equation



$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

Braiding Operators are Universal Quantum Gates

Let V be a
two complex dimensional vector space.

Universal gates can be constructed
from certain solutions to the
Yang-Baxter Equation

$$R\colon\ V \bigotimes V \longrightarrow V \bigotimes V$$

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

Representative Examples of Unitary Solutions to the Yang-Baxter Equation that are Universal Gates.

$$R = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

Bell Basis Change Matrix

$$R' = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \qquad R'' = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Swap Gate with Phase

Issues

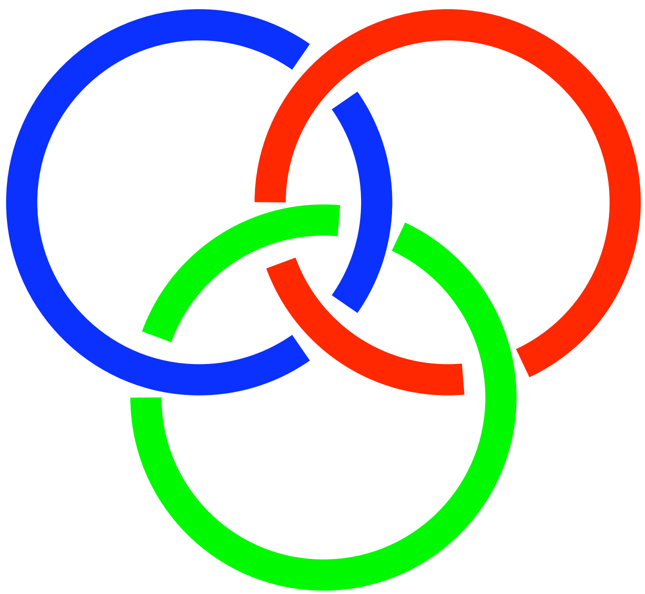
1. Giving a Universal Gate that is topological does not create "topological quantum computing" because the U(2) local operations have not been made topological.

2. Nevertheless, Yang-Baxter gates are interesting to construct and help to discuss Topological Entanglement versus Quantum Entanglement.

Quantum Entanglement and Topological Entanglement

An example of Aravind [1] makes the possibility of such a connection even more tantalizing. Aravind compares the Borromean rings (see figure 2) and the GHZ state

$$|\psi\rangle = (|\beta_1\rangle|\beta_2\rangle|\beta_3\rangle - |\alpha_1\rangle|\alpha_2\rangle|\alpha_3\rangle)/\sqrt{2}.$$



Is the Aravind analogy only superficial?!

Consider this state.

$$|\psi\rangle = (1/2)(|000\rangle + |001\rangle + |101\rangle + |110\rangle)$$

Observation in any coordinate
yields entangled and unentangled
states with equal probability.

e.g.

$$|\psi\rangle = (1/2)(|0\rangle(|00\rangle + |01\rangle) + |1\rangle(|01\rangle + |10\rangle)$$

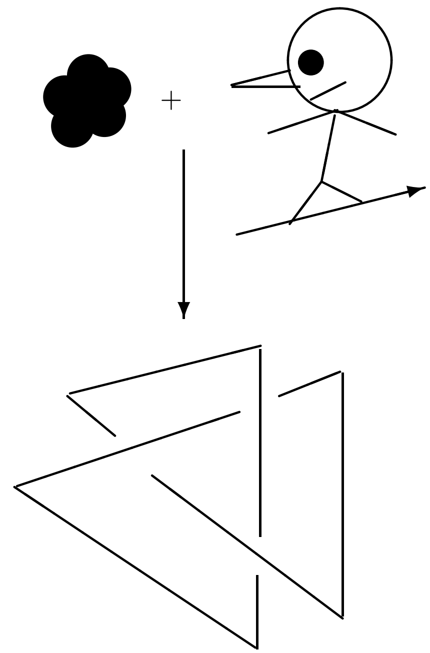
First coordinate measurement
gives
|00> + |01> and
|01> + |10>
with equal probability.

Do we need Quantum Knots?

$a|K> + b|K'>$

K: probability $|a|^2$

K':probability $|b|^2$



Observing a Quantum Knot


Air on the
Dirac Strings

$SU(2)$ Representations of the Artin Braid Group

Theorem. If $g = a + bu$ and $h = c + dv$ are pure unit quaternions,then, without loss of generality, the braid relation $ghg = hgh$ is true if and only if $h = a + bv$, and $\phi_g(v) = \phi_{h^{-1}}(u)$. Furthermore, given that $g = a + bu$ and $h = a + bv$, the condition $\phi_g(v) = \phi_{h^{-1}}(u)$ is satisfied if and only if $u \cdot v = \frac{a^2 - b^2}{2b^2}$ when $u \neq v$. If $u = v$ then then $g = h$ and the braid relation is trivially satisfied.

g = a + bu
h= a + bv

u•v = (a^2 - b^2)/2b^2

An Example. Let

$$g = e^{i\theta} = a + bi$$

where $a = cos(\theta)$ and $b = sin(\theta)$. Let

$$h = a + b[(c^2 - s^2)i + 2csk]$$

where $c^2 + s^2 = 1$ and $c^2 - s^2 = \frac{a^2 - b^2}{2b^2}$. Then we can reexpress g and h in matrix form as the matrices G and H. Instead of writing the explicit form of H, we write $H = FGF^*$ where F is an element of $SU(2)$ as shown below.

$$G = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$F = \begin{pmatrix} ic & is \\ is & -ic \end{pmatrix}$$

SU(2) Fibonacci Model

$$\tau^2 + \tau = 1.$$

$$g = e^{7\pi i/10}$$

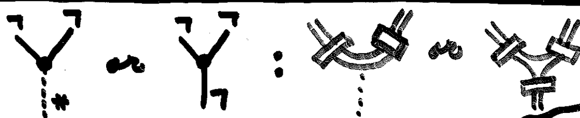
$$f = i\tau + k\sqrt{\tau}$$

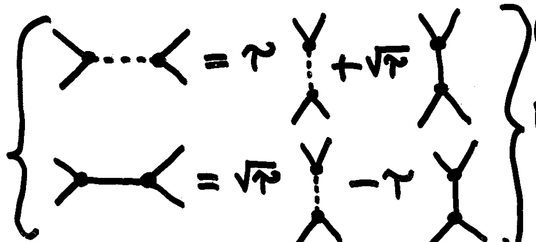
$$h = frf^{-1}$$

{g,h} represents 3-strand braids, generating a dense subset of SU(2).

We shall see that the representation
labeled "SU(2) Fibonacci Model"
in the last slide
extends beyond SU(2) to
representations of many-stranded
braid groups rich enough
to generate quantum computation.

Fibonacci Model in a Nutshell



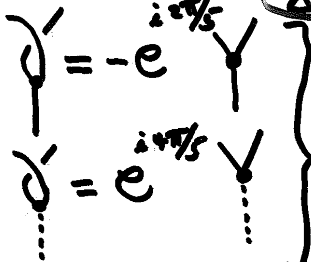


$R + FRF$ gen rep of B_3

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

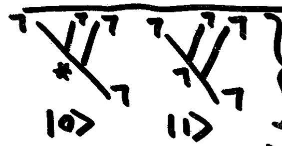
$\tau^2 + \tau = 1$, $\tau = \frac{1}{\Delta}$, $\Delta^2 = 1 + \Delta$

$\Delta = (1+\sqrt{5})/2$.



$$R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}$$

$A = e^{3\pi i/5}$, $\delta = -A^2 - A^{-2} = -2\cos(6\pi/5) = \frac{1+\sqrt{5}}{2}$



$V_{\tau}^{\tau\tau\tau} = V$ gives

rep: $B_3 \longrightarrow U(2)$

$\mathrm{rep}(\chi|) = R$
$\mathrm{rep}(|\chi) = FRF$
} gens dense subset of $U(2)$.

But first, a digression:
We show how to make a
quantum computation of the
the trace of a unitary matrix.
This is

1. A good example of a quantum algorithm.

2. Useful for the quantum computation of
knot polynomials such as the Jones polynomial.

Quantum Computation of the Trace of a Unitary Matrix

The Hadamard Test

In order to (quantum) compute the trace of a unitary matrix U, one can use the *Hadamard test* to obtain the diagonal matrix elements $\langle\psi|U|\psi\rangle$ of U. The trace is then the sum of these matrix elements as $|\psi\rangle$ runs over an orthonormal basis for the vector space. We first obtain

$$\frac{1}{2} + \frac{1}{2} Re\langle\psi|U|\psi\rangle$$

as an expectation by applying the Hadamard gate H

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

to the first qubit of

$$C_U \circ (H \otimes 1)|0\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle.$$

Here C_U denotes controlled U, acting as U when the control bit is $|1\rangle$ and the identity mapping when the control bit is $|0\rangle$. We measure the expectation for the first qubit $|0\rangle$ of the resulting state

$$\frac{1}{2}(H|0\rangle \otimes |\psi\rangle + H|1\rangle \otimes U|\psi\rangle) = \frac{1}{2}((|0\rangle + |1\rangle) \otimes |\psi\rangle + (|0\rangle - |1\rangle) \otimes U|\psi\rangle)$$

$$= \frac{1}{2}(|0\rangle \otimes (|\psi\rangle + U|\psi\rangle) + |1\rangle \otimes (|\psi\rangle - U|\psi\rangle)).$$

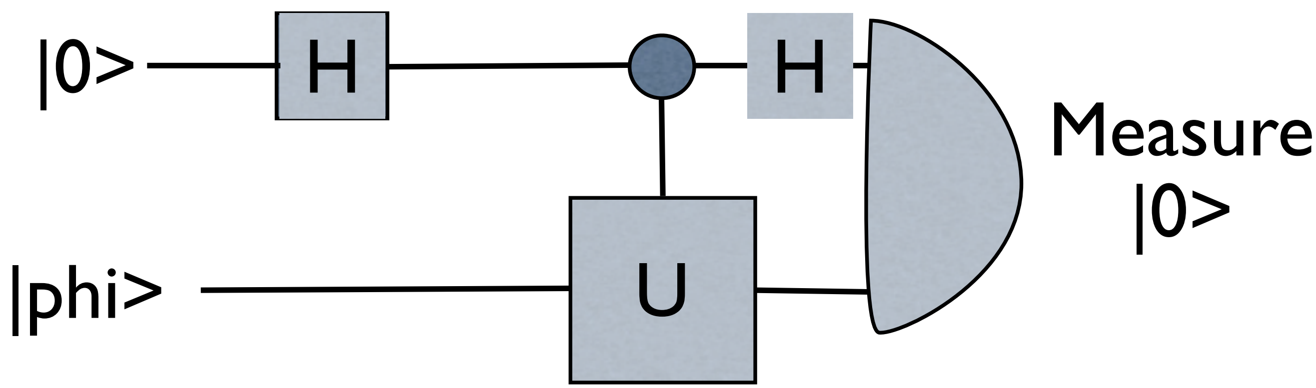
This expectation is

$$\frac{1}{2}(\langle\psi| + \langle\psi|U^\dagger)(|\psi\rangle + U|\psi\rangle) = \frac{1}{2} + \frac{1}{2}Re\langle\psi|U|\psi\rangle.$$

The imaginary part is obtained by applying the same procedure to

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle - i|1\rangle \otimes U|\psi\rangle$$

Hadamard Test



|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2

Quantum Hall Effect

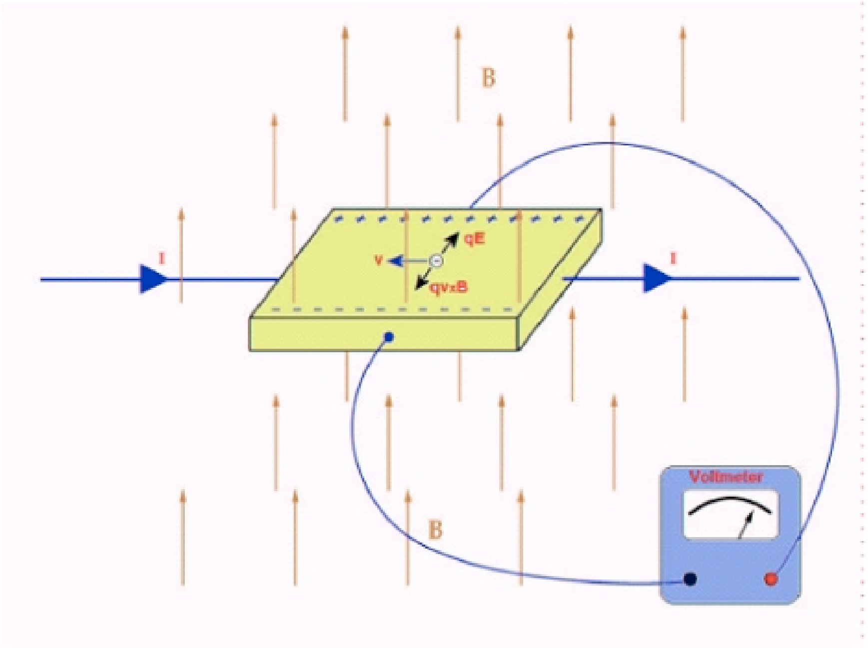
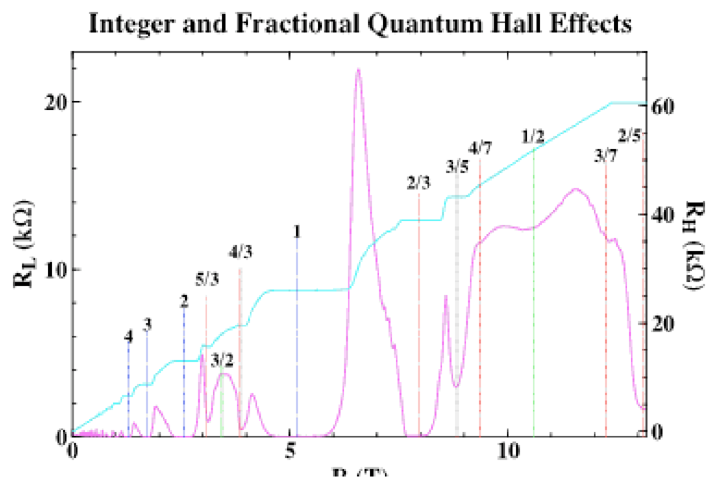


Figure 1: A schematics of the experimental setup of the Hall effect. A current driven through the conductor, drawn as a prism, leads to the emergence of voltage in the perpendicular direction. This is the Hall voltage, which Maxwell erroneously predicted to be zero.

Fractional Quantum Hall Effect (Cambridge Univ Website)

The fractional quantum Hall effect (FQHE) is a fascinating manifestation of simple collective behaviour in a two-dimensional system of strongly interacting electrons. At particular magnetic fields, the electron gas condenses into a remakable state with liquid-like properties. This state is very delicate, requiring high quality material with a low carrier concentration, and extremely low temperatures. As in the integer Quantum Hall Effect, a series of plateaux forms in the Hall resistance. Each particular values of magnetic field corresponds to a filling factor (the ratio of electrons to magnetic flux quanta) nu=*p*/*q*, where *p* and *q* are integers with no common factors). *q* always turns out to be an odd number. The principal series of such fractions are 1/3, 2/5, 3/7 etc, and 2/3, 3/5, 4/7, etc.



There are two main theories of the FQHE:

- **Fractionally-charged quasiparticles**. This theory, proposed by Laughlin, hides the interactions by constructing a set of quasiparticles with charge $e^*=e/q$, where the fraction is *p*/*q* as above.
- **Composite Fermions**. This theory was proposed by Jain, and Halperin, Lee and Read. In order to hide the interactions, it attaches two (or, in general, an even number) flux quanta *h*/*e* to each electron, forming integer-charged quasiparticles called Composite Fermions. The fractional states are mapped to the Integer QHE. This makes electrons at a filling factor 1/3, for example, behave in the same way as at filing factor 1. A remarkable result is that filling factor 1/2 corresponds to zero magnetic field. Experiments support this.

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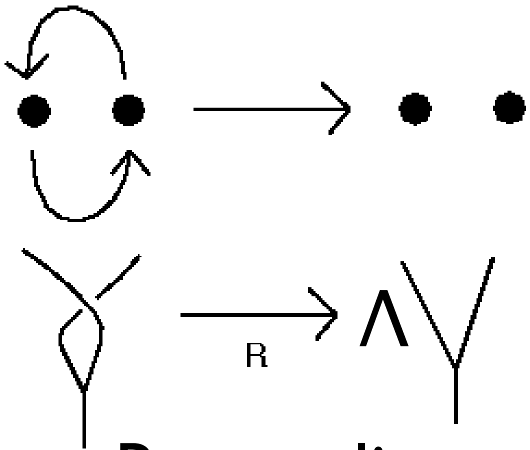
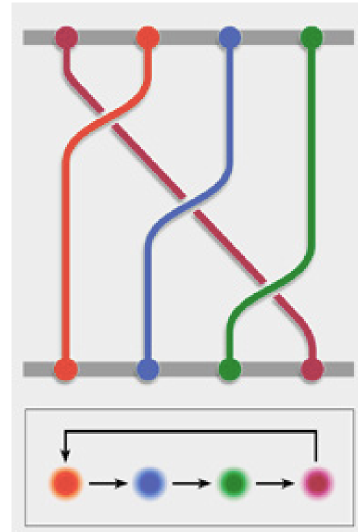
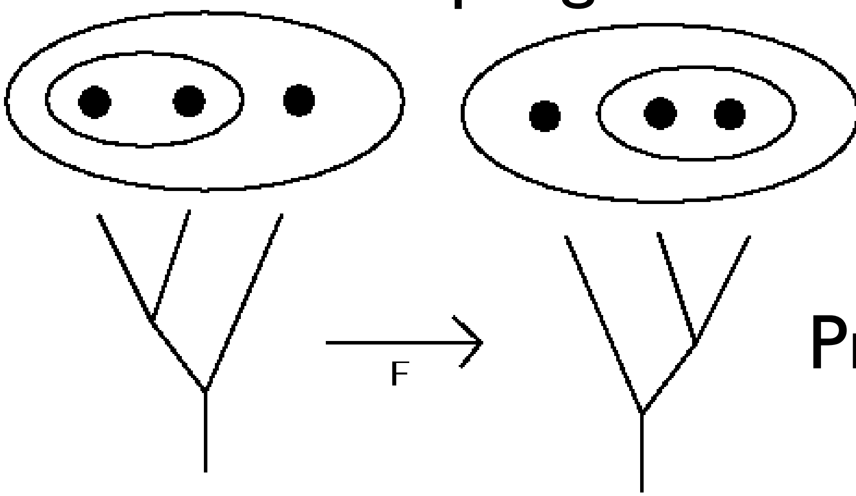
The quasi-particle theory is connected with Chern-Simons Theory and it explains the FQHE on the basis of "anyons": particles that have non-trivial (not +1 or -1) phase change when they exchange places in the plane.

Braiding Anyons



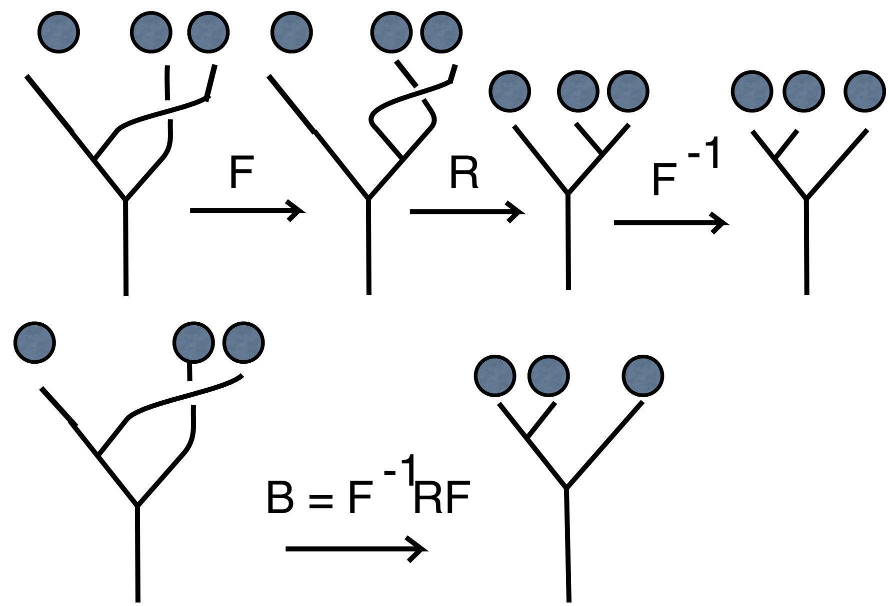


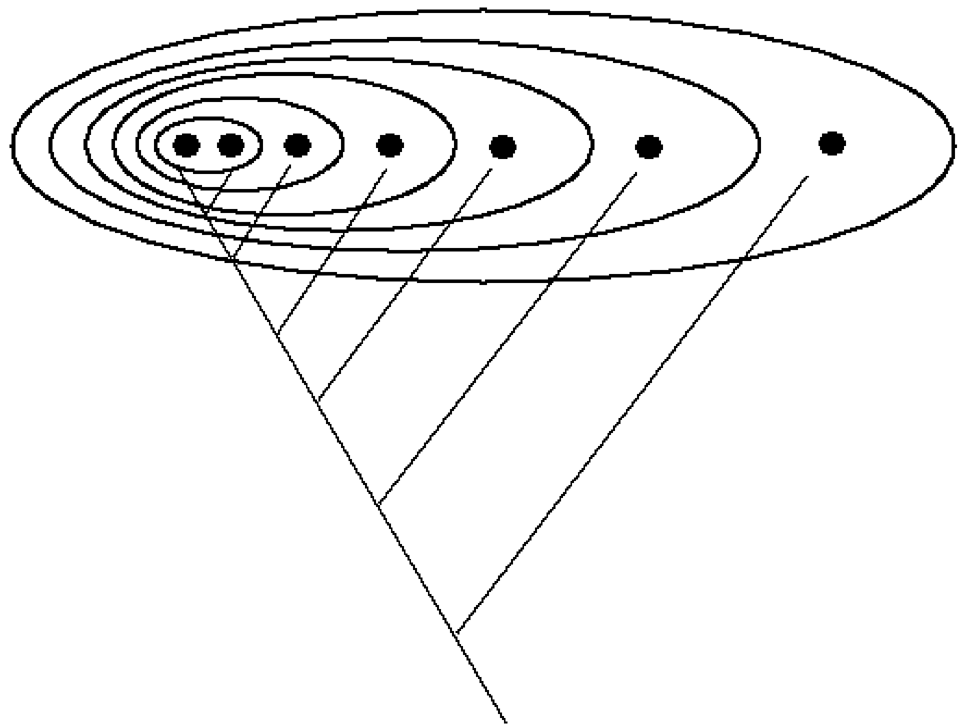
Recoupling



Process Spaces

Non-Local Braiding is Induced via Recoupling





Process Spaces Can be Abitrarily Large.
With a coherent recoupling theory, all transformations are in the representation of one braid group.

Mathematical Models for Recoupling Theory with Braiding come from a Combination of Penrose Spin Networks and Knot Theory.

See "Temperley Lieb Recoupling Theory and Invariants of Three-Manifolds" by L. Kauffman and S. Lins, PUP, 1994.

Penrose Spin Networks

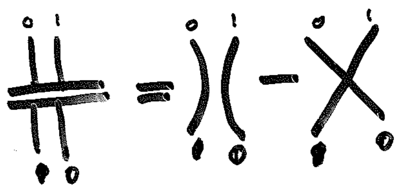
$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$SL(2,\mathbb{C}) = \left\{ A \;\middle|\; A \text{ 2x2 matrix } \& \; A\epsilon A^T = \epsilon \right\}$$

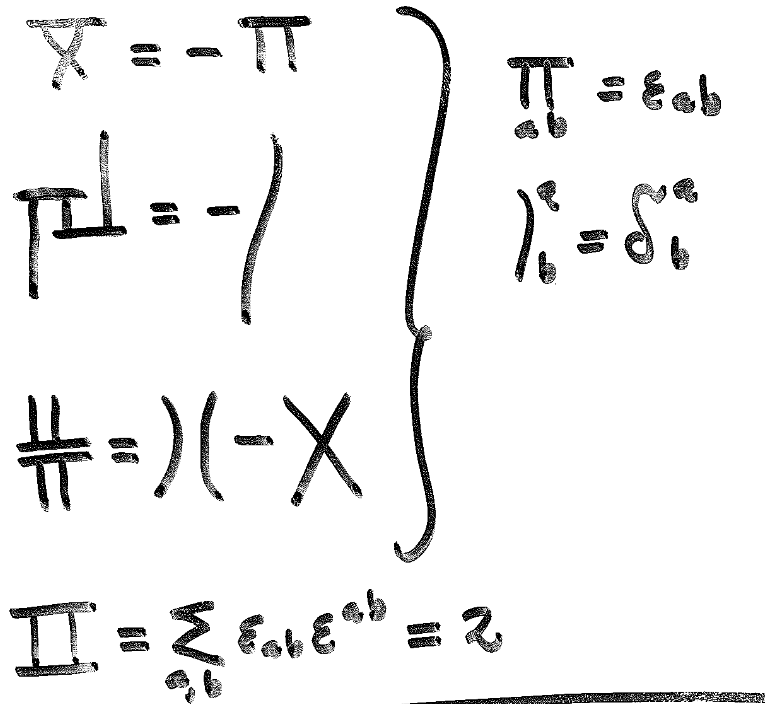
Let  $= \epsilon_{ab}$

 = vector v^a

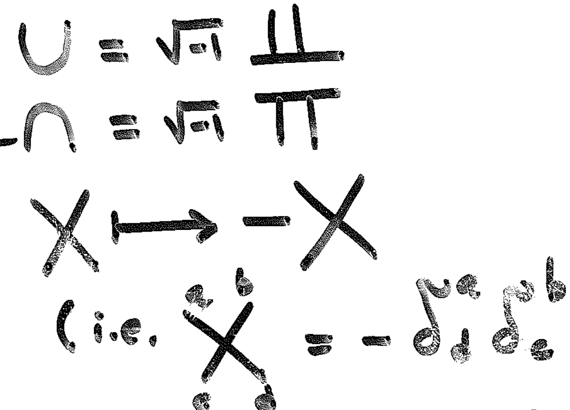
$\langle v, w \rangle$ =  $= \epsilon_{ab} v^a w^b$

 : $\epsilon^{ab}\epsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c$

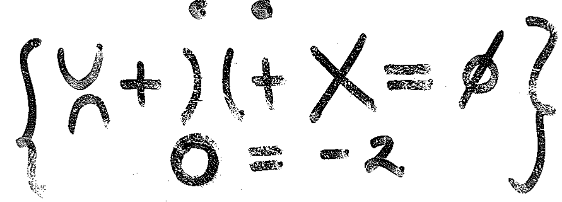
Wanted diagrammatic representation that is convenient and topologically invariant in the plane.



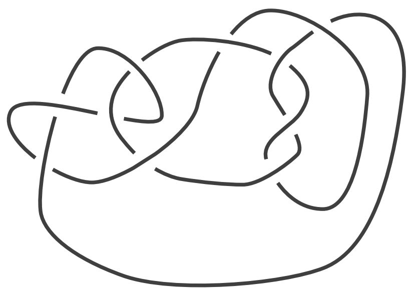
Penrose: **adjusts the tensors**

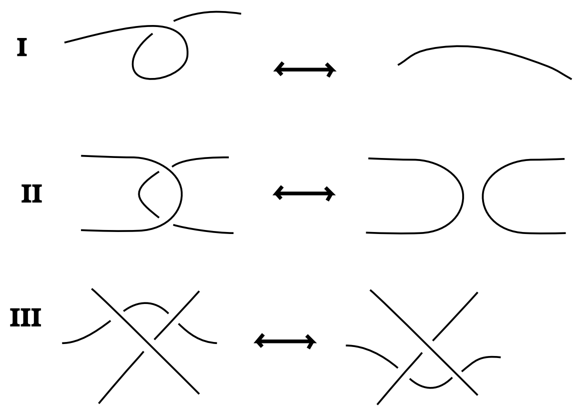


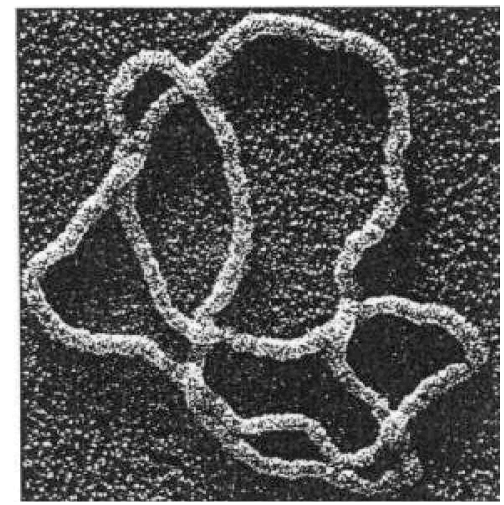
Then:

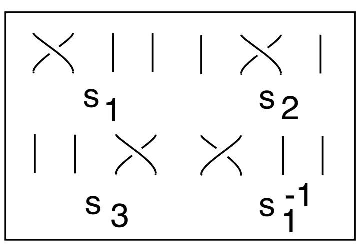


Knots and Links

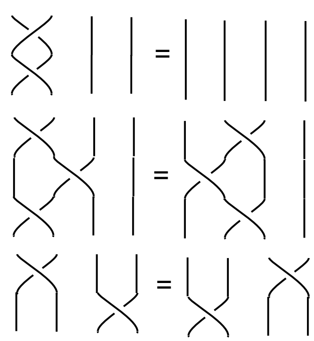








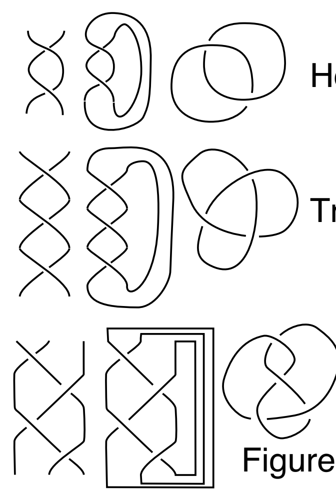
Braid Generators



$s_1^{-1} s_1 = 1$

$s_1 s_2 s_1 = s_2 s_1 s_2$

$s_1 s_3 = s_3 s_1$

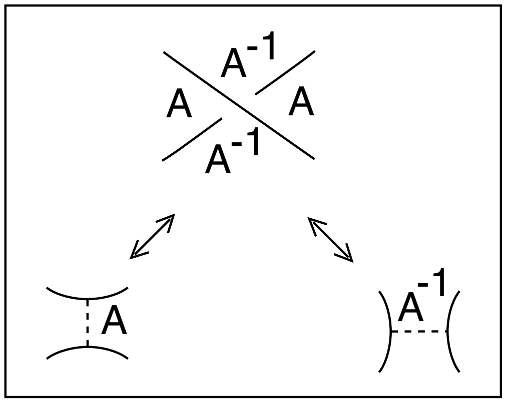


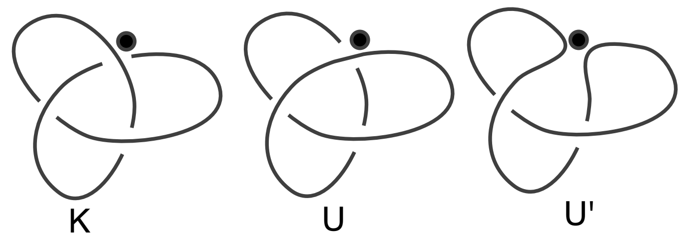
Hopf Link

Trefoil Knot

Figure Eight Knot

Bracket Polynomial Model for Jones Polynomial



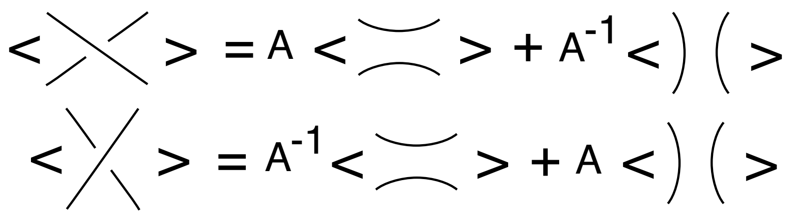


$$A^{-1} < K > -A < U > = (A^{-2} - A^{2}) < U' >$$

$$< U > = -A^{3}$$

$$< U' > = (-A^{-3})^{2} = A^{-6}$$

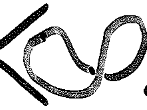
$$< K > = -A^{5} - A^{-3} + A^{-7}.$$

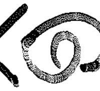


$$< K > = \sum_{S} < K|S > \delta^{||S||-1}.$$

(16.1)

$\langle \bigcirc \bigcirc \rangle = \delta = -A^{2} - A^{-2}$

$\langle$  $\rangle = A \langle$  $\rangle$

$+ A^{-1} \langle$  $\rangle$

$= A(-A^{3}) + A^{-1}(-A^{-3})$

$= -A^{4} - A^{-4}$

K^{*} = mirror image

$\Rightarrow$

$f_{K^{*}}(A) = f_{K}(A^{-1})$

$w(K) = 3$

K

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

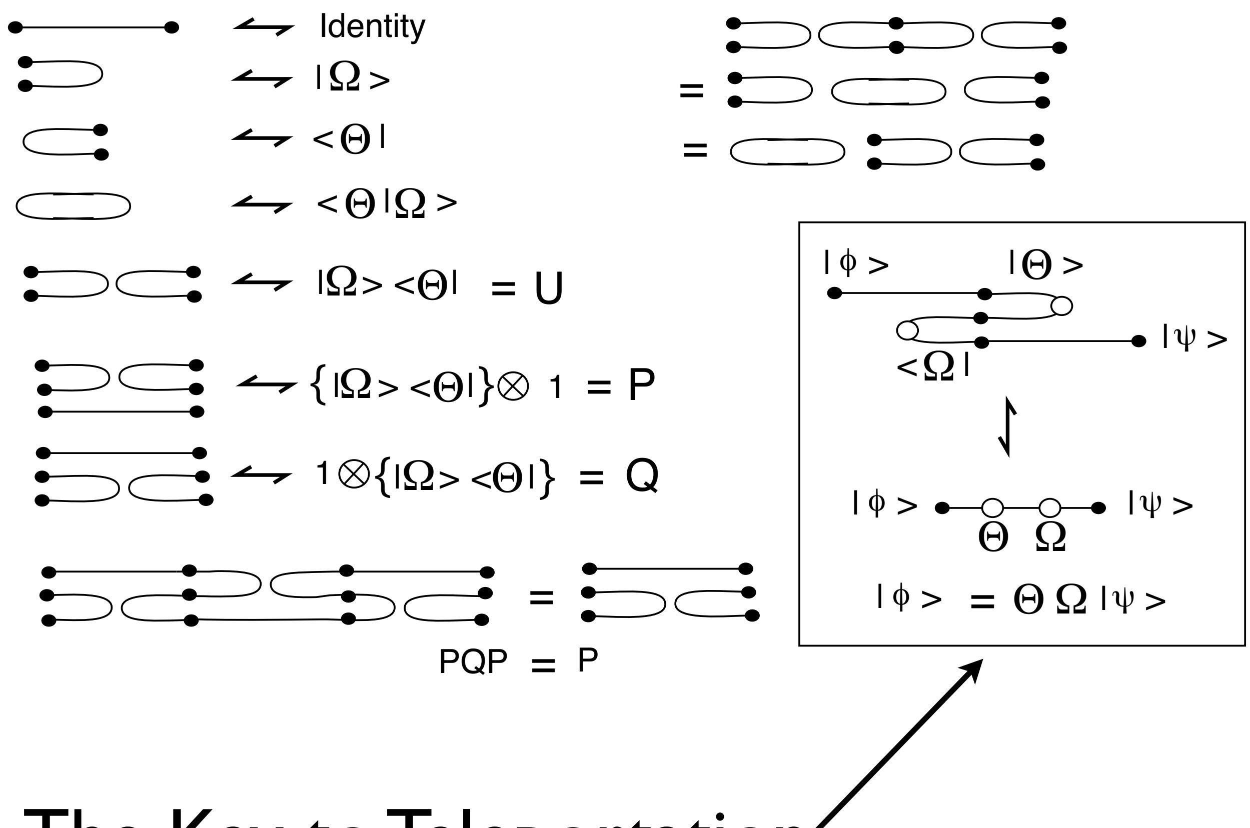
K^{*} 

$= A(-A^{4} - A^{-4}) + A^{-1}(-A^{-3})^{2}$

$= -A^{5} - A^{-3} + A^{-7}$

$f_{K} = (-A^{3})^{-3}(-A^{5} - A^{-3} + A^{-7}) = A^{-4} + A^{-12} - A^{-16}$

Temperley Lieb Category



The Key to Teleportation

Any two one-dimensional projectors generate a Temperley-Lieb algebra.

P = ><, Q =][

PP = ><>< = <> >< = <> P

QQ = [] Q

PQP = ><][>< = <][> >< = <][> P

QPQ = [><] Q

This trick can be used to manufacture unitary representations of the three-strand braid group.

(b)

$$U_1 = \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U_2 = \delta \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

$$a^2 + b^2 = 1$$

$$\delta = a^{-2}$$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \left|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|$$

$$e_2 = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} = \left|\begin{pmatrix} a \\ b \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} a \\ b \end{pmatrix}\right|$$

$$e_1^2 = e_1, \quad e_2^2 = e_2$$

$$e_1 e_2 e_1 = a^2 e_1$$

$$e_1 e_2 = \begin{pmatrix} a^2 & ab \\ 0 & 0 \end{pmatrix}$$

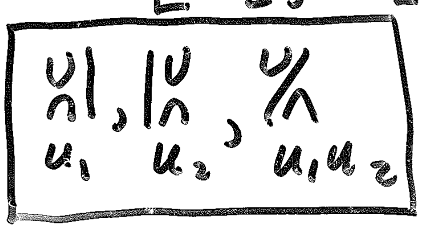
$$e_2 e_1 = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$$

$$Tr(U_1) = \delta$$

$$Tr(U_2) = \delta$$

$$Tr(U_1 U_2) = 1$$

$$U_1 = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} \delta^{-1} & \sqrt{1-\delta^{-2}} \\ \sqrt{1-\delta^{-2}} & \delta - \delta^{-1} \end{bmatrix}$$



$$\rho(\times\,|) = A U_1 + A^{-1} I$$

$$\rho(|\,\times) = A U_2 + A^{-1} I$$

$$\delta = -A^2 - A^{-2}, \quad A = e^{i\theta}$$

$$= -2\cos(2\theta)$$

Need $\delta^2 \geq 1$: $\cos^2(2\theta) \geq \frac{1}{4}$

ρ gives unitary rep of

3-strand braids $\longrightarrow U(2)$

$B_3 \longrightarrow$

Diagrammatic Matrices, Knots and Teleportation

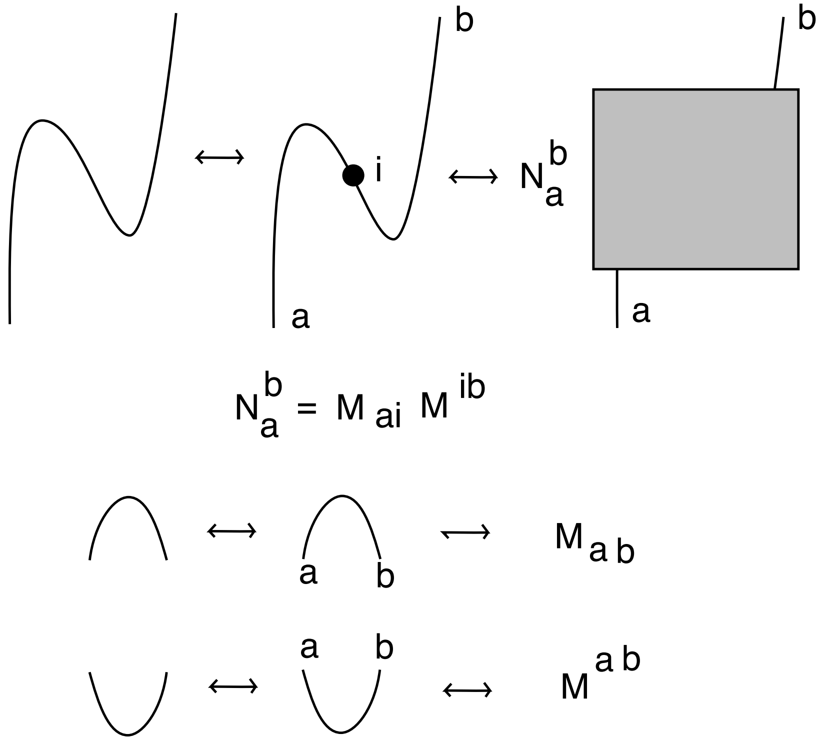
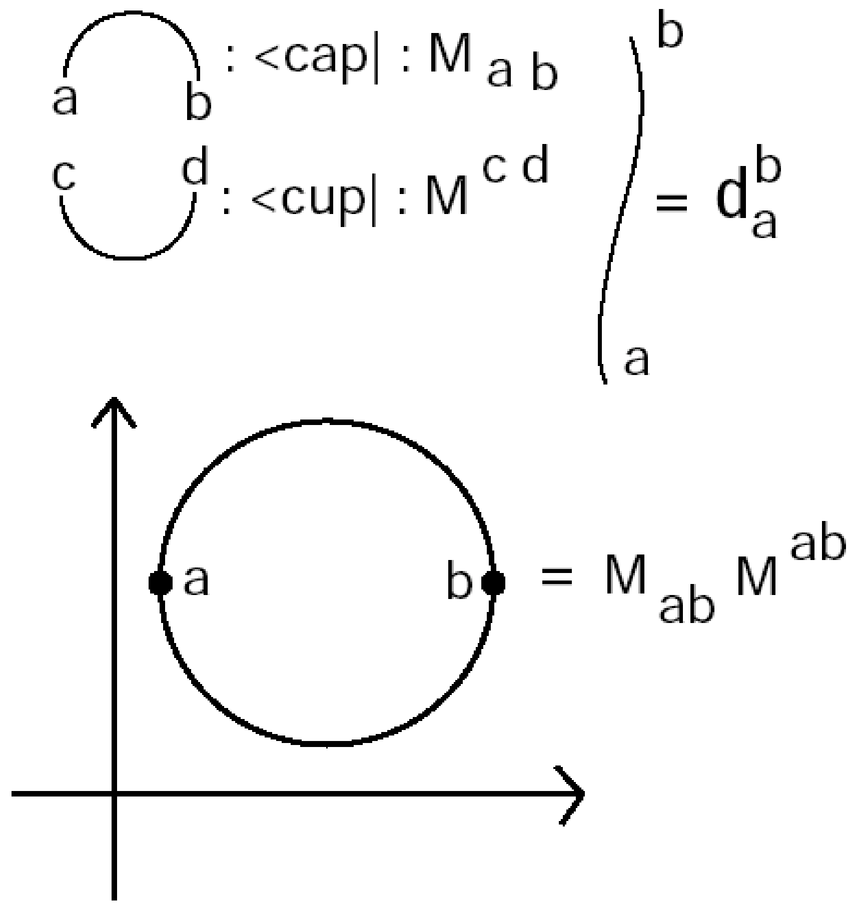
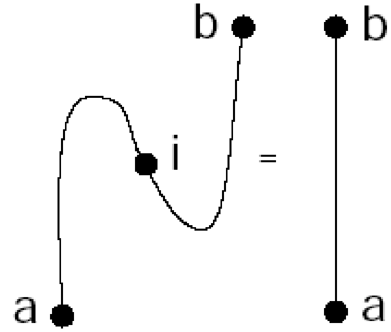


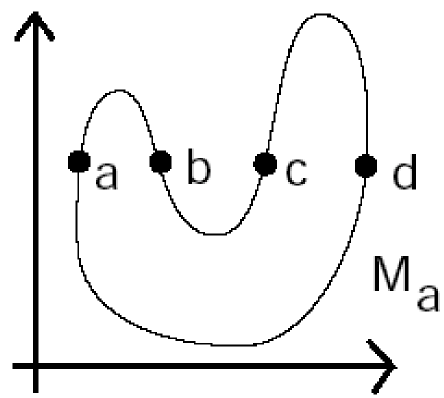
Figure 5 - Matrix Composition



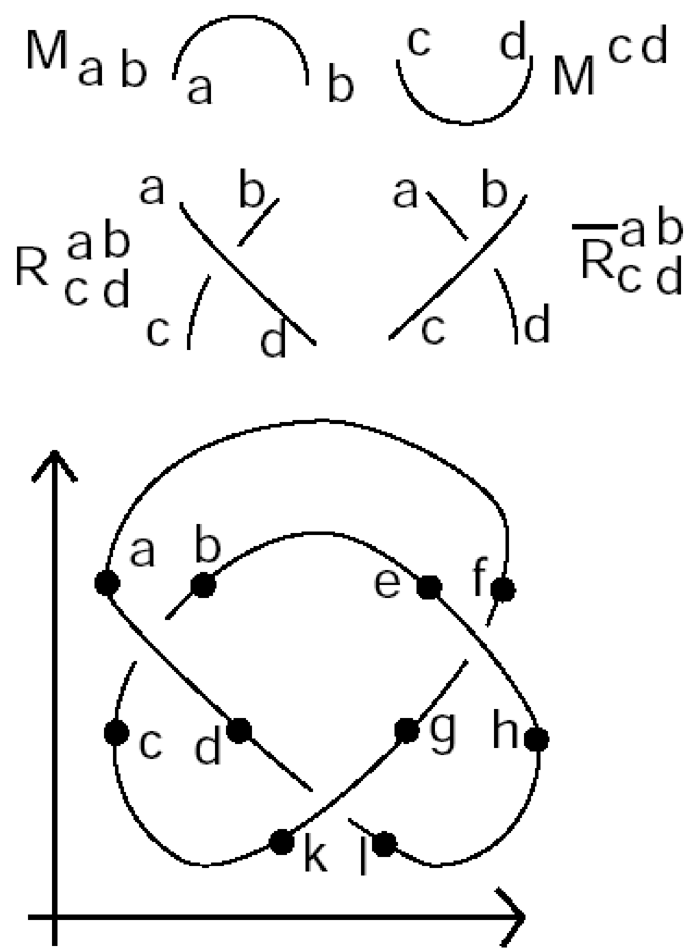
a
b
: <cap| : M a b
c
d
: <cup| : M c d
b
= d_a^b
a
a
b
= $M_{ab} M^{ab}$



$$M_{ai}\, M^{ib} = d_a^b$$

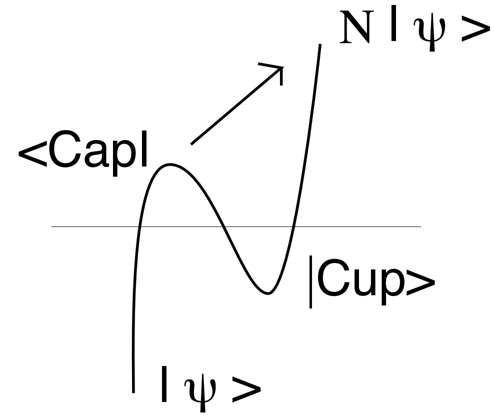


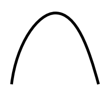
$$M_{ab} M^{bc} M_{cd} M^{ad}$$

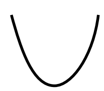


$$Z_K = M_{af} M_{be} M^{ck} M^{lh} R^{ab}_{cd} R^{ef}_{gh} \overline{R}^{dg}_{kl}$$

State and Matrix Duality



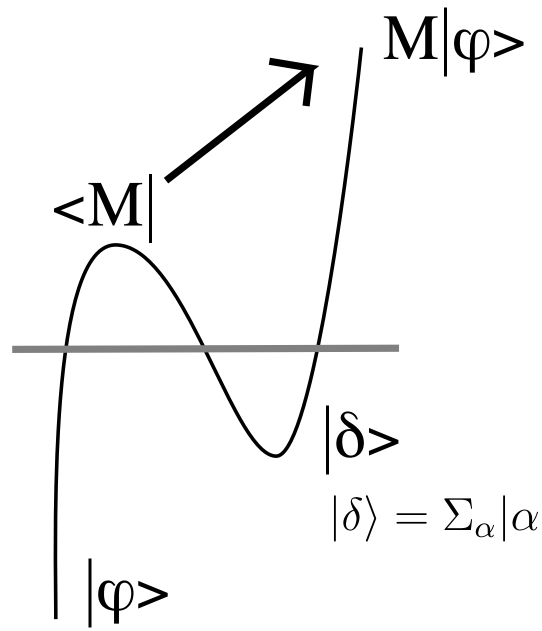
 $\longleftrightarrow \quad \langle Cap| = \sum_{a,i} M_{a,i} \langle a| \langle i|$

 $\longleftrightarrow \quad |Cup\rangle = \sum_{i,b} M^{i,b} |i\rangle |b\rangle$

$$N_a^b = \sum_i M_{ai} \, M^{ib}$$

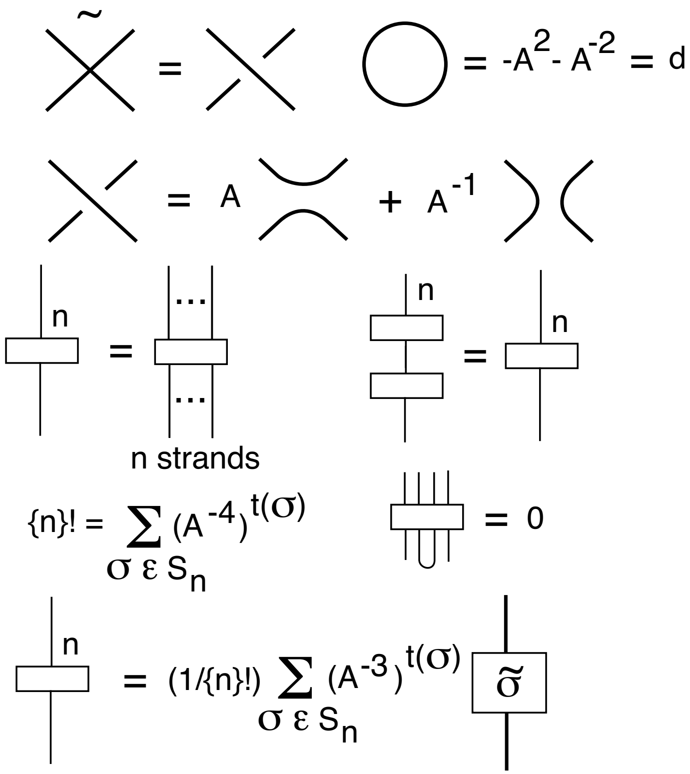
The Topology of Teleportation

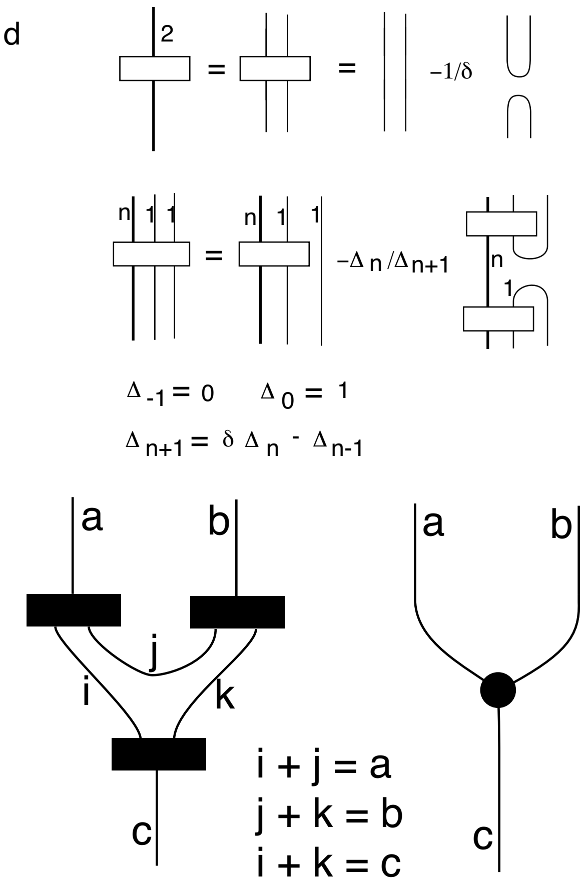
|00> + |11>   <----> $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

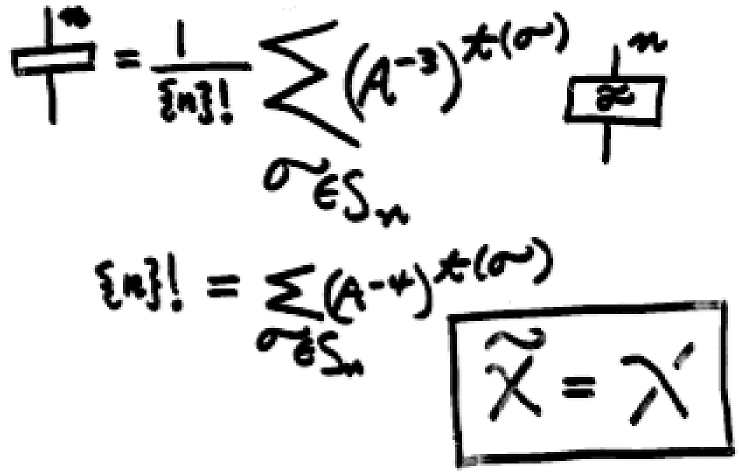


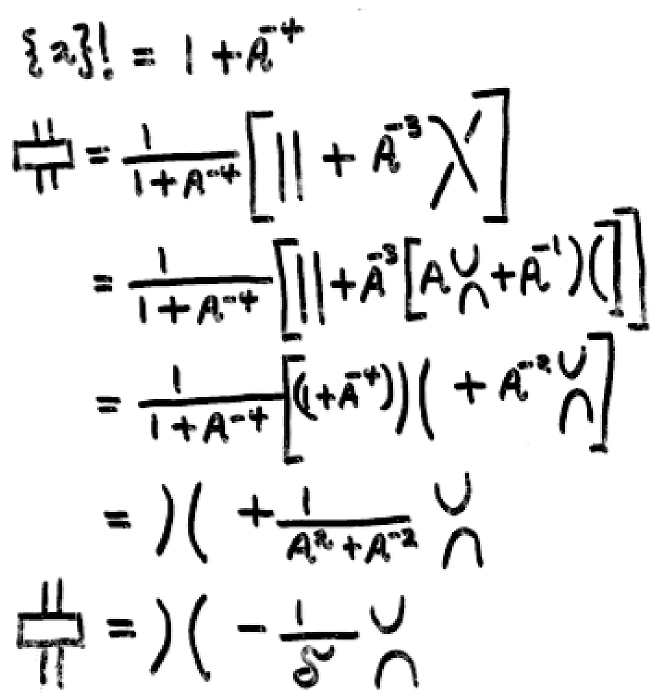
$$|\delta\rangle = \Sigma_{\alpha}|\alpha, \alpha\rangle \in H \otimes H.$$

q-Deformed Spin Networks



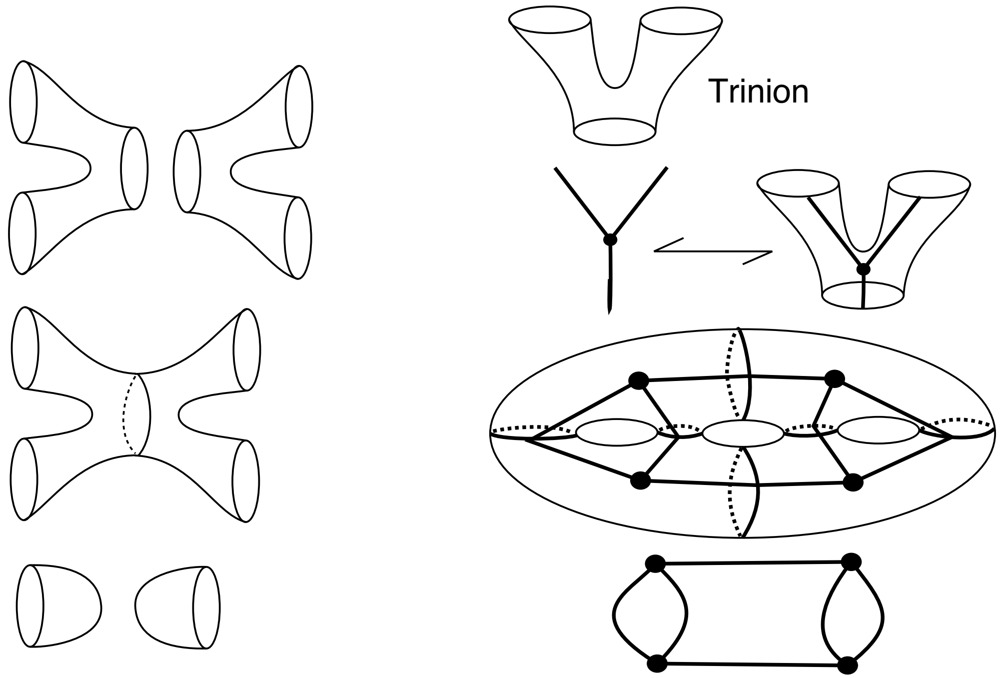






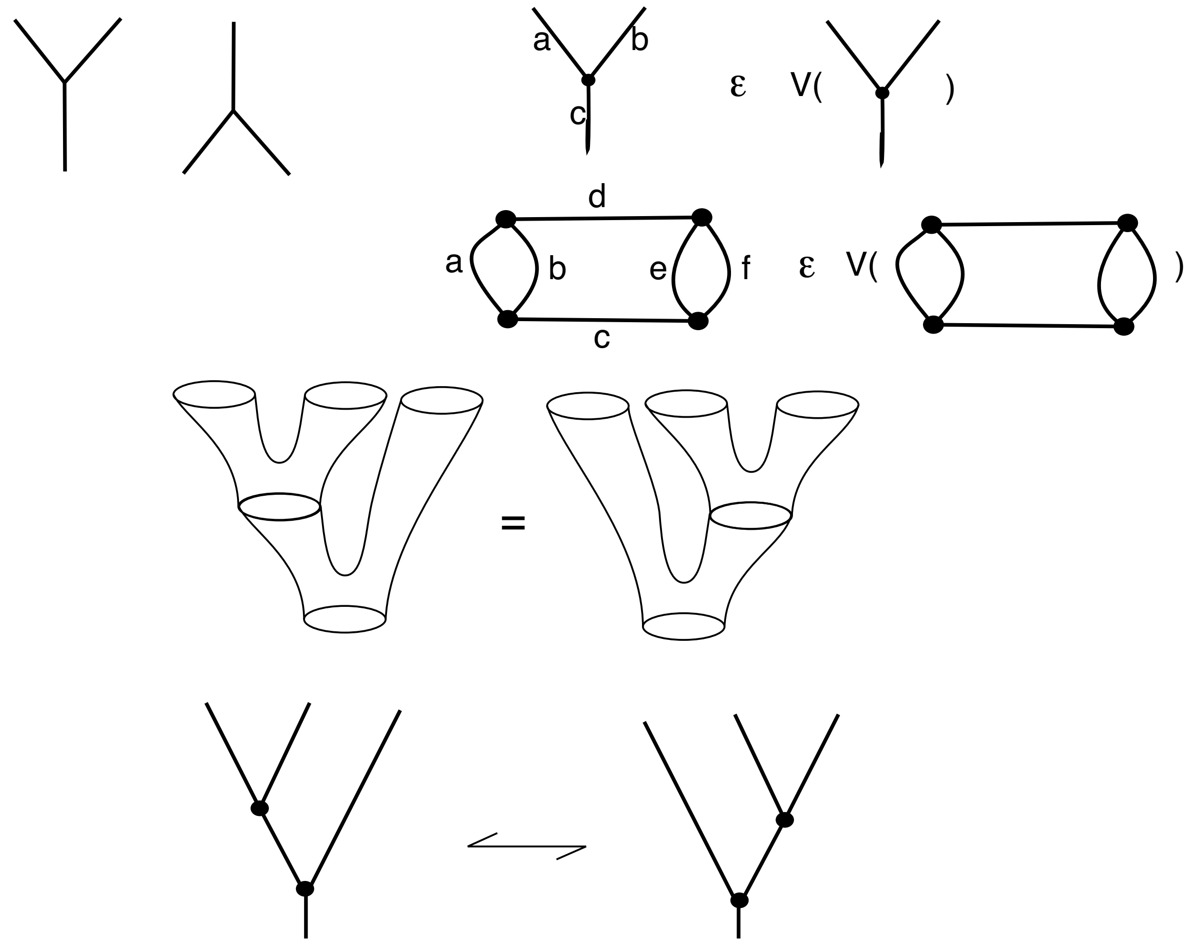
Projectors are Sums over Permuations, Lifted to Braids and Expanded via the Bracket into the Temperley Lieb Algebra

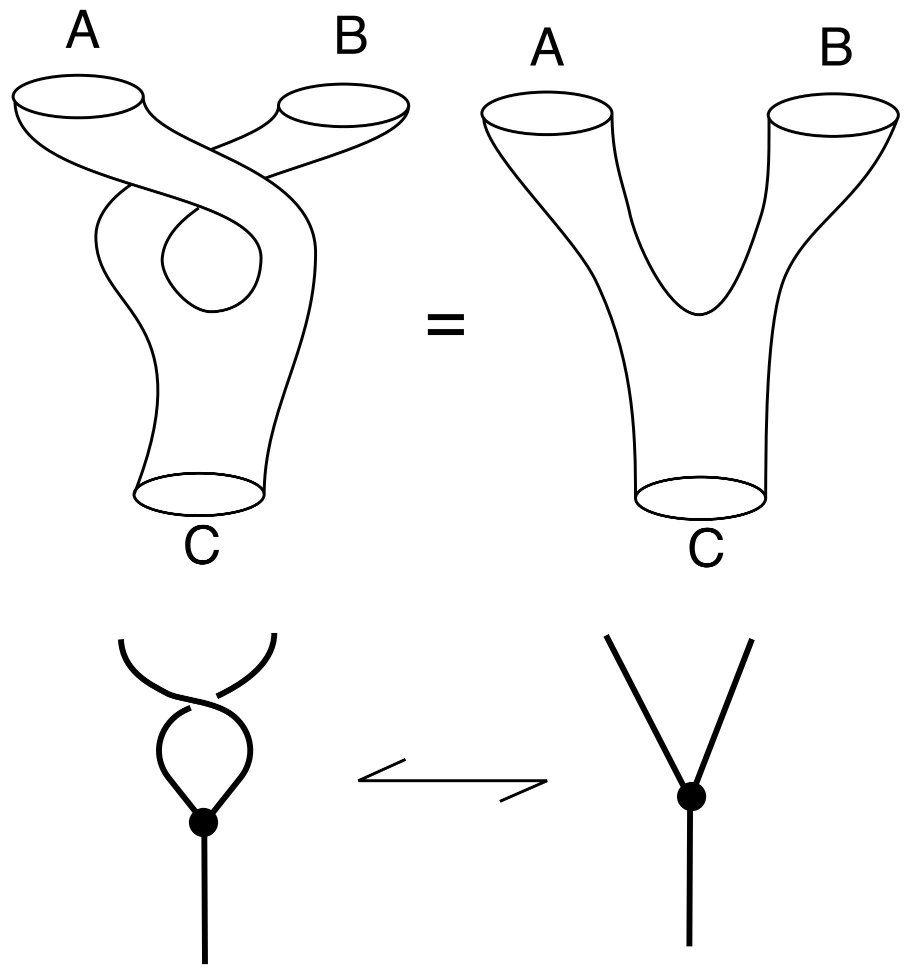
Topological Quantum Field Theory



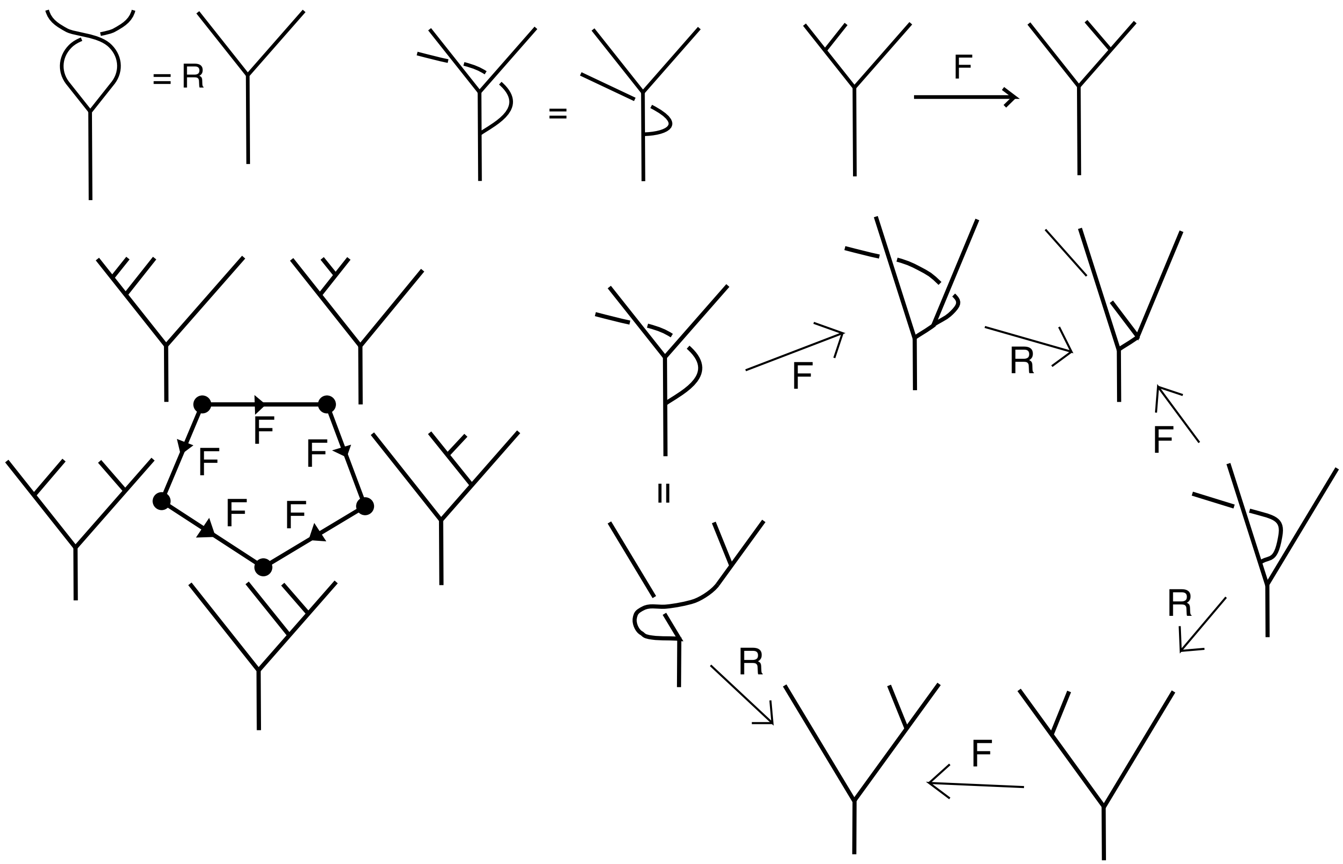
Process Spaces on Surfaces Lead to Three-Manifold Invariants.

Process Vector Spaces and Recoupling

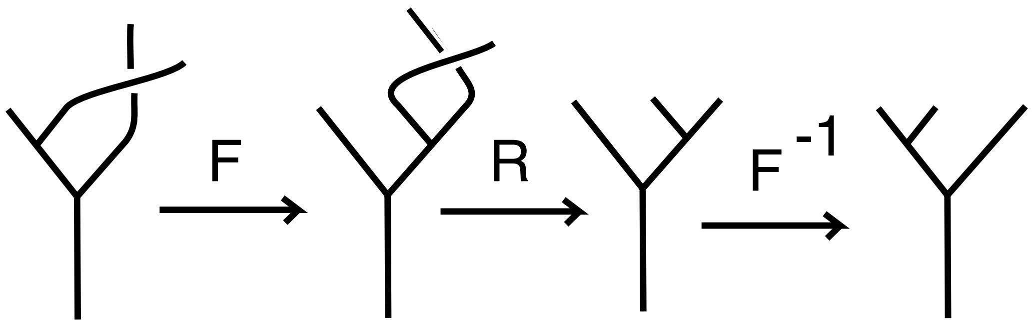


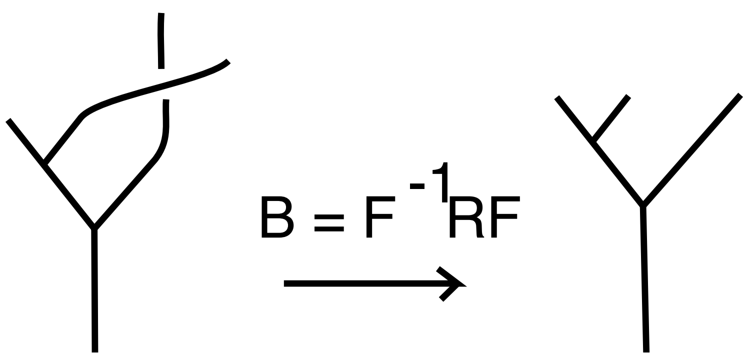

A
B
C
A
B
C

Braiding, Naturality, Recoupling, Pentagon and Hexagon

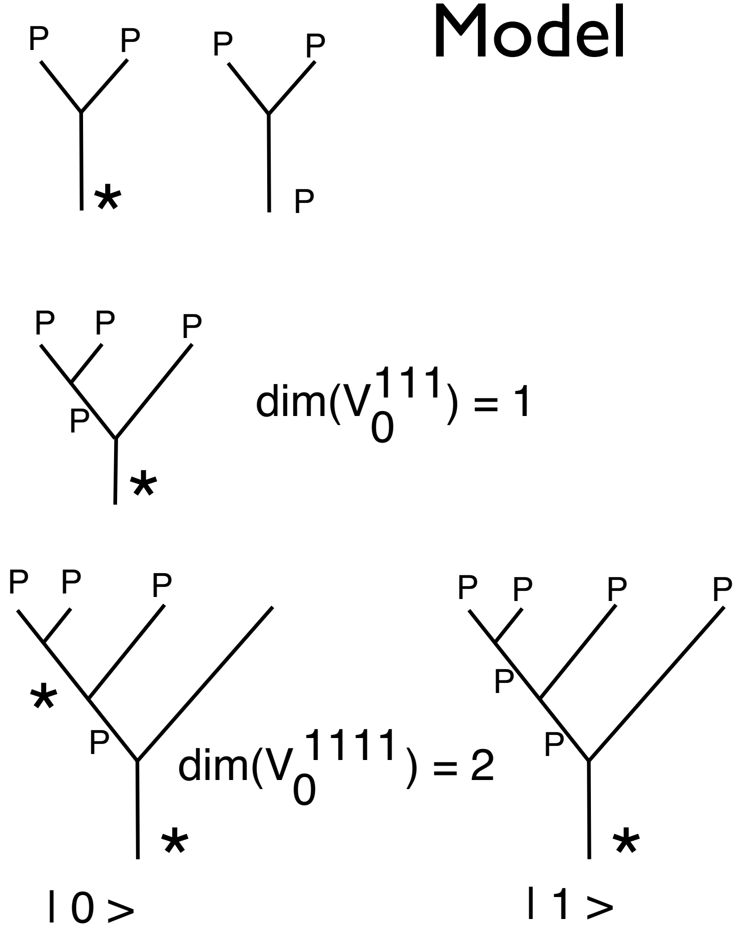


Non-Local Braiding is Induced via Recoupling

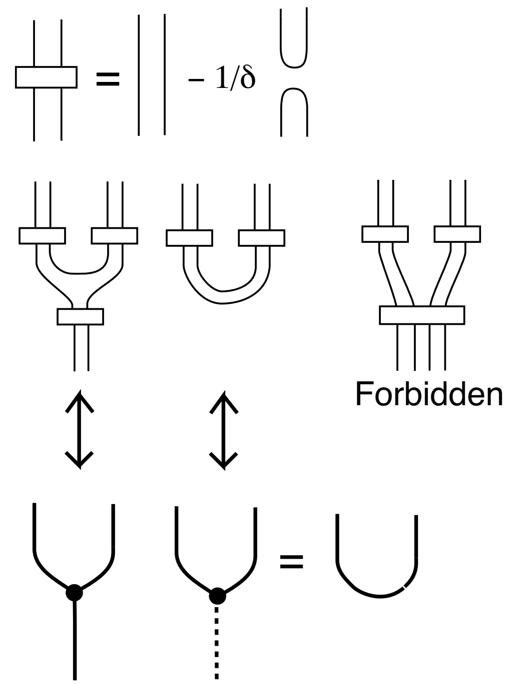




Fibonacci Model



$$A = e^{3\pi i/5}.$$

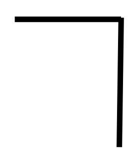


Temperley Lieb Representation of Fibonacci Model

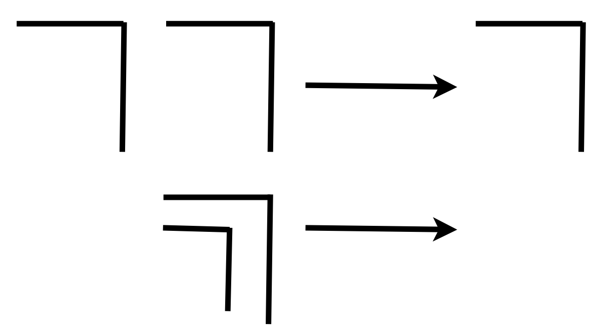
Iconics

In the Fibonacci Model we have
one "particle" P that interacts
itself to produce either P or * (nothing).

This is analogous to the logical particle

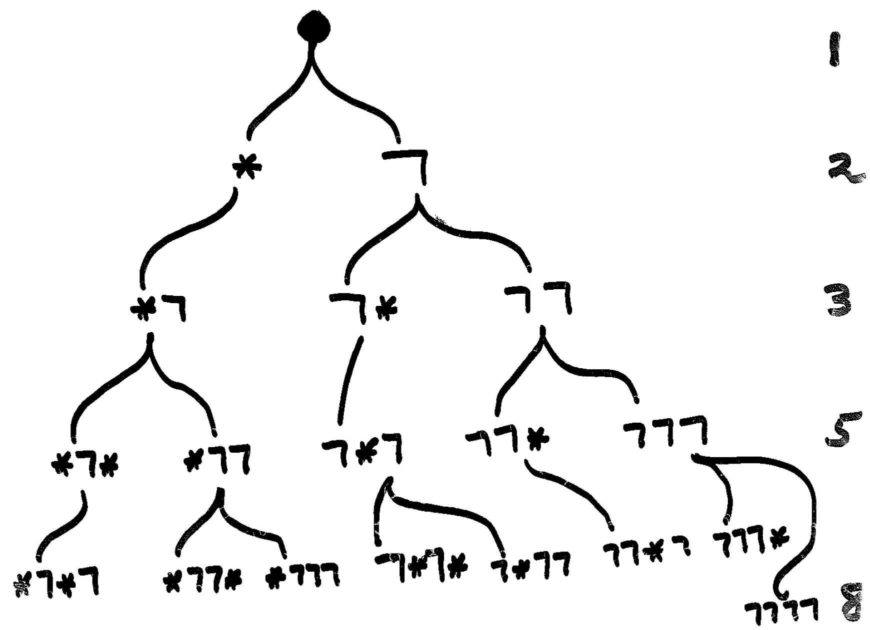


of G. Spencer-Brown
that interacts with itself in two ways:

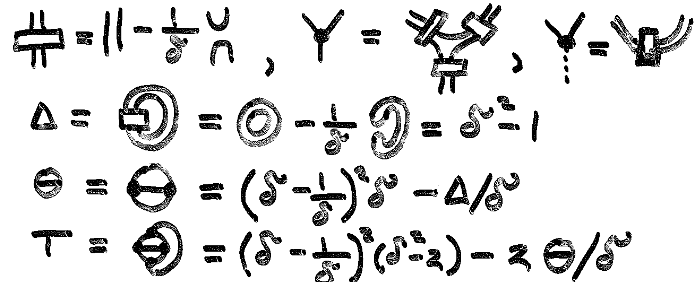


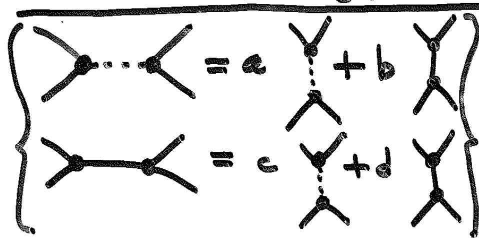
Seq in ¬, *

** forbidden



A sketch of the derivation





$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow F = \begin{pmatrix} 1/\Delta & \Delta/\Theta \\ \Theta/\Delta^2 & \Delta T/\Theta^2 \end{pmatrix}$$

$\Delta^2 = \Delta + 1$
So $\Delta^2 = \delta^2$.
Take $\Delta = \delta$.

$$F^2 = I \Rightarrow \frac{1}{\Delta} + \frac{1}{\Delta^2} = 1$$

With $\delta^2 = \delta + 1$ (So $\delta = \frac{1+\sqrt{5}}{2}$)

$\Delta = \delta^2 - 1 = \delta$ and above ok.

Then $F = \begin{pmatrix} 1/\Delta & \Theta/\Delta^2 \\ \Delta/\Theta & -1/\Delta \end{pmatrix}$.

Replace each vertex v by $\alpha \cdot v$ where $\alpha^4 = \Delta^3/\Theta^2$. Then

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}, \; \tau = \frac{1}{\Delta}.$$

The Simple, yet Quantum Universal, Structure of the Fibonacci Model

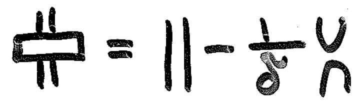
$$A = e^{3\pi i/5}.$$

$$\delta = -A^2 - A^{-2}$$

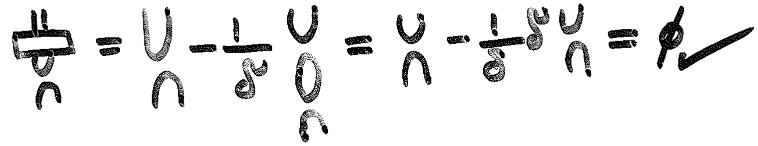
$$\Delta = \delta = (1 + \sqrt{5})/2.$$

$$F = \begin{pmatrix} 1/\Delta & 1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & -1/\Delta \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

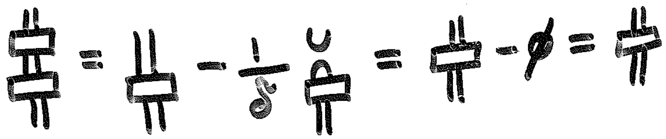
$$R = \begin{pmatrix} -A^4 & 0 \\ 0 & A^8 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}.$$



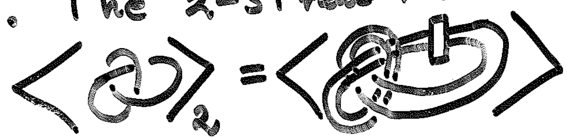
1.



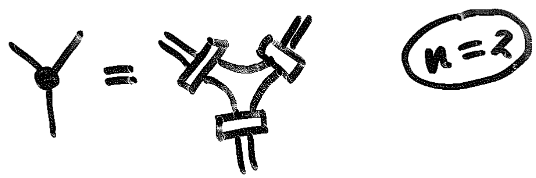
2.

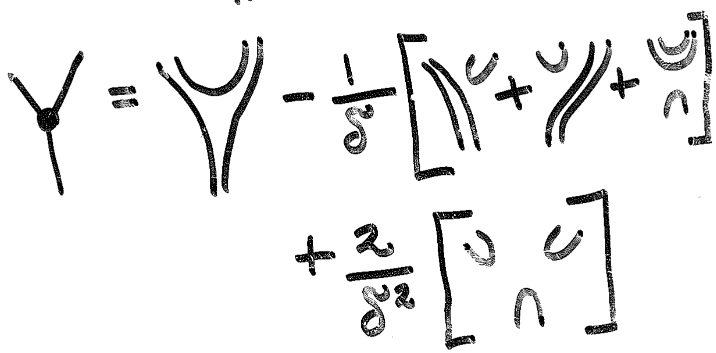


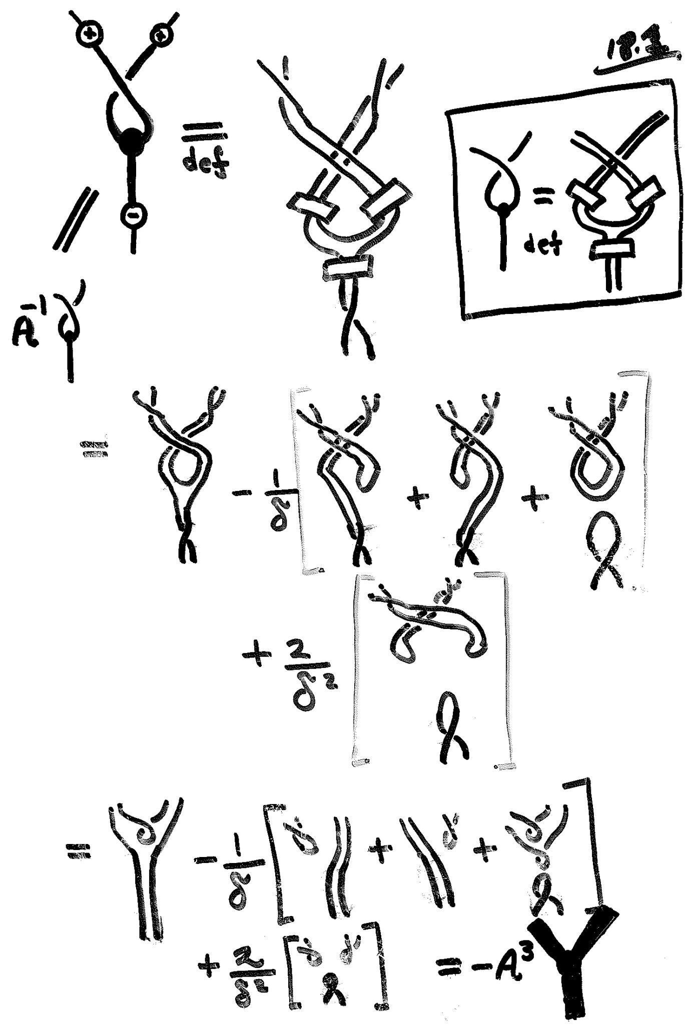
3. The 2-strand invariant

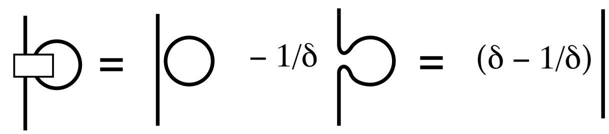


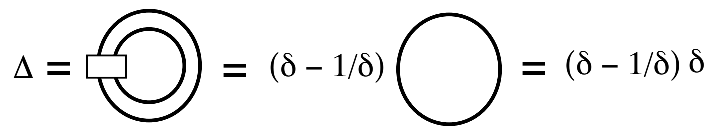
4.



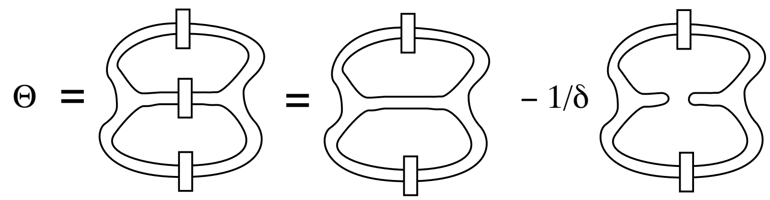



18.2
def
det
A⁻¹
− 1/δ
+ 2/δ²
= −A³

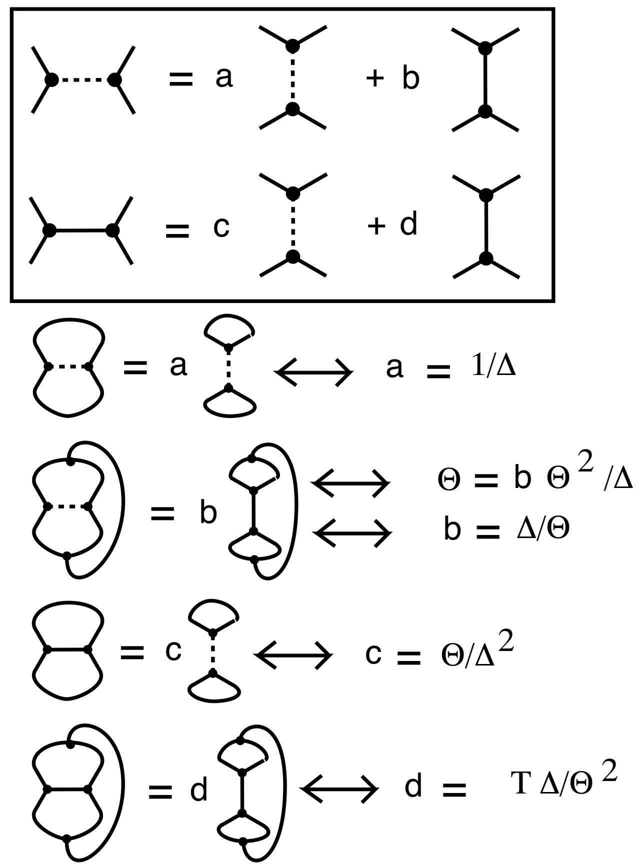




$$\Delta = \delta^2 - 1$$

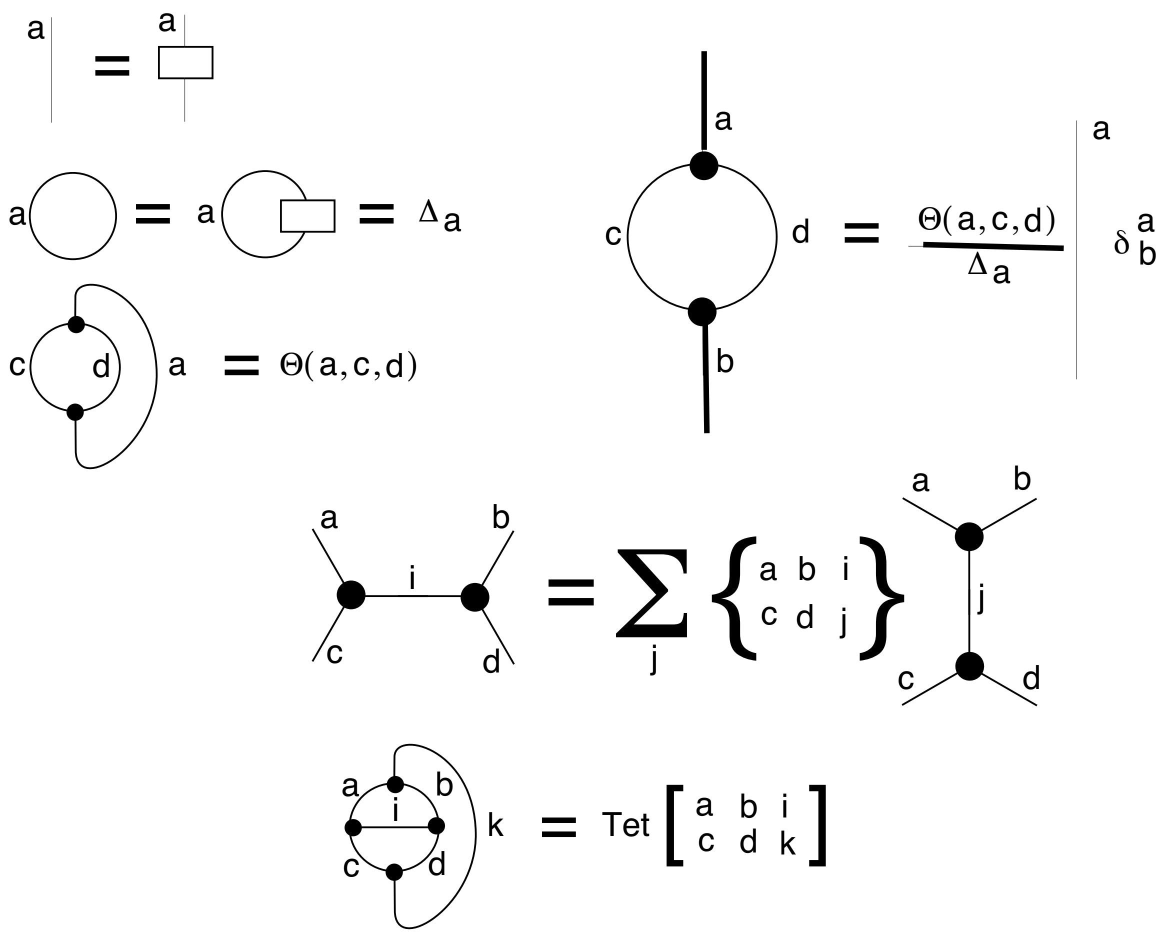


$$\Theta = (\delta - 1/\delta)^2\,\delta - \Delta/\delta$$

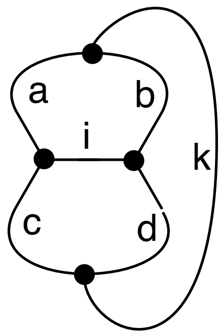


= a
+ b
= c
+ d
= a
a = 1/Δ
= b
Θ = b Θ 2 /Δ
b = Δ/Θ
= c
c = Θ/Δ 2
= d
d = T Δ/Θ 2

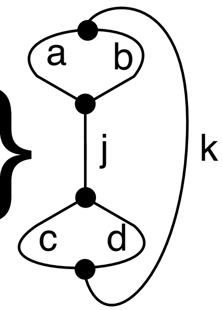
Closure, Bubble and Recoupling



The 6-j Coefficients



$$= \sum_j \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$$

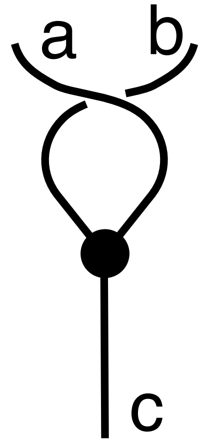


$$= \sum_j \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} \frac{\Theta(a,b,j)}{\Delta_j} \frac{\Theta(c,d,j)}{\Delta_j} \Delta_j \, \delta_j^k$$

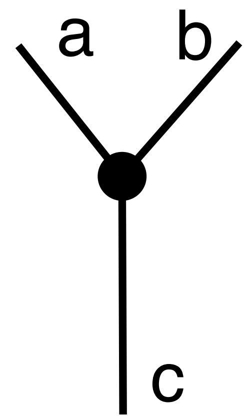
$$= \begin{Bmatrix} a & b & i \\ c & d & k \end{Bmatrix} \frac{\Theta(a,b,k)\,\Theta(c,d,k)}{\Delta_k}$$

$$\begin{Bmatrix} a & b & i \\ c & d & k \end{Bmatrix} = \frac{\mathrm{Tet}\begin{bmatrix} a & b & i \\ c & d & k \end{bmatrix} \Delta_k}{\Theta(a,b,k)\;\Theta(c,d,k)}$$

Local Braiding



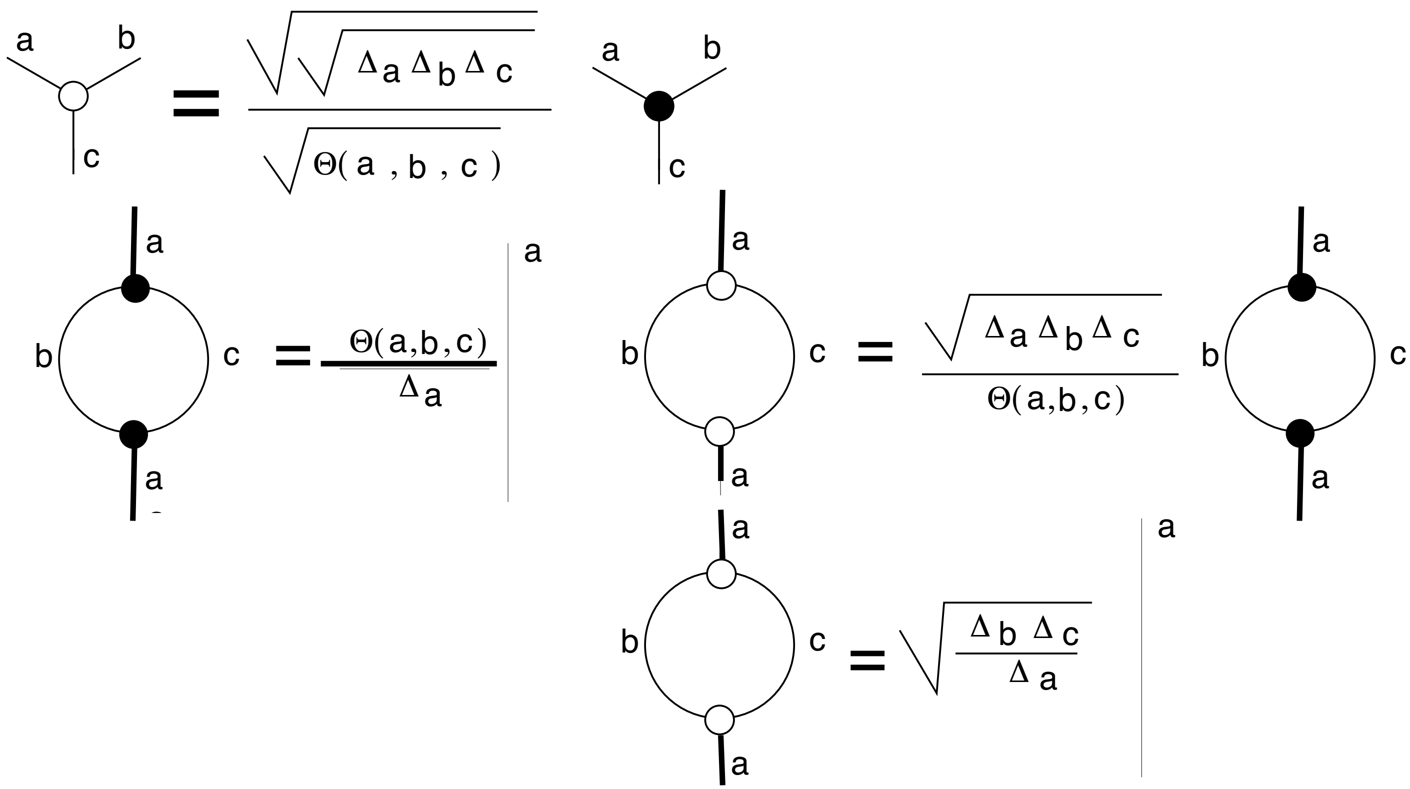
$= \lambda_c^{ab}$



$$\lambda_c^{ab} = (-1)^{(a+b-c)/2} A^{(a'+b'-c')/2}$$

$$x' = x(x+2)$$

Redefining the Vertex is the key to obtaining Unitary Recoupling Transformations.



$$\Delta_n = (-1)^n \frac{sin((n+1)\pi/r)}{sin(\pi/r)}.$$

Here the corresponding quantum integer is

$$[n] = \frac{sin(n\pi/r)}{sin(\pi/r)}.$$

Note that $[n+1]$ is a positive real number for $n = 0, 1, 2, ...r-2$ and that $[r-1] = 0$.

The evaluation of the theta net is expressed in terms of quantum integers by the formula

$$\Theta(a,b,c) = (-1)^{m+n+p} \frac{[m+n+p+1]![n]![m]![p]!}{[m+n]![n+p]![p+m]!}$$

where

$$a = m+p, b = m+n, c = n+p.$$

Note that

$$(a+b+c)/2 = m+n+p.$$

When $A = e^{i\pi/2r}$, the recoupling theory becomes finite with the restriction that only three-vertices (labeled with a, b, c) are *admissible* when $a+b+c \leq 2r-4$. All the summations in the formulas for recoupling are restricted to admissible triples of this form.

Lemma. For the bracket evaluation at the root of unity $A = e^{i\pi/2r}$ the factor

$$f(a,b,c) = \frac{\sqrt{\sqrt{\Delta_a \Delta_b \Delta_c}}}{\sqrt{\Theta(a,b,c)}}$$

is real, and can be taken to be a positive real number for (a,b,c) admissible (i.e. $a+b+c \leq 2r-4$).

Proof. By the results from the previous subsection,

$$\Theta(a,b,c) = (-1)^{(a+b+c)/2}\hat{\Theta}(a,b,c)$$

where $\hat{\Theta}(a,b,c)$ is positive real, and

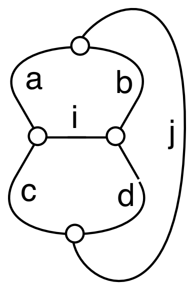
$$\Delta_a \Delta_b \Delta_c = (-1)^{(a+b+c)}[a+1][b+1][c+1]$$

where the quantum integers in this formula can be taken to be positive real. It follows from this that

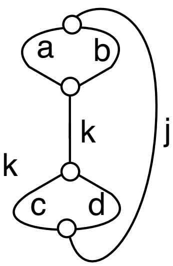
$$f(a,b,c) = \sqrt{\frac{\sqrt{[a+1][b+1][c+1]}}{\hat{\Theta}(a,b,c)}},$$

showing that this factor can be taken to be positive real. □

New Recoupling Formula



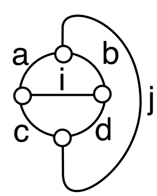
$$= \sum_k \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{i\,k}$$



$$= \sum_k \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{i\,k} \sqrt{\frac{\Delta_a \Delta_b}{\Delta_j}} \sqrt{\frac{\Delta_c \Delta_d}{\Delta_j}} \; \Delta_j \, \delta_j^k$$

$$= \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{i\,j} \sqrt{\frac{\Delta_a \Delta_b}{\Delta_j}} \sqrt{\frac{\Delta_c \Delta_d}{\Delta_j}} \; \Delta_j$$

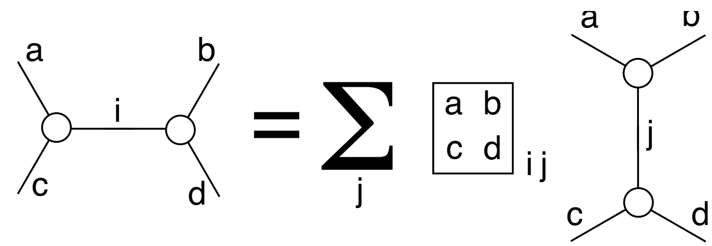
$$\boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{i\,j} =$$

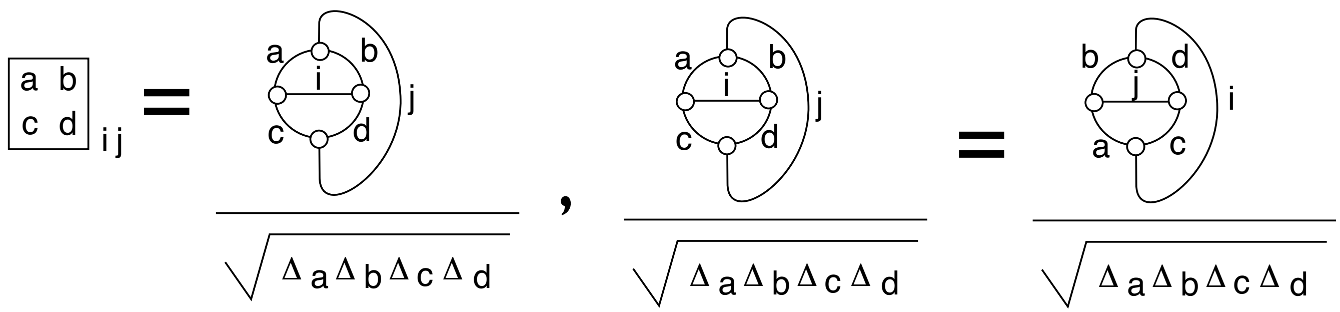


$$\overline{\sqrt{\frac{\Delta_a \Delta_b}{\Delta_j}} \sqrt{\frac{\Delta_c \Delta_d}{\Delta_j}} \; \Delta_j}$$

$$= \frac{\mathrm{ModTet}\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix}}{\sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}}$$

The Recoupling Matrix is Real Unitary at Roots of Unity.





$M[a,b,c,d]_{ij} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}_{ij}$

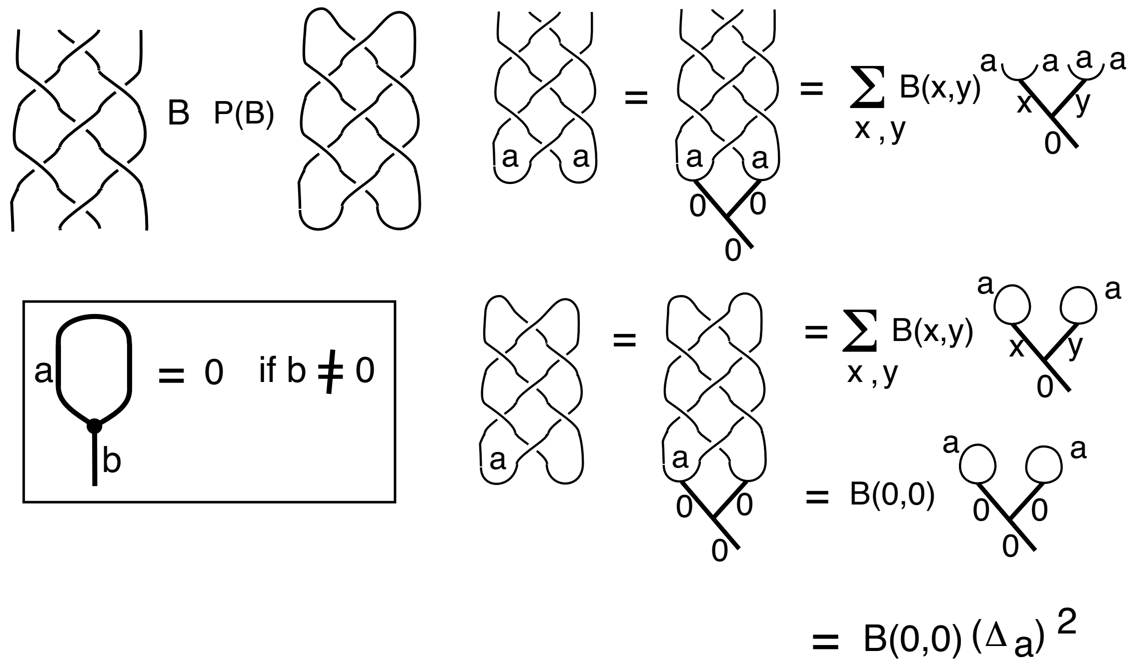
$\Rightarrow \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{T} = \boxed{\begin{matrix} a & b \\ c & d \end{matrix}}^{-1}$

Theorem. Unitary Representations of the Braid Group come from Temperley Lieb Recoupling Theory at roots of unity.

$$A = e^{i\pi/2r}$$

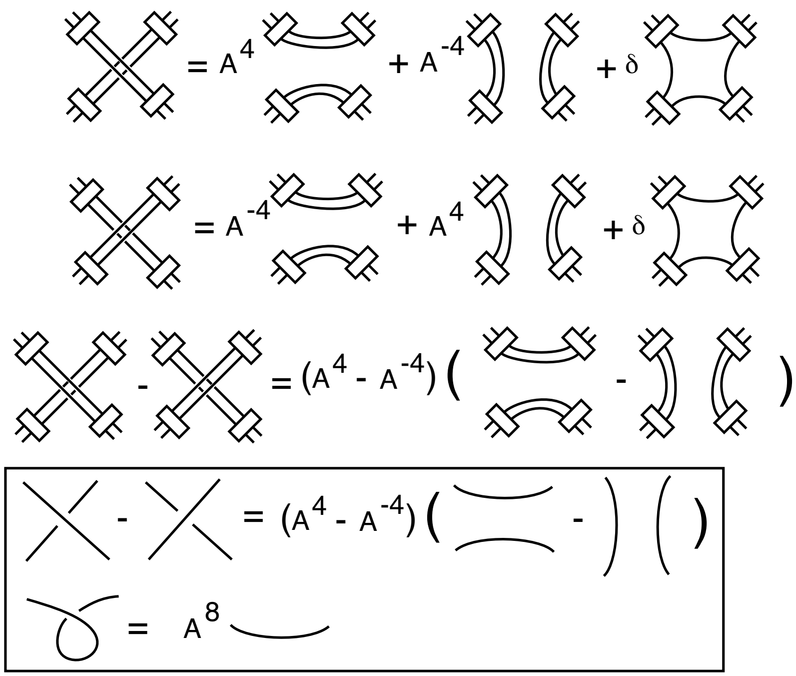
Sufficient to Produce Enough Unitary Transformations for Quantum Computing.

Quantum Computation of Colored Jones Polynomials and WRT invariants.



Need to compute a diagonal element of a unitary transformation.
Use the Hadamard Test.

Colored Jones Polynomial for n = 2 is Specialization of the Dubrovnik version of Kauffman polynomial.



Will these models actually be used
for quantum computation?
Will quantum computation actually happen?
Will topology play a key role?
Time will tell.