Discreteness of Asymptotic Tensor Ranks

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IPAM

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- These are of the form $\underset{\sim}{F}(T) = \lim_{n \to \infty} F(T^{\otimes n})^{1/n}$

- Play central role in algebraic complexity theory (fast matrix multiplication), quantum information (entanglement cost and distillation) and combinatorics (cap sets, sunflower-free sets).

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- Our result: We prove for several parameters and regimes that the set of possible values is discrete.

1. Asymptotic ranks, applications and context

2. Smallest values

3. Discreteness theorem

4. Proof ingredients

5. General result

1. Asymptotic ranks and applications

Warm-up: Matrix rank

$$M = $$

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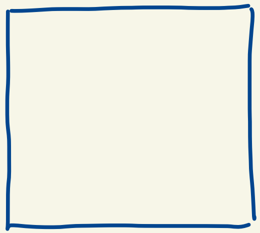
$M =$ 

(1) decomposition into rank-1 matrices

minimize

$$M = \sum_{i=1}^{r} u_i \otimes v_i$$

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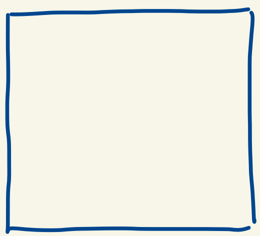
$M =$ 

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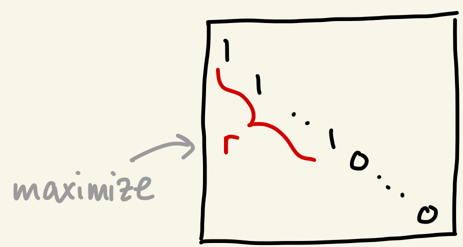
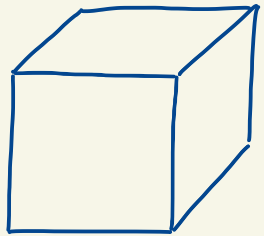


Tensor ranks

$T =$ 

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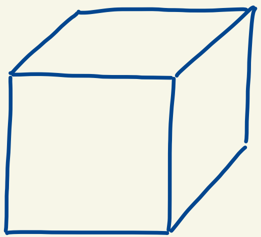
$T =$ 

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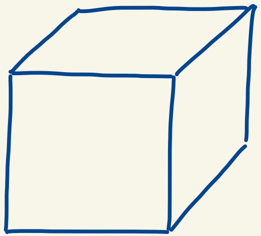
$T =$ 

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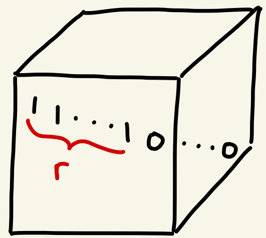

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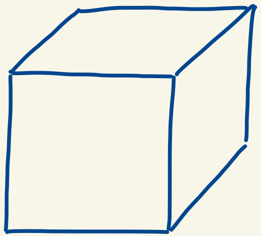
T = 

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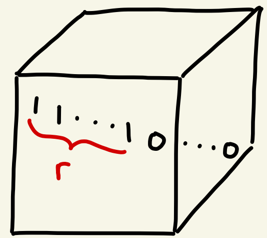
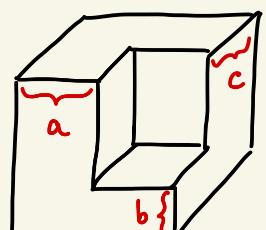


(3) slicerank



$$\underbrace{a+b+c}_{\text{minimize}}$$

Asymptotic ranks

"Rank" $\rightsquigarrow$ "Asymptotic rank"

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Tensor rank	R	Asymptotic tensor rank	$\underset{\sim}{R}$
Subrank	Q	Asymptotic subrank	$\underset{\sim}{Q}$
Slice rank	SR	Asymptotic slice rank	$\underset{\sim}{SR}$

Applications and context

<u>Asymptotic tensor rank</u> $\underset{\sim}{R}$

measures the "rate" at which a tripartite pure quantum state T can be obtained from $GHZ = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$ via SLOCC

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Asymptotic subrank $\underset{\sim}{Q}$

$$GHZ^{\otimes rk - o(k)} \leq T^{\otimes k} \qquad \underset{\sim}{Q}(T) = 2^{r} \qquad \text{"value"}$$

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Tensor characterization:

$$MaMu_n = \langle n,n,n\rangle = \sum_{i,j,k=1}^{n} e_{i,j}\otimes e_{j,k}\otimes e_{k,i}$$

$$\underset{\sim}{R}(MaMu_2) = 2^{\omega}$$ [Strassen]

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Known: Closed under applying any (univariate) polynomial with non-negative integer coefficients [Wigderson–Zuiddam 23]

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Important tools:

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Gives information on the power of the slice rank method

Known: Closed under polynomials, as before [Wigderson-Zuiddam 23]

2. Smallest values of $\underset{\sim}{Q}$ and $\underset{\sim}{SR}$

[Strassen 88]

[Costa-Dalai 21]

[Christandl-Gesmundo-Z 23]

0 $\quad$ 1 $\quad$ $2^{h(1/3)}$ $\quad$ 2 $\quad$ ≈ 2.68 $\quad$?

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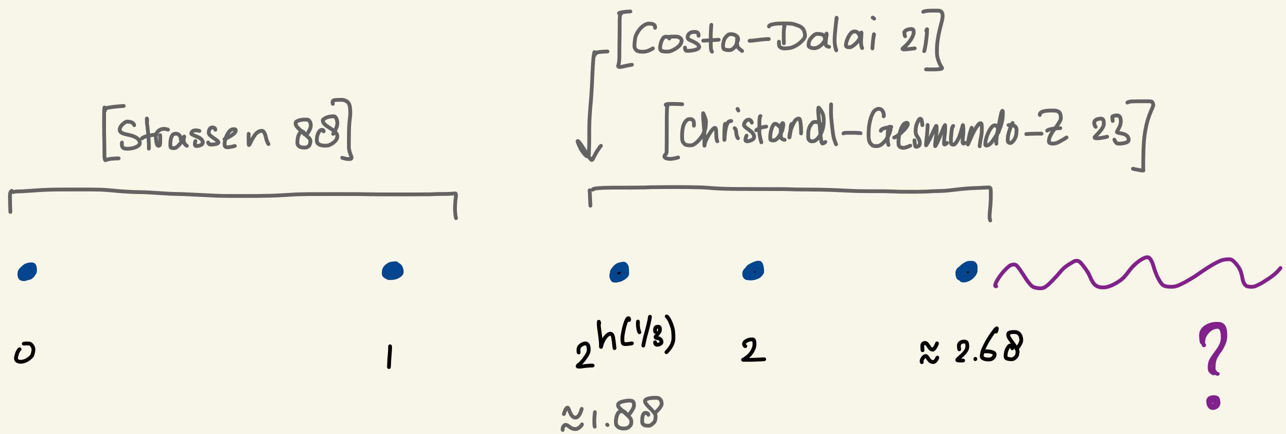

- Countably many values over $\mathbb{C}$
  [Blatter-Draisma-Rupniewski 22a]
- Well-ordered over finite fields (no accumulation points from above)
  [Blatter-Draisma-Rupniewski 22b]

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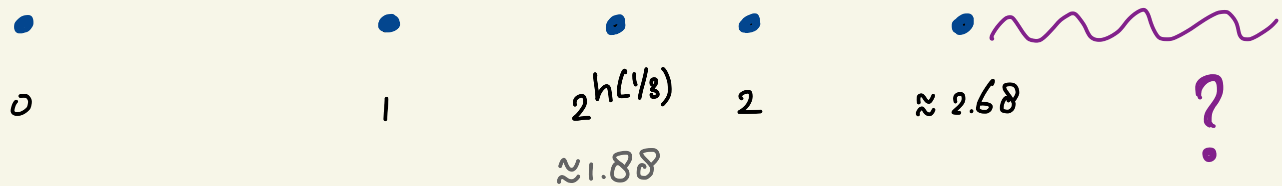

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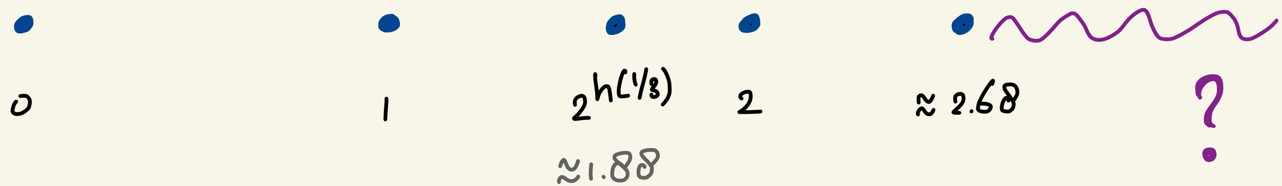

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iii) $T \geq I_2 = e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$ and has a flattening of rank two

iv) all flattenings of T have rank at least three, and

a) $T \trianglerighteq e_0 \otimes e_0 \otimes e_0 + \sum_{i=1}^{2} e_0 \otimes e_i \otimes e_i + e_i \otimes e_0 \otimes e_i$ "null algebra"

b) T is equivalent to $D = e_1 \wedge e_2 \wedge e_3$

3. Discreteness theorem (simple to explain version)

Theorem Over any finite set of coefficients $S \subseteq \mathbb{F}$, the set

$$\left\{ \underset{\sim}{Q}(T) : T \in S^{n_1} \otimes S^{n_2} \otimes S^{n_3},\ n_1, n_2, n_3 \in \mathbb{N} \right\}$$

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Lemma 1 (Big tensors) If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$ is concise, then $\underset{\sim}{Q}(T) \geqslant \min(n_1, n_2, n_3)^{1/3}$.

Lemma 2 (Thin tensors)

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Lemma 2 (Thin tensors) If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{c}$ is concise and $n_1 \geqslant N(c)$, then $\underset{\sim}{Q}(T) = c$.

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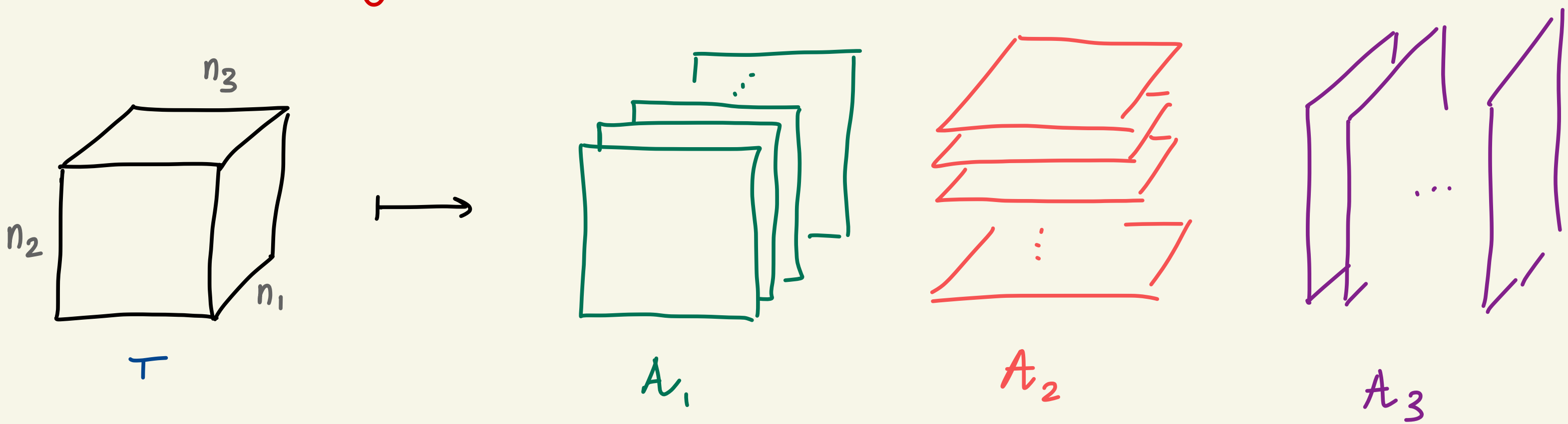
Proof sketch of main result:

Consider infinite sequence $\underset{\sim}{Q}(T_i)$ with $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$ concise.

- If $\min(a_1, b_i, c_1) \to \infty$, then $\underset{\sim}{Q}(T_i) \to \infty$
- If $\max_i c_i = c$, then $a_i \to \infty$ so $\underset{\sim}{Q}(T_i)$ eventually constant $\square$

Lemma 1 Proof ingredient



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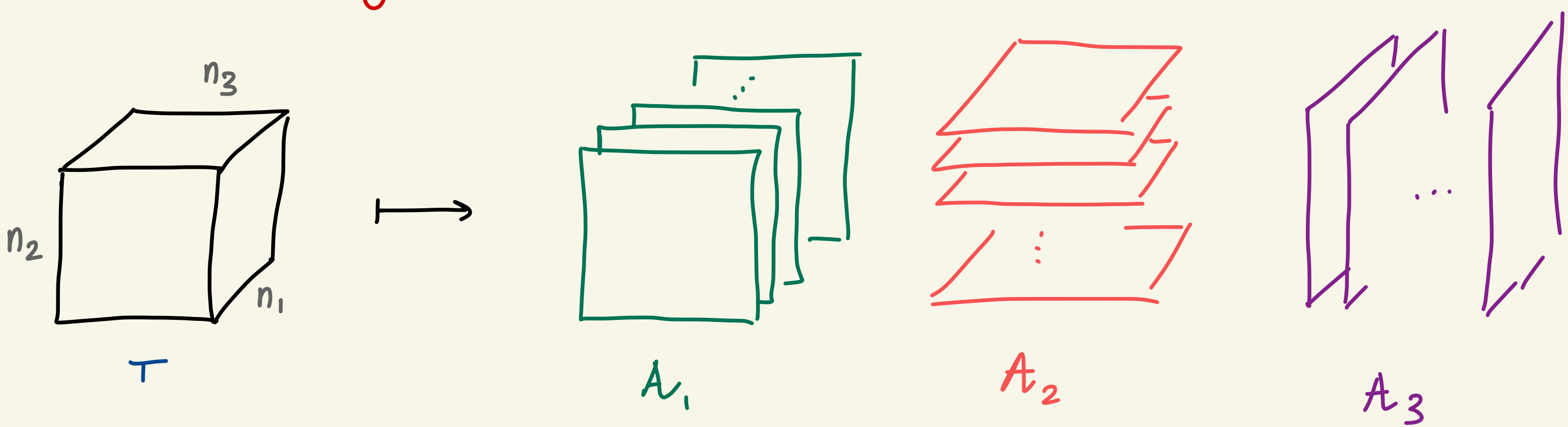

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<u>Lemma</u> For concise $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$, and any distinct $i, j, k \in [3]$,

$$Q_i(T)\, Q_j(T) \geq n_k .$$

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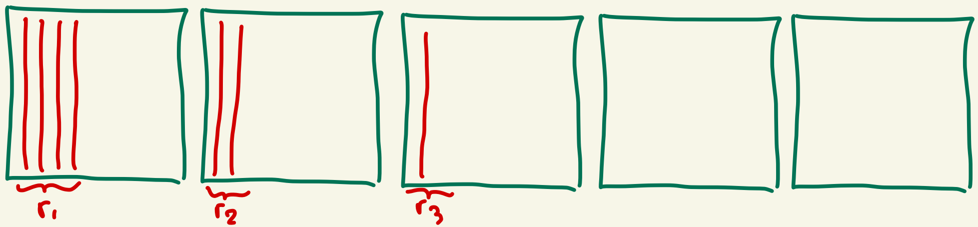
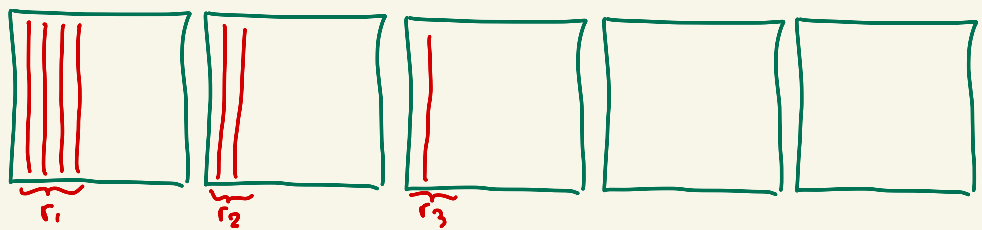


We may assume the first $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$ columns are linearly independent.

Lemma For concise $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$, and any distinct $i, j, k \in [3]$,

$$Q_i(T)\, Q_j(T) \geq n_k .$$

Proof sketch: Apply random basis transformation to T.



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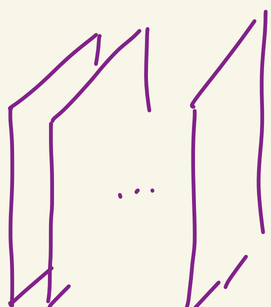

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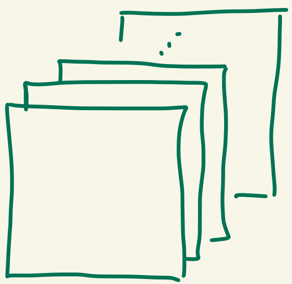

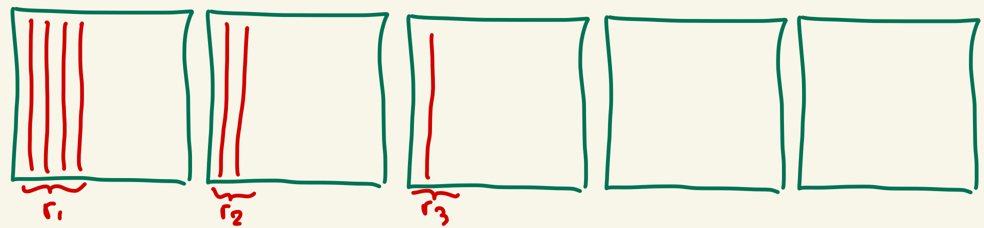

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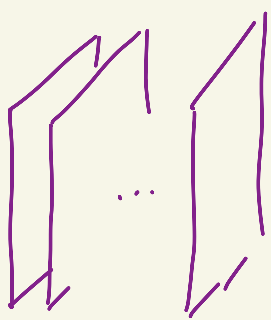

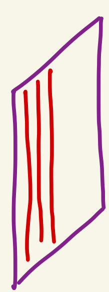

This slice has at least $\#\{i : r_i \neq 0\}$ many lin. independent columns.

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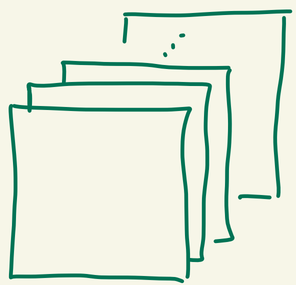

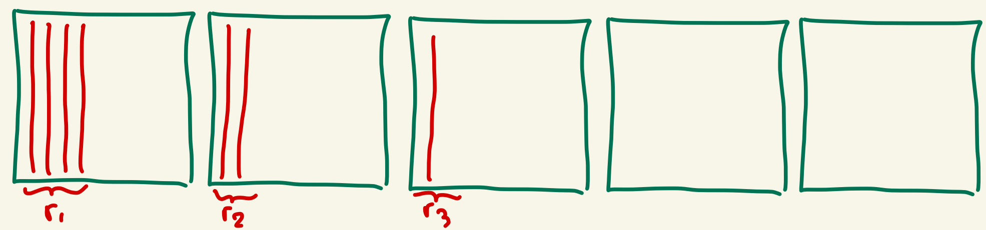

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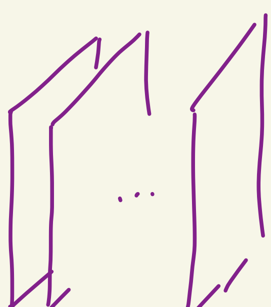

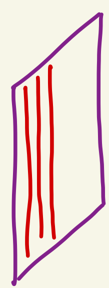

This slice has at least $\#\{i : r_i \neq 0\}$ many lin. independent columns.

$$Q_1(T) \cdot Q_3(T) \geq r_1 \cdot \#\{i : r_i \neq 0\} \geq \textstyle\sum_i r_i = n_2 .$$

□

Lemma 2 Proof ingredient

- $\mathrm{minrank}(\mathcal{A}_i) = \min\{\mathrm{rank}(A) : 0 \neq A \in \mathcal{A}_i\}$
- relation between minrank and subrank
- tensor power tricks

5. General result

Theorem: We have discreteness when

- finite $S \subseteq \mathbb{F}$
  - asymptotic subrank
  - asymptotic slice rank
  - asymptotic tensor rank (simple proof)
- $\mathbb{F} = \mathbb{C}$ for asymptotic slice rank (uses moment polytopes, quantum functionals)
- $\mathbb{F}$ arbitrary
  - asymptotic subrank and asymptotic slice rank for "tight" tensors
  - asymptotic slice rank for "oblique" tensors.

Open problems

1. Is $\underset{\sim}{Q}(T) \geq n^{1/3}$ optimal for concise $n \times n \times n$ tensors?

   For symmetric T, we have a better lower bound $n^{1/2}$.

2. Values of $\underset{\sim}{Q}(T)$

3. Higher order tensors

4. Arbitrary fields